Review Problems for the NLA Qual Exam

1. **Let the eigenvalues of an** $n \times n$ **real symmetric matric A be ordered from the largest to the smallest. Prove that for any** $1 \le k \le n$**,**

$$\lambda_k = \max_{S^k} \min_{0 \neq x \in S^k} r(x, A),$$

where S^k **is any** k **dimensional subspace of** $\mathbb{R}^n$**, and** $r(x,A)$ **is the Rayleigh quotient** $\frac{x^T A x}{x^T x}$**.**

2.

(a) **Prove that the growth factor $\rho = \frac{\|U\|_{max}}{\|A\|_{max}}$ is unbounded for LU factorization without pivoting.**

(b) **Prove that the growth factor is bounded by 2^{m-1} for LU factorization of $A \in \mathbb{R}^{m \times m}$ with row pivoting.**

3. A **is a diagonalizable matrix with one eigenvalue being** -1, **and others residing in the unit disk centered at** 2 **in the complex plane. Prove that the solution to** $Ax = b$ **through GMRES algorithm has error**

$$\|e_n\| \leq K2^{-n}$$

for some constant K.

4.

(a) **State and prove the Bauer-Fike Theorem.**

(b) **Show that the eigenvalue problem for Hermitian matrices is well conditioned.**

(c) **Give an example that this is not true for the non-Hermitian matricies.**

5. **Let**

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad A = P\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$

Determine the least square solution to the over-determined linear system $Ax = b$**.**

6. **Prove that the convergence of Rayleigh quotient iteration for a hermitian matrix is ultimately cubic.**

7. **Suppose A is a real symmetric matrix with eigenvalues more or less uniformly distributed over $[2, 18]$ together with an outlier at $\lambda = 50$. How many steps of the conjugate gradient iteration must be taken to be sure of reducing the initial error $\|e_0\|_A$ by a factor of 20^{20}?**

8. **Derive the asymptotic operation count of Gaussian elimination applied on an $m \times m$ real matrix A.**

9.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \qquad \epsilon = 10^{-9}$$

(a) **Find A^*A and the 2-norm $\kappa(A)$.**

(b) **MATLAB returns** $\text{rank}(A) = 3$**, but** $\text{rank}(A^*A) = 1$**. Explain.**

10. **Derive the asymptotic operation count for the following algorithms applied on a full-rank $m \times n (n \leq m)$ matrix A.**

   (a) **Reduced QR factorization by modified Gram-Schmidt orthogonalization.**

   (b) **Reduced QR factorization by Householder triangularization(without forming Q).**

1. **Derive the asymptotic operation count for the following algorithms applied on a full-rank $m \times n\,(n \leq m)$ matrix A.**

   (a) **Reduced QR factorization by modified Gram-Schmidt orthogonalization.**

   (b) **Reduced QR factorization by Householder triangularization(without forming Q).**

2.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \qquad \epsilon = 10^{-9}$$

(a) **Find A^*A and the 2-norm $\kappa(A)$.**

(b) **MATLAB returns** $\text{rank}(A) = 3$**, but** $\text{rank}(A^*A) = 1$**. Explain.**

3. **Let**

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & -0.6 & 0.8 \end{pmatrix}, \qquad A = P \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Determine the least square solution to the over-determined linear system $Ax = b$.

4. A **is a diagonalizable matrix with one eigenvalue being** -1**, and others residing in the unit disk centered at** 2 **in the complex plane. Prove that the solution to** $Ax = b$ **through GMRES algorithm has error**

$$\|e_n\| \le K2^{-n}$$

for some constant K.

5.

