Sec 4.1

Definition of a Complex Number

If a and b are real numbers, the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

Example

Adding and Subtracting Complex Numbers

a. $(4 + 7i) + (1 - 6i) = 4 + 7i + 1 - 6i$ — Remove parentheses.

$= (4 + 1) + (7i - 6i)$ — Group like terms.

$= 5 + i$ — Write in standard form.

b. $(1 + 2i) - (4 + 2i) = 1 + 2i - 4 - 2i$ — Remove parentheses.

$= (1 - 4) + (2i - 2i)$ — Group like terms.

$= -3 + 0$ — Simplify.

$= -3$ — Write in standard form.

c. $3i - (-2 + 3i) - (2 + 5i) = 3i + 2 - 3i - 2 - 5i$

$= (2 - 2) + (3i - 3i - 5i)$

$= 0 - 5i$

$= -5i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2i - i - i)$

$= 0 + 0i$

$= 0$

Multiplication of Complex Numbers

$$
\begin{aligned}
(a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive Property} \\
&= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property} \\
&= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1 \\
&= ac - bd + (ad)i + (bc)i && \text{Commutative Property} \\
&= (ac - bd) + (ad + bc)i && \text{Associative Property}
\end{aligned}
$$

Example

Multiplying Complex Numbers

a. $4(-2 + 3i) = 4(-2) + 4(3i)$	Distributive Property
$= -8 + 12i$	Simplify.
b. $(2 - i)(4 + 3i) = 2(4 + 3i) - i(4 + 3i)$	Distributive Property
$= 8 + 6i - 4i - 3i^2$	Distributive Property
$= 8 + 6i - 4i - 3(-1)$	$i^2 = -1$
$= (8 + 3) + (6i - 4i)$	Group like terms.
$= 11 + 2i$	Write in standard form.
c. $(3 + 2i)(3 - 2i) = 3(3 - 2i) + 2i(3 - 2i)$	Distributive Property
$= 9 - 6i + 6i - 4i^2$	Distributive Property
$= 9 - 6i + 6i - 4(-1)$	$i^2 = -1$
$= 9 + 4$	Simplify.
$= 13$	Write in standard form.
d. $(3 + 2i)^2 = (3 + 2i)(3 + 2i)$	Square of a binomial
$= 3(3 + 2i) + 2i(3 + 2i)$	Distributive Property
$= 9 + 6i + 6i + 4i^2$	Distributive Property
$= 9 + 6i + 6i + 4(-1)$	$i^2 = -1$
$= 9 + 12i - 4$	Simplify.
$= 5 + 12i$	Write in standard form.

Multipling Complex Conjugates

$$
\begin{aligned}
(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\
&= a^2 - b^2(-1) \\
&= a^2 + b^2
\end{aligned}
$$

Try as a Class

Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $4 - 3i$

Solution

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i)(4 + 3i) = 4^2 - (3i)^2 = 16 - 9i^2 = 16 - 9(-1) = 25$$

Simplifying Fractions with Complex Denominators

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right)$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

Example

Writing a Quotient of Complex Numbers in Standard Form

$$\frac{2+3i}{4-2i} = \frac{2+3i}{4-2i}\left(\frac{4+2i}{4+2i}\right)$$ Multiply numerator and denominator by complex conjugate of denominator.

$$= \frac{8+4i+12i+6i^2}{16-4i^2}$$ Expand.

$$= \frac{8-6+16i}{16+4}$$ $i^2 = -1$

$$= \frac{2+16i}{20}$$ Simplify.

$$= \frac{1}{10} + \frac{4}{5}i$$ Write in standard form.

Principal Square Root of a Negative Number

If a is a positive number, the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}\,i.$$

Examples

Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}\,i\sqrt{12}\,i = \sqrt{36}\,i^2 = 6(-1) = -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}\,i = 4\sqrt{3}i - 3\sqrt{3}i = \sqrt{3}\,i$

c. $\left(-1 + \sqrt{-3}\right)^2 = \left(-1 + \sqrt{3}\,i\right)^2$

$$= (-1)^2 - 2\sqrt{3}\,i + \left(\sqrt{3}\right)^2(i^2)$$

$$= 1 - 2\sqrt{3}i + 3(-1)$$

$$= -2 - 2\sqrt{3}i$$

Try as a Class

Complex Solutions of a Quadratic Equation

Solve (a) $x^2 + 4 = 0$ and (b) $3x^2 - 2x + 5 = 0$.

