Review Problems for the Real Analysis Qual Exam

1. Give a definition of the Borel σ-algebra of $\mathscr{B}$ of $\mathbb{R}^n$. Let $\mathscr{I}$ be the smallest σ-algebra containing all the open cubes in $\mathbb{R}^n$. Show that $\mathscr{I} = \mathscr{B}$.

2. Let f be a function on $\mathbb{R}^n$. Show that f is measurable if and only if for every open set G in $\mathbb{R}^1$, the inverse image $f^{-1}(G)$ is a measurable subset of $\mathbb{R}^n$.

3. Suppose $E \subset \mathbb{R}^n$ with finite outer measure. Show that E is measurable if and only if given $\epsilon > 0$, $E = \left(S \cup N_1\right) \sim N_2$, where S is a finite union of non-overlapping intervals and the outer measure of N_1 and N_2 are less than ϵ. (Thrm 3.29, Wheeden pg 43)

4. Define convergence in measure. Let $\{f_k\}$ be a sequence of measurable functions on a measurable set $E \subset \mathbb{R}^n$. Suppose for any $\epsilon > 0$, there exist K such that $|\{|f_k - f| > \epsilon\}| < \epsilon$ if $k > K$, show that $f_k \to f$ in measure.

5. Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$. Suppose for each $x \in E$, there is a $\delta_x > 0$ such that $|E \sim F| < \epsilon$ and $f_k(x) > \delta$ for all k and all $x \in F$. (#15, Wheeden pg 62)

6. State Fatou's Lemma for measurable functions. Use Fatou's Lemma to prove Lebesgue's Dominated Convergence Theorem:
   Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}^n$ such that $f_k \to f$ a.e. in E, and there exists $\varphi \in L(E)$ such that $|f_k| \leq \varphi$ a.e. in E for all k. Then $\int_E f_k \to \int_E f$.

7. Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on a measurable set E. If $f_k \to f$ and $f_k \leq f$ a.e. on E, show that $\int_E f_k \to \int_E f_k$.

8. Let $f \in L(0,1)$. Prove the following:

$$\lim_{k\to\infty} k \int_0^1 \ln\left(1+\frac{|f(x)|^2}{k^2}\right) = 0.$$

9. Let $f_k, f \in L(E)$ with $|E|$ finite. Suppose $0 \le f_k$ and $f_k \to f$ a.e. in E. Prove

$$\lim_{k\to\infty} \int_E f_k e^{-f_k} = \int_E f e^{-f}.$$

10. Let f be a bounded function in $[a, b]$. Show that if f is Riemann integrable on $[a, b]$, then f is continuous a.e. in $[a, b]$.

11. Let $f(x,y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x,y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}$, $f(x,y)$ is finite for almost every x.

12. Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{\frac{n}{2}}$$

13. Let f be an essentially bounded function defined on a measurable set E with $|E| < \infty$. Show that $f \in L^p(E)$ for all $p > 0$ and $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. Also demonstrate by examples that this result fail if $|E| = \infty$.

14. Let f be measurable on $[0,1]$. For $1 \leq p < \infty$ define

$$g(p) = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Show that g is non-decreasing on $[1,\infty)$. Assume in addition that $f \notin L^\infty(0,1)$. Show that

$$\lim_{p \to \infty} g(p) = \infty.$$

15. Let f, $\{f_k\} \in L^p(E)$, where $E \subset \mathbb{R}^n$ is a measurable set. Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$. Conversely, if $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, show that $\|f - f_k\|_p \to 0$.

1. Give a definition of the Borel σ-algebra of $\mathscr{B}$ of $\mathbb{R}^n$. Let $\mathscr{I}$ be the smallest σ-algebra containing all the finite intervals of the type $(a, b]$. Show that $\mathscr{I} = \mathscr{B}$.

2. Define convergence in measure. Let $\{f_k\}$ be a sequence of measurable functions on a measurable set $E \subset \mathbb{R}^n$. Suppose for any $\epsilon > 0$, there exist K such that $|\{|f_k - f| > \epsilon\}| < \epsilon$ if $k > K$, show that $f_k \to f$ in measure.

3. Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on a measurable set E. If $f_k \to f$ and $f_k \le f$ a.e. on E, show that $\int_E f_k \to \int_E f_k$.

4. Let f be a bounded function in $[a,b]$. Show that if f is Riemann integrable on $[a,b]$, then f is continuous a.e. in $[a,b]$.