(a) **Prove that the growth factor $\rho = \frac{\|U\|_{max}}{\|A\|_{max}}$ is unbounded for LU factorization without pivoting.**

(b) **Prove that the growth factor is bounded by 2^{m-1} for LU factorization of $A \in \mathbb{R}^{m \times m}$ with row pivoting.**

NLA Practice Qual Exam #2

1. **Let the eigenvalues of an $n \times n$ real symmetric matric A be ordered from the largest to the smallest. Prove that for any $1 \leq k \leq n$,**

$$\lambda_k = \max_{S^k} \min_{0 \neq x \in S^k} r(x, A),$$

where S^k is any k dimensional subspace of $\mathbb{R}^n$, and $r(x, A)$ is the Rayleigh quotient $\frac{x^T A x}{x^T x}$.

2. A **is a diagonalizable matrix with one eigenvalue being** -1**, and others residing in the unit disk centered at** 2 **in the complex plane. Prove that the solution to** $Ax = b$ **through GMRES algorithm has error**

$$\|e_n\| \leq K2^{-n}$$

for some constant K.

3. **Prove that every eigenvalue of an** $n \times n$ **matrix** A **lies in one of the** n **circular disks in the complex plane with centers** A_{jj} **and radii** $\sum_{i \neq j} |A_{ij}|$**.**

4.

$$A = \begin{pmatrix} 3 & 3 & 3 \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \qquad \epsilon = 10^{-9}$$

(a) **Find A^*A and the 2-norm $\kappa(A)$.**

(b) **MATLAB returns** $\text{rank}(A) = 3$**, but** $\text{rank}(A^*A) = 1$**. Explain.**

5.

(a) **State and prove the Bauer-Fike Theorem.**

(b) **Show that the eigenvalue problem for Hermitian matrices is well conditioned.**

(c) **Give an example that this is not true for the non-Hermitian matricies.**

NLA Practice Qual Exam #3

1. **Let**

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad A = P\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$

Determine the least square solution to the over-determined linear system $Ax = b$**.**

2. **Suppose A is a real symmetric matrix with eigenvalues more or less uniformly distributed over $[2,18]$ together with an outlier at $\lambda = 65$. How many steps of the conjugate gradient iteration must be taken to be sure of reducing the initial error $\|e_0\|_A$ by a factor of 20^{20}?**

3. **Write a MATLAB function** $[Q,R] = msgs(A)$ **that computes the reduced QR factorization of an** $m \times n$ **matrix** A **using the modified Gram Schmidt process (**R **is** $n \times n$**).**

4. **Prove that the convergence of Rayleigh quotient iteration for a hermitian matrix is ultimately cubic.**

5.

(a) **Prove that the growth factor $\rho = \frac{\|U\|_{max}}{\|A\|_{max}}$ is unbounded for LU factorization without pivoting.**

(b) **Prove that the growth factor is bounded by 2^{m-1} for LU factorization of $A \in \mathbb{R}^{m\times m}$ with row pivoting.**

NLA Practice Qual Exam #4

1. **Let the eigenvalues of an $n \times n$ real symmetric matric A be ordered from the largest to the smallest. Prove that for any $1 \leq k \leq n$,**

$$\lambda_k = \max_{S^k} \min_{0 \neq x \in S^k} r(x, A),$$

where S^k is any k dimensional subspace of $\mathbb{R}^n$, and $r(x, A)$ is the Rayleigh quotient $\frac{x^T A x}{x^T x}$.

2. A **is a diagonalizable matrix with one eigenvalue being** -1**, and others residing in the unit disk centered at** 2 **in the complex plane. Prove that the solution to** $Ax = b$ **through GMRES algorithm has error**

$$\|e_n\| \leq K2^{-n}$$

for some constant K.

3. **Derive the asymptotic operation count for the following algorithms applied on a full-rank** $m \times n\,(n \le m)$ **matrix** A**.**

   (a) **Reduced QR factorization by modified Gram-Schmidt orthogonalization.**

   (b) **Reduced QR factorization by Householder triangularization(without forming Q).**

4.

(a) **State and prove the Bauer-Fike Theorem.**

(b) **Show that the eigenvalue problem for Hermitian matrices is well conditioned.**

(c) **Give an example that this is not true for the non-Hermitian matricies.**

5. **Construct the first Householder reflection matrix H_1 in the Householder triangularization of A.**

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$