Solution

Solution

a. $x^2 + 4 = 0$ — Write original equation.

$x^2 = -4$ — Subtract 4 from each side.

$x = \pm 2i$ — Extract square roots.

b. $3x^2 - 2x + 5 = 0$ — Write original equation.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$ Quadratic Formula

$$= \frac{2 \pm \sqrt{-56}}{6}$$ Simplify.

$$= \frac{2 \pm 2\sqrt{14}i}{6}$$ Write $\sqrt{-56}$ in standard form.

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i$$ Write in standard form.

Sec 4.2

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Examples

Solutions of Polynomial Equations

a. The first-degree equation $x - 2 = 0$ has exactly *one* solution: $x = 2$.

b. The second-degree equation

$$x^2 - 6x + 9 = 0 \qquad \text{Second-degree equation}$$
$$(x - 3)(x - 3) = 0 \qquad \text{Factor.}$$

has exactly *two* solutions: $x = 3$ and $x = 3$. (This is called a *repeated solution*.)

c. The third-degree equation

$$x^3 + 4x = 0 \qquad \text{Third-degree equation}$$
$$x(x - 2i)(x + 2i) = 0 \qquad \text{Factor.}$$

has exactly *three* solutions: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree equation

$$x^4 - 1 = 0 \qquad \text{Fourth-degree equation}$$
$$(x - 1)(x + 1)(x - i)(x + i) = 0 \qquad \text{Factor.}$$

has exactly *four* solutions: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac$, is called the **discriminant**,

The discriminant…

1. If $b^2 - 4ac < 0$, the equation has two complex solutions.
2. If $b^2 - 4ac = 0$, the equation has one repeated real solution.
3. If $b^2 - 4ac > 0$, the equation has two distinct real solutions.

Try as a Class

Using the Discriminant

Use the discriminant to find the number of real solutions of each equation.

a. $4x^2 - 20x + 25 = 0$ **b.** $13x^2 + 7x + 2 = 0$ **c.** $5x^2 - 8x = 0$

Solution(Pt.1)

Solution

a. For this equation, $a = 4$, $b = -20$, and $c = 25$. So, the discriminant is

$$b^2 - 4ac = (-20)^2 - 4(4)(25) = 400 - 400 = 0.$$

Because the discriminant is zero, there is one repeated real solution.

b. For this equation, $a = 13$, $b = 7$, and $c = 2$. So, the discriminant is

$$b^2 - 4ac = 7^2 - 4(13)(2) = 49 - 104 = -55.$$

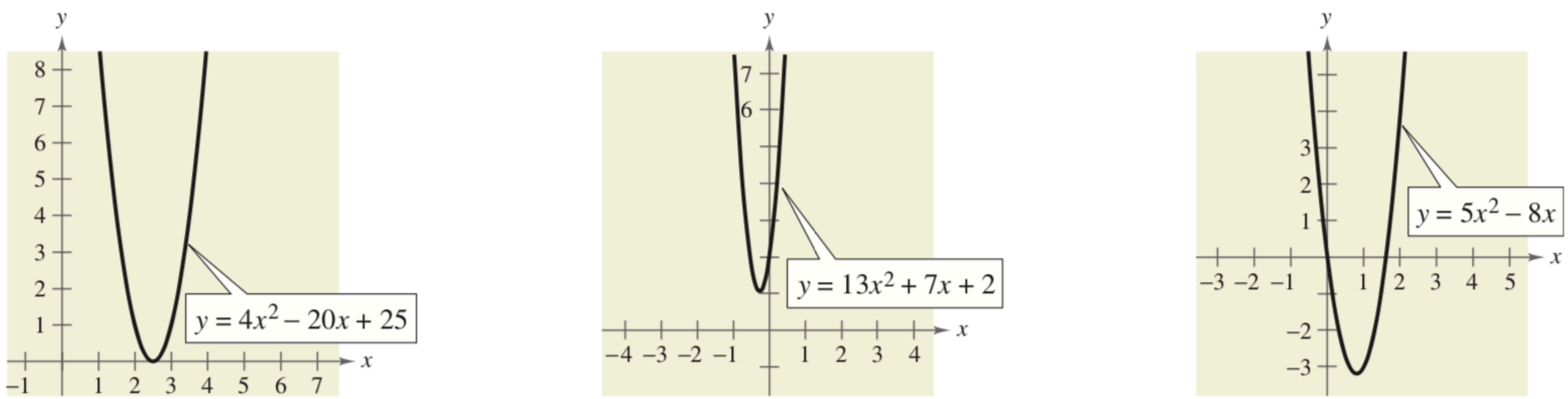
Because the discriminant is negative, there are two complex solutions.

c. For this equation, $a = 5$, $b = -8$, and $c = 0$. So, the discriminant is

$$b^2 - 4ac = (-8)^2 - 4(5)(0) = 64 - 0 = 64.$$

Because the discriminant is positive, there are two distinct real solutions.

