Theodorsen's Method

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Introduction

We will present the details of Theodorsen's method for computing the conformal map $Z = f(z)$ from the exterior of the unit disk $D := \{z : |z| > 1\}$ to the exterior of a NACA 2415 airfoil, call it Ω

$$f : D \to \Omega.$$

We will then calculate the streamlines of a steady flow, at varying angles of attack, around the unit circle and apply Theodorsen's method to confromally map them back to the physical domain.

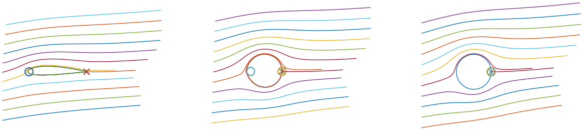


Figure: The following are graphs of the potential flow, assuming a 7° angle of attack.

Next, we will use those results to calculate the pressure curves in the physical domains. Finally, we will compare our results to outputs from a standard engineering code.

Since Theodorsen's Method requires that our curve be nearly circular and the NACA 2415 airfoil is slender and contains a corner at its trailing edge, so we can not apply Theodoren's Method directly.

So we apply Karman-Trefftz which smooths the corner at the trailing edge and maps the exterior of our airfoil to the exterior of a nearly circular curve.

In particular, we will be forming the conformal map

$$f(z) = k^{-1}\left(T(z)\right)$$

where

- $T(z)$ is the numerical conformal map formed by our application of Theodorsen's method from the exterior of the unit disk to the exterior of nearly circular image of the NACA 2415 airfoil under Karman-Trettz
- And, $k^{-1}(z)$ is the inverse map to the NACA 2415 airfoil.

Our contributions:

- Matlab code for starlike parametrization of boundary
- change in normalization of Theodorsen's method
- codes for calculation of streamlines and pressure curves
- assembling and testing code on NACA arifoil

Conformal Maps:

Analytic Functions in the Exterior of the Unit Disk

Theorem

If $h(z)$ *is analytic in* $|z| > 1$*, continuous for* $|z| \geq 1$*, and* $h(\infty) = 0$*, then for* $|z| > 1$

$$h(z) = -\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{h(\zeta)}{\zeta - z} d\zeta,$$

where the unit circle $|\zeta| = 1$ *is traversed in the counterclockwise direction.*

Which we can rewrite as,

$$h(z) = \sum_{k=-1} \hat{h}_{-k} z^{-k}$$

where $\hat{h}_k$ are the Fourier coefficients of $h(e^{i\theta})$.

Conformal Maps:

Karman-Trefftz

Karman-Trefftz Transformation:

$$\frac{\zeta - \zeta_1}{\zeta - \zeta_2} = \left(\frac{z - z_1}{z - z_2}\right)^{1/\beta}$$

- ζ_1 is the map of the corner, z_1
- ζ_2 is the map of an interior point, z_2, near the leading edge
- z_1 is the corner
- z_2 is an interior point near the leading edge
- $\beta\pi$ is the exterior angle at the corner, z_1.

Karman-Trefftz Map

$$k(z) = \frac{\zeta_1 - \zeta_2 \left(\frac{z - z_1}{z - z_2}\right)^{1/\beta}}{1 - \left(\frac{z - z_1}{z - z_2}\right)^{1/\beta}} = \zeta$$

Inverse Karman-Trefftz Map

$$k^{-1}(\zeta) = \frac{z_1 - z_2 \left(\frac{\zeta - \zeta_1}{\zeta - \zeta_2}\right)^{\beta}}{1 - \left(\frac{\zeta - \zeta_1}{\zeta - \zeta_2}\right)^{\beta}} = z$$

Derivative of Karman-Trefftz Map at Infinity

$$k'(\infty) = \beta \frac{\zeta_2 - \zeta_1}{z_2 - z_1}$$

Derivative of Karman-Trefftz Inverse Map at Infinity

$$(k^{-1})'(\infty) = \frac{1}{\beta} \frac{z_2 - z_1}{\zeta_2 - \zeta_1}$$

Numerical Analysis:

Parameterizing the Boundary

Theodorsen's method makes two demands of our data:

1. that the boundary be "star-like" with respect to an interior point(we will choose the origin). That is, that our boundary be of the form,

$$\gamma(\phi) = \rho(\phi)e^{i\phi}, \quad \rho(\phi) > 0, \quad \text{and } \phi \in [0, 2\pi]$$

2. that the boundary be sufficiently close to a circle for the method to converge.

