Review Problems

#1 Given a definition of the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}^n$. Let $\mathcal{I}$ be the smallest σ-algebra containing all open cubes in $\mathbb{R}^n$. Show that $\mathcal{I} = \mathcal{B}$.

Definition (Borel σ-Algebra):

The smallest σ-algebra containing all open sets.

Definition (σ-Algebra):

The collection of subsets of X, Σ, is called a σ-Algebra if,

i) $E^c \in \Sigma, \forall E \in \Sigma$

ii) $\bigcup_k E_k \in \Sigma, \forall E_k \in \Sigma \quad (k=1,2,\dots)$

iii) $X \in \Sigma$.

Proof:

(Next PG)

Proof:

$\Longrightarrow (\Sigma \subset \mathcal{B})$

As $\mathcal{B}$ contains all open sets of $\mathbb{R}^n$, it is clear, all open cubes. Furthermore as Σ is the smallest σ-alg. containing said cubes, it follows by definition that $\Sigma \subset \mathcal{B}$.

$\Longleftarrow (\mathcal{B} \subset \Sigma)$

Since Σ is a σ-alg containing all open cubes in $\mathbb{R}^n$, it suffices to show $\mathbb{R}^n$ is second-countable.

Since, $\mathbb{R}^n$ is the countable product of $\mathbb{R}$, a second-countable space, it follows that $\mathbb{R}^n$ is second-countable.

Hence by the property $\bigcup_k E_k \in \Sigma$ of σ-algebras we get that the basis of open sets are open cubes in Σ.

Thus $\mathcal{B} \subset \Sigma$

Review Problem

#2 Let f be a function on $\mathbb{R}^n$. Show that f is measurable if and only if for every open set G in $\mathbb{R}^1$, the inverse image $f^{-1}(G)$ is a measurable subset of $\mathbb{R}^n$.

(pg 51)

Proof:

$\Leftarrow$

If $G = (a, \infty)$, then $f^{-1}(G) = \{x \mid a < f < \infty\}$. Since $f^{-1}(G)$ is measurable for every such G, then f is measurable.

$\Rightarrow$

Suppose f is measurable and G be any open subset of $\mathbb{R}^1$. Since every open set in $\mathbb{R}^1$ can be written as a countable union of disjoint open intervals (by Thrm 1.10), we have that

$$G = \bigcup_k (a_k, b_k).$$

However $f^{-1}((a_k, b_k)) = \{x \mid a_k < f < b_k\}$. Thus, by definition, $f^{-1}((a_k, b_k))$ is measurable for each k. Since

$$f^{-1}(G) = \bigcup_k f^{-1}((a_k, b_k))$$

it follows that $f^{-1}(G)$ is measurable.

Review Problems

#3 Suppose $E \subset \mathbb{R}^n$ with finite outer measure. Show that E is measurable iff given $\varepsilon > 0$, $E = (S \cup N_1) \sim N_2$, where S is a finite union of non-overlapping intervals & the outer measure of N_1 & N_2 are less than ε.

Proof:

$\Rightarrow$

Assume E is measurable, and let $\varepsilon > 0$. There is an open G containing E s.t. $|G - E|_e < \varepsilon$.

Then G can be written $G = \bigcup_k I_k$ where I_k are non-overlapping intervals. Hence,

$$|G| = \sum_{k=0}^{\infty} |I_k| < \infty.$$

There is an $N \in \mathbb{N}$ s.t. $|\bigcup_{k=N+1}^{\infty} I_k| < \varepsilon$.

Now, let's define $S = \bigcup_{k=0}^{N} I_k$, $N_1 = \bigcup_{k=N+1}^{\infty} I_k$ and $N_2 = G - E$. Then it follows that $|N_1|_e, |N_2|_e < \varepsilon$ & we have

$$E = G - (G - E) = (S \cup N_1) - N_2$$

as $E \subseteq G$.

$\Leftarrow$

(Next Pg)

$\Leftarrow$

Suppose for each $\varepsilon > 0$, $E = (S \cup N_1) - N_2$ with S a finite union of intervals, and $|N_1|_e, |N_2|_e < \varepsilon/4$.