Solution(Pt.2)



(a) Repeated real solution **(b) No real solution** **(c) Two distinct real solutions**

Try as a Class

Solving a Quadratic Equation

Solve $x^2 + 2x + 2 = 0$. Write complex solutions in standard form.

Solution

Solution

Using $a = 1$, $b = 2$, and $c = 2$, you can apply the Quadratic Formula as follows.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$ Substitute 1 for a, 2 for b, and 2 for c.

$$= \frac{-2 \pm \sqrt{-4}}{2}$$ Simplify.

$$= \frac{-2 \pm 2i}{2}$$ Simplify.

$$= -1 \pm i$$ Write in standard form.

Complex Solutions Occur in Conjugate Pairs

If $a + bi$, $b \neq 0$, is a solution of a polynomial equation with real coefficients, the conjugate $a - bi$ is also a solution of the equation.

Try as a Class

Solving a Polynomial Equation

Solve $x^4 - x^2 - 20 = 0$.

Solution

Solution

$$x^4 - x^2 - 20 = 0 \qquad \text{Write original equation.}$$

$$(x^2 - 5)(x^2 + 4) = 0 \qquad \text{Partially factor.}$$

$$(x + \sqrt{5})(x - \sqrt{5})(x + 2i)(x - 2i) = 0 \qquad \text{Factor completely.}$$

Setting each factor equal to zero yields the solutions $x = -\sqrt{5}$, $x = \sqrt{5}$, $x = -2i$, and $x = 2i$.

Try as a Class

Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Solution

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$[x - (1 + 3i)] \quad \text{and} \quad [x - (1 - 3i)]$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r}
x^2 - x - 6 \\
x^2 - 2x + 10 \,\big)\,\overline{x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2} \\
-x^3 - 4x^2 + 2x \\
\underline{-x^3 + 2x^2 - 10x} \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

Try as a Class

Finding a Polynomial with Given Zeros

Find a fourth-degree polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9. \end{aligned}$$

Try as a Class

Finding a Polynomial with Given Zeros

Find a cubic polynomial function f with real coefficients that has 2 and $1 - i$ as zeros, such that $f(1) = 3$.

Solution

Solution

Because $1 - i$ is a zero of f, so is $1 + i$. So,

$$\begin{aligned} f(x) &= a(x - 2)[x - (1 - i)][x - (1 + i)] \\ &= a(x - 2)[(x - 1) + i][(x - 1) - i] \\ &= a(x - 2)[(x - 1)^2 - i^2] \\ &= a(x - 2)(x^2 - 2x + 2) \\ &= a(x^3 - 4x^2 + 6x - 4). \end{aligned}$$

To find the value of a, use the fact that $f(1) = 3$ and obtain

$$\begin{aligned} f(1) &= a[1^3 - 4(1)^2 + 6(1) - 4] \\ 3 &= -a \\ -3 &= a. \end{aligned}$$

So, $a = -3$ and it follows that

$$\begin{aligned} f(x) &= -3(x^3 - 4x^2 + 6x - 4) \\ &= -3x^3 + 12x^2 - 18x + 12. \end{aligned}$$