For that reason, prior to applying Theodorsen's Method, we first

1. smoothed the data by applying Karman-Trefftz,
2. centered the image of the data and transformed it from Cartesian to polar coordinates.

We then

1. fitted periodic splines through the data,
2. and formed $\rho(\phi)$ by interpolating the periodic splines.

Numerical Analysis:

Conjugate Periodic Functions

Given the conjugate periodic functions

$$\begin{cases} \phi(\theta) = u(1, \theta) \\ \psi(\theta) = v(1, \theta) - b_0, \end{cases}$$

the *conjugation operator* K is defined

$$\begin{cases} \phi(\theta) = a_0 + \sum_{n=1} a_n \cos(n\theta) + b_n \sin(n\theta) \\ -K\phi(\theta) := \psi(\theta) = \sum_{n=1} -a_n \sin(n\theta) + b_n \cos(n\theta) \end{cases}$$

So $v(1,\theta) - b_0 = -K\phi(\theta) = -F^{-1}\hat{K}Fu(1,\theta)$ and $K = F^{-1}\hat{K}F$, where F and F^{-1} are the Fourier and inverse Fourier transforms and

$$\hat{K} = \begin{cases} a_n \to -b_n \\ a_0 \to 0 \\ b_n \to a_n. \end{cases}$$

For $h(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, K still factors as $K = F^{-1}\hat{K}F$. But in this case

$$\hat{K} = \begin{cases} c_n \to -ic_n & n > 0 \\ c_0 \to 0 & \\ c_n \to ic_n & n < 0, \end{cases}$$

so that $Kh(\theta) = \sum_{n=-\infty}^{-1} ic_n e^{in\theta} - \sum_{n=1}^{\infty} ic_n e^{in\theta}$

Numerical Analysis:

Discrete Fourier Transform(DFT) vs Fast Fourier Transform(FFT)

It it well known that the operation cost of the DFT is $O(N^2)$ compared to that of the FFT $O(N\log(N))$.

It is because of this computational advantage that we used the FFT when computing the Fourier coefficients of $h(e^{i\theta})$.

Theodorsen's Method

Recall, we stated that we require the boundary be represented by the star-like parameterization $\gamma(\phi) = \rho(\phi)e^{i\phi}$ w.r.t. the origin, $\phi \in [0, 2\pi]$, and $0 < \rho(\phi)$.

Theodorsen's method finds the boundary correspondance $\phi = \phi(\theta)$ for a conformal map $Z = f(z)$ from the exterior of the unit disk to the exterior of Ω under the Karman-Trefftz map by successive conjugation.

Using the normalization $f(\infty) = \infty$ and $0 < a_1 = f'(\infty) \in \mathbb{R}$, let's define $h(z)$ to be the *auxiliary function*,

$$h(z) := \log\left(\frac{f(z)}{z}\right).$$

Since $f(e^{i\theta}) = \rho\left(\phi(\theta)\right) e^{i\phi(\theta)}$ we have that,

$$h(e^{i\theta}) = \log\left(\frac{\rho\left(\phi(\theta)\right) e^{i\phi(\theta)}}{e^{i\theta}}\right)$$

$$= \log \rho\left(\phi(\theta)\right) + i\left(\phi(\theta) - \theta\right)$$

Notice, this means $h(e^{i\theta})$ is of the form $u(1,\theta) + iv(1,\theta)$ where

$$\begin{cases} u(1,\theta) = \log \rho\left(\phi(\theta)\right) \\ v(1,\theta) = \phi(\theta) - \theta \end{cases} .$$