Since $|N_1|_e < \varepsilon/4$, then there exist a set U s.t. $N_1 \subset U$ + $|U| < 3\varepsilon/4$. Let $U \supset G \supset S$ so that $U = G \cup S$, then

$$G - E = (S \cup G) - ((S \cup N_1) - N_2)$$

$$\subseteq (U - N_1) \cup N_2.$$

So, $|G - E|_e \leq |(U - N_1) \cup N_2|_e$ and we get

$$|G - E|_e \leq |U - N_1|_e + |N_2|_e < \varepsilon.$$

Thus E is measurable.

Review Problems

#4 Define convergence in measure. Let $\{f_k\}$ be a sequence of measurable functions on a measurable set $E \subset \mathbb{R}^n$. Suppose for any $\varepsilon > 0$, there exist K s.t.

$$|\{|f_k - f| > \varepsilon\}| < \varepsilon, \quad \text{if } k > K,$$

show that $f_k \to f$ in measure.

Definition:

Let f & f_k be measurable functions which are defined a.e. in a set E. Then $\{f_k\}$ is said to converge in measure on E to f if for every $\varepsilon > 0$,

$$\lim_{k \to \infty} |\{x \in E : |f(x) - f_k(x)| > \varepsilon\}| = 0$$

Written as,

$$f_k \xrightarrow{m} f.$$

#

Proof:

Let $\alpha > 0$. Then, let K_α be such that

$$|\{x : |f(x) - f_k(x)| > \alpha\}| < \alpha \quad \text{if } k > K_\alpha.$$

We're given $\varepsilon > 0$; so, let K_ε be such that

$$|\{x : |f(x) - f_k(x)| > \varepsilon\}|, \quad \text{if } k > K_\varepsilon.$$

Let $\delta = \min(\alpha, \varepsilon)$ and $k > \max(K_\alpha, K_\varepsilon)$. Then

$$\{x : |f(x) - f_k(x)| > \alpha\} \subset \{x : |f(x) - f_k(x)| > \varepsilon\}.$$

Thus we have

$$|\{x : |f(x) - f_k(x)| > \alpha\}| \leq |\{x : |f(x) - f_k(x)| > \varepsilon\}| \leq \delta \leq \varepsilon.$$

Therefore, $f_k \xrightarrow{m} f$.

Review Problems

#5 Let $\{f_K\}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$. Suppose for each $x \in E$, there is a $\delta_x > 0$ s.t. $f_K(x) > \delta_x$ for all K, show that given $\varepsilon > 0$, there is a closed $F \subset E$ + $\delta > 0$ s.t. $|E - F| < \varepsilon$ and $f_K(x) > \delta$ for all K + all $x \in F$.

Proof:

Let $F(x) = \sup_K f_K(x)$ + define

$$E_m = \{x \in E : \delta_x < F(x) < m, m \in \mathbb{N}\}$$

Notice, as $\{f_K\}$ is a sequence of measurable functions, it follows $F(x)$ is measurable. Since $F(x)$ is finite valued, then E_m increases to E. Hence $|E_m|$ increases to $|E|$, + $|E - E_m| \to 0$ as $m \to \infty$.

Given $\varepsilon > 0$. There exist $\delta_x > 0$ s.t. $|E - E_m| < \varepsilon/2$. Further, there is closed F in E_m s.t. $|E_m - F| < \varepsilon/2$. Hence $|E - F| \leq |E - E_m| + |E_m - F| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus $|E - F| < \varepsilon$ and $f_K(x) > \delta_x$ for all K + all $x \in E$.

Review Problems

#6a State Fatou's Lemma for measurable functions.

Definition:

If $\{f_k\}$ is a sequence of measurable functions on E, then

$$\liminf \int_E f_k \geq \int_E \liminf f_k.$$

Review Problems

#6b Use Fatou's Lemma to prove the Lebesgue's Dominated Convergence Theorem

Proof:

Review Problems

#7 Let $\{f_k\}$ be a sequence of nonnegative measurable defined on a measurable set E. If $f_k \to f$ and $f_k \le f$ a.e. on E, $\int_E f_k \to \int_E f$.

Proof:

By Fatou's,

$$\int_E f = \int_E \liminf f_k \le \liminf \int_E f_k$$

So we need only show that

$$\limsup \int_E f_k \le \int_E f.$$

Since $f_k \le f$ a.e. on E, then $f - f_k \ge 0$; and,

$$\int_E \liminf (f - f_k) \le \liminf \int_E f - f_k.$$

And as $f_k \to f$ a.e. on E, then

$$\int_E \liminf (f - f_k) \le \liminf \int_E (f - f_k)$$

$$\Rightarrow 0 \le \int_E f - \limsup \int_E f_k.$$

Hence $\limsup \int_E f_k \le \int_E f$ and our proof.