Sec 4.3

The Complex Plane

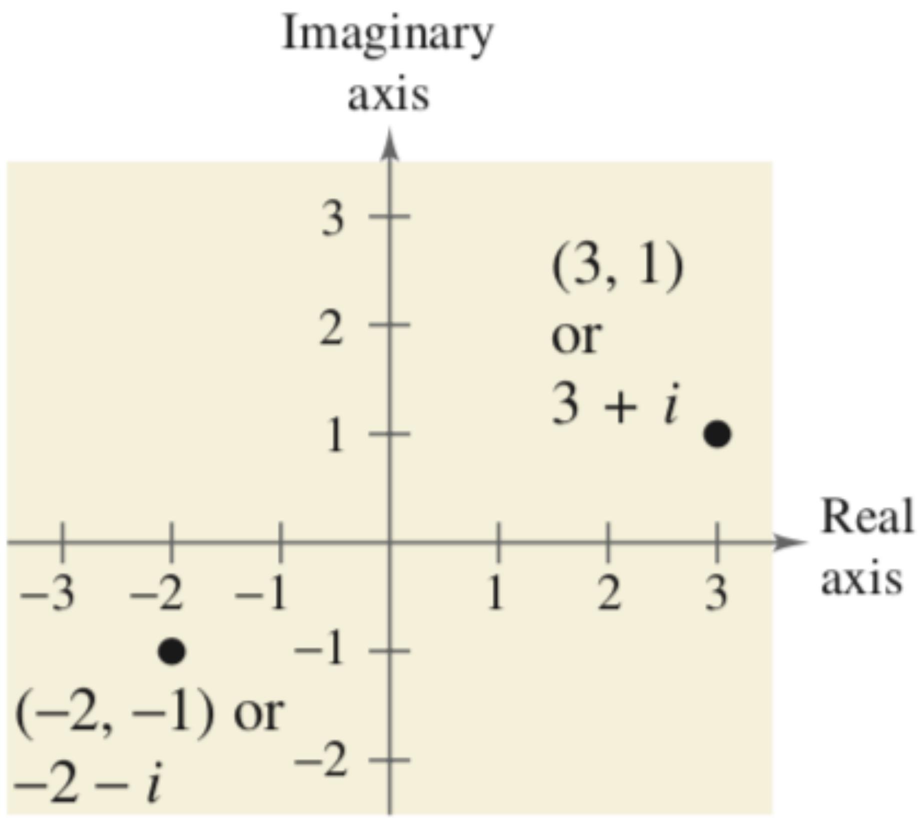


FIGURE 4.5

Definition of the Absolute Value of a Complex Number

The **absolute value** of the complex number $z = a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}.$$

Example

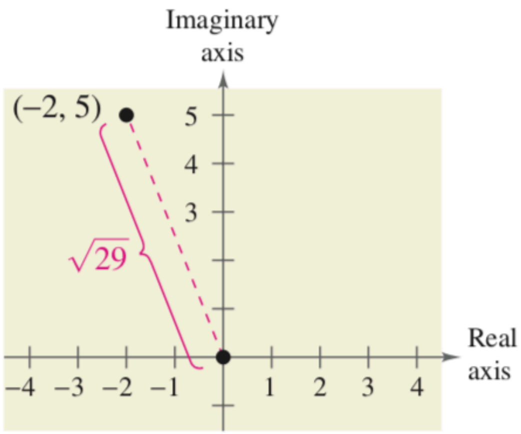


FIGURE 4.6

Finding the Absolute Value of a Complex Number

Plot $z = -2 + 5i$ and find its absolute value.

Solution

The number is plotted in Figure 4.6. It has an absolute value of

$$\begin{aligned}|z| &= \sqrt{(-2)^2 + 5^2}\\ &= \sqrt{29}.\end{aligned}$$

CHECKPoint Now try Exercise 9.