Now, we apply the conjugation operator K to the real and imaginary parts of $h(e^{i\theta})$. And, as $v(1,\theta) = 0 = b_0$, we get *Theodorsen's equation* for the exterior case,

$$\phi(\theta) - \theta = -K \log \rho\left(\phi(\theta)\right) \qquad (1)$$

Theodorsen's Iteration:

$$\phi^{(0)}(\theta) - \theta = 0 \qquad \text{(initial guess)}$$

$$\phi^{(n+1)}(\theta) - \theta = -K \log \rho \left(\phi^{(n)}(\theta) \right) \tag{2}$$

Theodorsen's Method:

Normalization of Theodorsen's Method

The normalization for the Theodorsen map $\zeta = f(z)$ from the exterior of the disk to the exterior of the closed curve, $\gamma(\phi) = \rho(\phi)e^{i\phi}$, is

$$f(z) = a_1 z + \sum_{j=0}^{\infty} \frac{a_{-j}}{z^j},$$

where $a_1 > 0$.

In order to apply the Kutta condition, we need to rotate the unit circle so that $f(1)$ maps to the image of the trailing edge on the smooth curve $f(1) = \zeta_1 - c = \rho(1)$, where c is the center of the smooth curve, since our starlike parametrization of the boundary has $\rho(0) = \rho_1$. We, therefore, need to find the angle θ_{tr}, where

$$f(e^{i\theta_{tr}}) = \rho_1.$$

We tried two methods:

- Solving $f(e^{i\theta_{tr}}) = \rho_1$ for $e^{i\theta_{tr}}$ by Newton's method
- interpolating the inverse $\theta = \theta(\phi)$ of the boundary correspondence at the Fourier points, $(\theta_i = 2\pi(i-1)/N, \phi_i)$, $i = 1, \ldots, N$, $(2\pi, \phi_N + 2\pi)$, computed by the Theordorsen iterations, with a cubic spline and setting $\theta_{tr} = \theta(0)$, if $\phi_1 < 0$, and $\theta_{tr} = \theta(2\pi)$, if $\phi_1 > 0$.

Both methods yielded similar results:

N	Spline	Newton
32	-0.018200117390517	-0.018182051778078
64	-0.018388548886676	-0.018425772325956
128	-0.018419768716229	-0.018416359631915
256	-0.018416494188328	-0.018416443147069
512	-0.018416413704615	-0.018416419384364

Table: The values of the angle of the image of the trailing edge on the unit circle

Note that the number of correct digits increases by about 1 each time N is double indicating the roughly $O(N^{-3})$ accuracy expected by the spline fits.

Plotting Exterior of the Unit Circle to the Exterior of a NACA 2415 Airfoil using Theodorsen's Method

Now, we will apply all this together on data provided by a graduate student in the areospace department using 128 Fourier points and a maximum of 100 iterations

Convergence of Theodorsen's Method of Data:

```
For n = 2 ^ m Fourier points, m = 7
Enter maximum number of iterations, itmax = 100
Iteration no.  Error between successive iterates
     1          9.03959053963086e-02
     2          1.88101169449113e-02
     3          2.85558454886958e-03
     4          6.86770436595818e-04
     5          1.67327439827059e-04
     6          4.27456081064648e-05
     7          1.09123161293745e-05
     8          2.71404209550852e-06
     9          7.42707724654679e-07
    10          1.75019060577597e-07
    11          5.16105633785457e-08
    12          1.14784564075876e-08
    13          3.63532715041970e-09
    14          7.65816743353298e-10
    15          2.58713939160771e-10
    16          5.74198466551934e-11
    17          1.85642612393622e-11
    18          4.32054392263126e-12
    19          1.34070532453734e-12
    20          3.23741033980696e-13
    21          9.76996261670138e-14
    22          2.44249065417534e-14
    23          7.10542735760100e-15
    24          1.77635683940025e-15
    25          8.88178419700125e-16
    26          8.88178419700125e-16
    27          8.88178419700125e-16
    28          8.88178419700125e-16
```

Figure: Theodorsen's method converged to within machine epsilon in 25 iterations on our data.