5. Let f be an essentially bounded function defined on a measurable set E with $|E| < \infty$. Show that $f \in L^p(E)$ for all $p > 0$ and $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. Also demonstrate by examples that this result fail if $|E| = \infty$.

6. Give an example of a sequence of functions $\{f_k\}$ on $[0,1]$ so that f_k converges in measure but f_k diverges poitwisely at every point of $[0,1]$.

RA Practice Qual Exam #2

1. Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{\frac{n}{2}}$$

2. Let $f(x,y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x,y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}$, $f(x,y)$ is finite for almost every x.

3. State Fatou's Lemma for measurable functions. Use Fatou's Lemma to prove Lebesgue's Dominated Convergence Theorem:
   Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}^n$ such that $f_k \to f$ a.e. in E, and there exists $\varphi \in L(E)$ such that $|f_k| \leq \varphi$ a.e. in E for all k. Then $\int_E f_k \to \int_E f$.

4. Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$. Suppose for each $x \in E$, there is a $\delta_x > 0$ such that $|E \sim F| < \epsilon$ and $f_k(x) > \delta$ for all k and all $x \in F$. (#15, Wheeden pg 62)

5. Let $f_k, f \in L(E)$ with $|E|$ finite. Suppose $0 \le f_k$ and $f_k \to f$ a.e. in E. Prove

$$\lim_{k\to\infty} \int_E f_k e^{-f_k} = \int_E f e^{-f}.$$

6. Give a definition of the Borel σ-algebra of $\mathscr{B}$ of $\mathbb{R}^n$. Let $\mathscr{I}$ be the smallest σ-algebra containing all the finite intervals of the type (a, b). Show that $\mathscr{I} = \mathscr{B}$.

RA Practice Qual Exam #3

1. Let f be a bounded function in $[a,b]$. Show that if f is Riemann integrable on $[a,b]$, then f is continuous a.e. in $[a,b]$.

2. Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x, y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}$, $f(x, y)$ is finite for almost every x.

3. Let $f \in L(0,1)$. Prove the following:

$$\lim_{k\to\infty} k \int_0^1 \ln\left(1 + \frac{|f(x)|^2}{k^2}\right) = 0.$$

4. Give a definition of the Borel σ-algebra of $\mathscr{B}$ of $\mathbb{R}^n$. Let $\mathscr{I}$ be the smallest σ-algebra containing all the finite intervals of the type $[a, b)$. Show that $\mathscr{I} = \mathscr{B}$.

5. Let $\{f_k\}$ be a sequence of measurable functions defined on a ,easurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$. Suppose for each $x \in E$, there is a $\delta_x > 0$ such that $|E \sim F| < \epsilon$ and $f_k(x) > \delta$ for all k and all $x \in F$. (#15, Wheeden pg 62)

6. Let f be a function on $\mathbb{R}^n$. Show that f is measurable if and only if for every open set G in $\mathbb{R}^1$, the inverse image $f^{-1}(G)$ is a measurable subset of $\mathbb{R}^n$.

RA Practice Qual Exam #4

1. Let f, $\{f_k\} \in L^p(E)$, where $E \subset \mathbb{R}^n$ is a measurable set. Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$. Conversely, if $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, show that $\|f - f_k\|_p \to 0$.

2. Define convergence in measure. Let $\{f_k\}$ be a sequence of measurable functions on a measurable set $E \subset \mathbb{R}^n$. Suppose for any $\epsilon > 0$, there exist K such that $|\{|f_k - f| > \epsilon\}| < \epsilon$ if $k > K$, show that $f_k \to f$ in measure.

3. Suppose $E \subset \mathbb{R}^n$ with finite outer measure. Show that E is measurable if and only if given $\epsilon > 0$, $E = \left(S \cup N_1\right) \sim N_2$, where S is a finite union of non-overlapping intervals and the outer measure of N_1 and N_2 are less than ϵ. (Thrm 3.29, Wheeden pg 43)

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5. Give a definition of the Borel σ-algebra of $\mathscr{B}$ of $\mathbb{R}^n$. Let $\mathscr{I}$ be the smallest σ-algebra containing all the finite intervals of the type $[a, b]$. Show that $\mathscr{I} = \mathscr{B}$.

6. Let $\{\varphi_k\}$ be an orthonormal basis in $L^2(I)$, where $I = (0,1)$, and $f \in L^2(I)$. Show that

$$\lim_{k\to\infty} \int_0^1 f(x)\varphi_k = 0$$