#

Review Problems

#8 Let $f \in \mathcal{L}(0,1)$. Prove the following

$$\lim_{k\to\infty} k\int_0^1 \ln\left(1+\frac{|f(x)|^2}{k^2}\right) = 0.$$

Proof:

Observe, $(1+t) \le e^{2\sqrt{t}}$ for $0 \le t$, which yields $\ln(1+t) \le 2\sqrt{t}$.

From our observation we find

$$k\ln\left(1+\frac{|f(x)|^2}{k^2}\right) \le 2|f(x)|.$$

And since $\left(1+\frac{x}{n}\right)^n \le e^x$ we find that

$$k\ln\left(1+\frac{|f(x)|^2}{k^2}\right) \le \frac{1}{k}\ln\left(e^{|f(x)|^2}\right) = \frac{1}{k}|f(x)|^2$$

Hence,

$$k\ln\left(1+\frac{|f(x)|^2}{k^2}\right) \le \frac{1}{k}|f(x)|^2,$$

and as $k\to\infty$ $\frac{1}{k}|f(x)|^2 \longrightarrow 0$. So by the Lebesgue Dominated Conv. Thm we find,

$$\lim_{k\to\infty} k\int_0^1 \ln\left(1+\frac{|f(x)|^2}{k^2}\right) = \int_0^1 \lim_{k\to\infty}\frac{1}{k}|f(x)|^2 = 0.$$

Review Problems

#9 Let $f_k, f \in \mathcal{L}(E)$ with $|E|$ finite. Suppose $0 \le f_k$ & $f_k \to f$ a.e. in E. Prove

$$\lim_{k\to\infty}\int_E f_k e^{-f_k} = \int_E f e^{-f}.$$

Proof:

First observe for $0 \le t$ we have $0 \le t e^{-t} < 1/e$. So, as $0 \le f_k$ we have $0 \le f_k e^{-f_k} < 1$. Hence the integrand is bounded & L.D.C.T. can be applied. So, since $f_k \to f$

$$\lim_{k\to\infty}\int_E f_k e^{-f_k} = \int_E \lim_{k\to\infty} f_k e^{-f_k} = \int_E f e^{-f}.$$

Review Problems

#10 Let f be a bounded function in $[a,b]$. Show that if f is Riemann integrable on $[a,b]$, then f is cont. a.e. in $[a,b]$.

Review Problems

#11 Let $f(x,y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x,y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}$ $f(x,y)$ is finite for almost every x.

Proof:

Let $E = \{(x,y) : f(x,y) = \infty\}$, $E_x = \{y : f(x,y) = \infty\}$, and $E^y = \{x : f(x,y) = \infty\}$. Note, by setup of the problem $|E_x| = 0$. And so, by Tonellie's theorem we find

$$\int_E \chi_E(x,y) = \int_{E_x} \left(\int_{E^y} \chi(x,y)\, dx \right) dy =$$

$$\int_{E^y} \left(\int_{E_x} \chi(x,y)\, dy \right) dx =$$

$$= \int_{E^y} |\{y : f(x,y) = \infty\}|\, dx = \int_{E^y} 0\, dx = 0.$$

But if $\int_E \chi_E(x,y) = 0$ then

$$\int_{E_x} \left(\int_{E^y} \chi(x,y)\, dx \right) dy = \int_{E_x} |\{x : f(x,y) = \infty\}|\, dy = 0.$$

Hence we find E^y has measure 0, and our proof.

Review Problem

#12 Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}.$$

Proof:

We set, $I_n = \int_{\mathbb{R}^n} e^{-|x|^2} dx$.

- $n=1$

$$I_1^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2-y^2} dx dy = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} dr d\theta$$

Setting $\rho = r^2$,

$$I_1^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\rho} \tfrac{1}{2} d\rho d\theta = \pi.$$

Hence $I_1 = \pi^{1/2}$.