The Data Plotted:

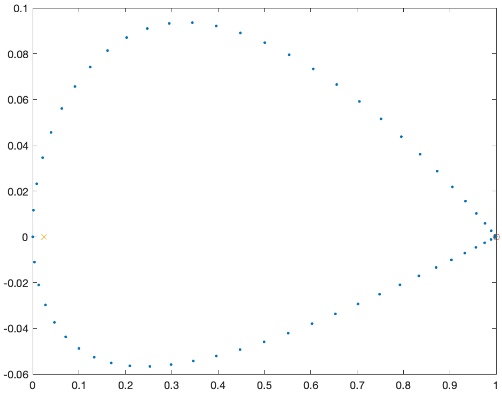


Figure: “x” is the interior point by Karman-Trefftz and “o” is the *sharp* boundary point

Karman-Trefftz Applied to the Interpolated Data:

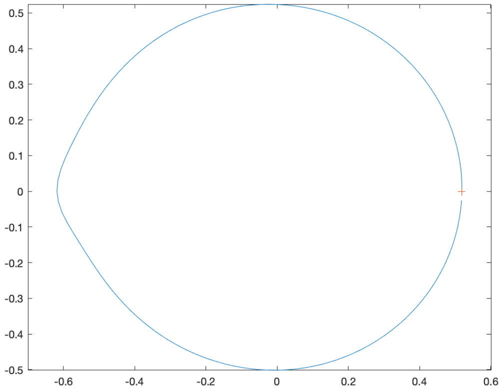
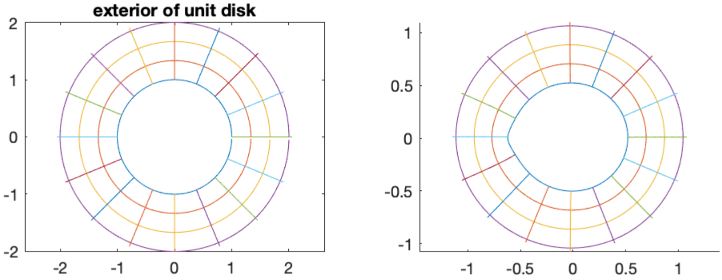
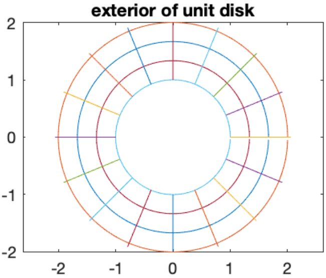


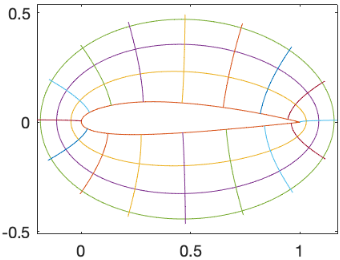
Figure: "+" is the mapped *sharp* boundary point

The Result of Theodorsen's Method from the Exterior of the Unit Disk to the Data Under Karman-Trefftz:



The Result of Theodorsen's Method and Inverse Karman-Trefftz from the Exterior of the Unit Disk to the Data:





Potential Flow and Pressure Curves

Where $w(z)$ is the complex velocity potential, using `ode45`, we will now numerically solve the system

$$\begin{cases} \frac{dx}{dt} = \text{Re}\{w'(z)\} \\ \frac{dy}{dt} = -\text{Im}\{w'(z)\} \end{cases}$$

and plot the resulting streamlines.

To calculate pressure curves we will also apply Bernoulli's equation,

Theorem

As the velocity of a fluid $\left|\frac{dw}{dz}\right|$ *increases the pressure* P *within the fluid decreases according to the following relation,*

$$P + \frac{\rho |dw/dz|^2}{2} = C.$$

Potential Flow and Pressure Curves:

Angle of attack 0°

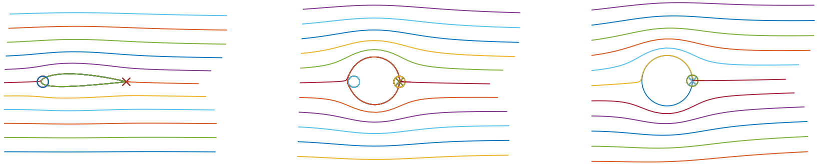


Figure: Assuming a 0° angle of attack.

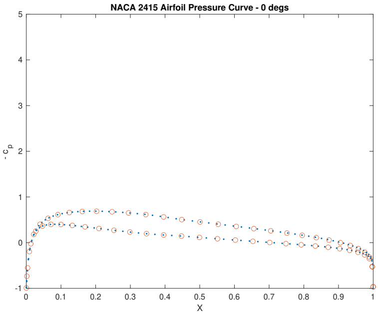


Figure: The following graph is of the predicted(·) pressure about the airfoil vs the measured(∘) pressure about the airfoil given a 0° angle of attack

Potential Flow and Pressure Curves:

Angle of attack 2°

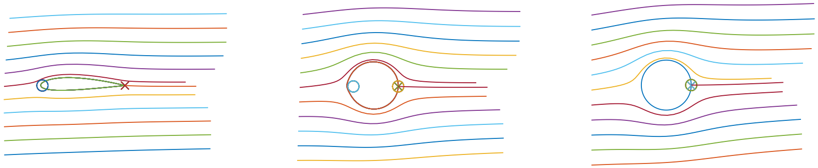


Figure: Assuming a 2° angle of attack.

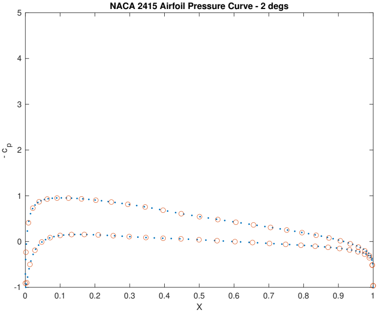


Figure: The following graph is of the predicted(·) pressure about the airfoil vs the measured(○) pressure about the airfoil given a 2° angle of attack

Potential Flow and Pressure Curves:

Angle of attack 5°

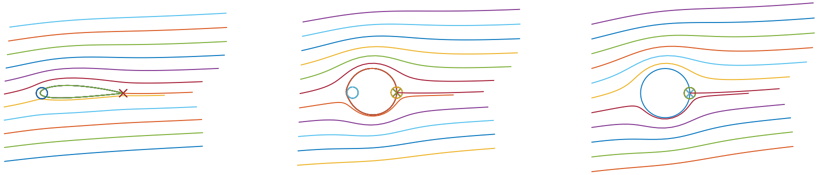


Figure: Assuming a 5° angle of attack.

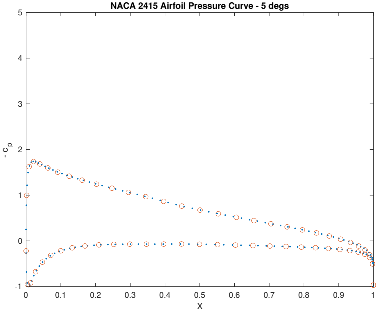


Figure: The following graph is of the predicted(·) pressure about the airfoil vs the measured(○) pressure about the airfoil given a 5° angle of attack

Potential Flow and Pressure Curves:

Angle of attack 10°

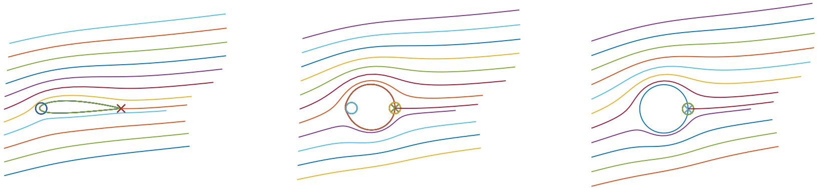


Figure: Assuming a 10° angle of attack.

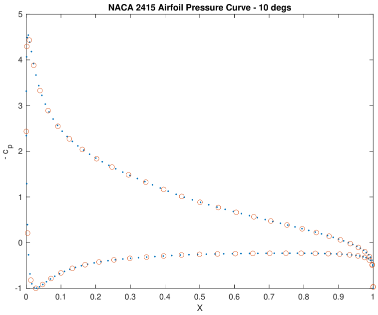


Figure: The following graph is of the predicted(·) pressure about the airfoil vs the measured(○) pressure about the airfoil given a 10° angle of attack

Thank You