- $n = k-1$

Assume,

$$I_{k-1} = \pi^{\frac{k-1}{2}} = \prod_1^{k-1} I_n.$$

- $n = k$

$$I_n = \int_{\mathbb{R}^k} \prod_1^k (e^{-x_n^2} dx_n) =$$

$$= \int_{\mathbb{R}} e^{-x_k^2} \left(\prod_1^{k-1} I_n\right) dx_k = \pi^{k/2}.$$

#

Review Problems

#13 Let f be an essentially bounded function defined on a measurable set E with $|E|<\infty$. Show that $f \in L^p(E)$ for all $p>0$ and $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. Also demonstrate by examples that this result fail if $|E|=\infty$.

- Proof: ($f \in L^p(E)$, $0<p$)

Since f is essentially bounded on E we have

$$\operatorname{ess\,sup}_E |f| = \inf\{\alpha : |\{x\in E : f(x)>\alpha\}| = 0\} = M.$$

And, as $|E|$ is finite it follows that

$$\int_E |f|^p \le \int_E M^p < \infty, \quad p\in(0,\infty).$$

Hence, $f \in L^p(E)$ for $p>0$.

#

Proof: ($\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$)

Let $M = \|f\|_\infty$. If $M' < M$, then

$$A = \{x \in E : |f(x)| > M'\}$$

has positive measure. Moreover,

$$\left(\int_E |f|^p\right)^{1/p} \geq \left(\int_A |f|^p\right)^{1/p} \geq \left(\int_A M'^p\right)^{1/p}$$
$$= M' |A|^{1/p}.$$

Since $|A|^{1/p} \to 1$ as $p \to \infty$, then $\liminf \|f\|_p \geq M'$ so that $\liminf \|f\|_p \geq M$.
Since $f \leq M$ a.e., then

$$\|f\|_p = \left(\int_E |f|^p\right)^{1/p} \geq \left(\int_A M^p\right)^{1/p} = M|A|^{1/p}.$$

Hence, $\limsup \|f\|_p \leq M$ and $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Proof: (Example of fail for $|E| = \infty$)

As a specific example, let $f(x) = 1$ $\forall E = \mathbb{R}$.

$\|1\|_\infty = \text{ess sup}(1) = 1 \neq \left(\int_{\mathbb{R}} |1|^p\right)^{1/p} = \infty$.

Hence $\|1\|_\infty \neq \lim_{p \to \infty} \|1\|_\infty$.

Review Problems

\#14 Let f be measurable on $[0,1]$. For $p \in [1, \infty)$ define

$$g(p) = \left(\int_0^1 |f(x)|^p dx\right)^{1/p}.$$

Show that g is nondecreasing on $[1,\infty)$. Assume in addition that $f \notin L^\infty(0,1)$. Show that $\lim_{p\to\infty} g(p) = \infty$.

Proof: (g nondecreasing)

Observe for each $p + q$, $p<q$, there exist a p' s.t. $p/q + 1/p' = 1 \Rightarrow p' = \frac{q-p}{q}$. From this and an application of Hölder's Inequalities we find,

$$\int_0^1 |f(x)|^p \cdot 1 \, dx \leq \left(\int_0^1 (|f(x)|^p)^{q/p}\right)^{p/q} \left(\int_0^1 |1|^{p'}\right)^{1/p'} = \left(\int_0^1 |f(x)|^q\right)^{p/q}.$$

Hence

$$\left(\int_0^1 |f(x)|^p\right)^{1/p} \leq \left(\int_0^1 |f(x)|^q\right)^{1/q}$$

$$\Longrightarrow g(p) \leq g(q) \quad \text{on } [1, \infty).$$

\#

Proof: $\left(\lim_{p\to\infty} g(p) = \infty\right)$

Define $|E| = |\{x : |f(x)| > \varepsilon > 0\}|$. Then

$$g(p) \geq \left(\int_E |f(x)|^p\right)^{1/p} \geq \left(\int_E \varepsilon^p\right)^{1/p} = \varepsilon |E|^{1/p}.$$

Taking the limit of both sides

$$\lim_{p\to\infty} g(p) \geq \lim_{p\to\infty} \varepsilon |E|^{1/p} = \varepsilon.$$

And as ε is arbitrary let $\varepsilon \to \infty$ and we have our proof.

Review Problems

#15 Let f, $\{f_k\} \in L^p(E)$, where $E \subset \mathbb{R}^n$ is a measurable set. Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$. Conversely, if $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, show that $\|f - f_k\|_p \to 0$