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

where $r = \sqrt{a^2 + b^2}$. Consequently, you have

$$a + bi = (r\cos\theta) + (r\sin\theta)i$$

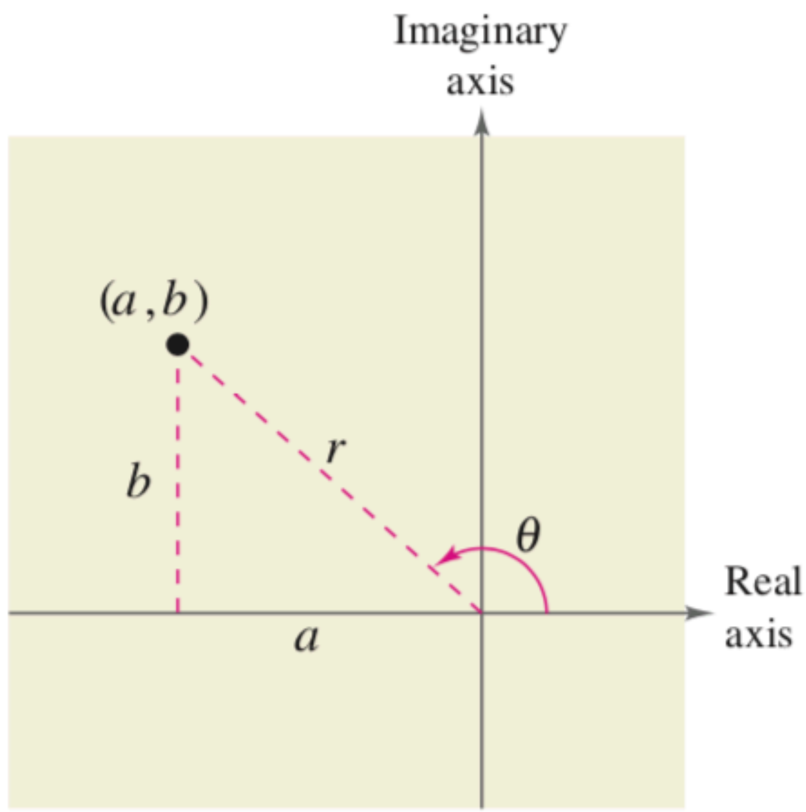


FIGURE 4.7

Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is called an **argument** of z.

Try as a Class

Writing a Complex Number in Trigonometric Form

Write the complex number $z = -2 - 2\sqrt{3}i$ in trigonometric form.

Solution(Pt.1)

Solution

The absolute value of z is

$$r = \left|-2 - 2\sqrt{3}i\right| = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$$

and the reference angle θ' is given by

$$\tan\,\theta' = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}.$$

Because $\tan(\pi/3) = \sqrt{3}$ and because $z = -2 - 2\sqrt{3}i$ lies in Quadrant III, you choose θ to be $\theta = \pi + \pi/3 = 4\pi/3$. So, the trigonometric form is

$$\begin{aligned} z &= r(\cos\,\theta + i\,\sin\,\theta) \\ &= 4\left(\cos\frac{4\pi}{3} + i\,\sin\frac{4\pi}{3}\right). \end{aligned}$$

See Figure 4.8.

Solution(Pt.2)

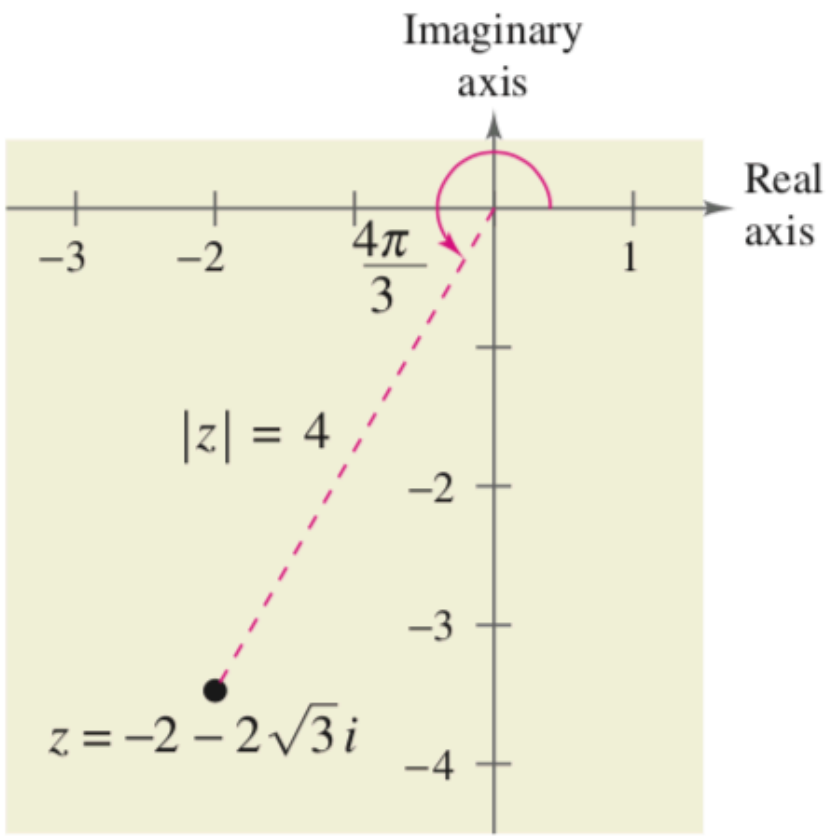


FIGURE 4.8

Try as a Class

Writing a Complex Number in Trigonometric Form

Write the complex number in trigonometric form.

$$z = 6 + 2i$$

Solution

Solution

The absolute value of z is

$$
\begin{aligned}
r &= |6 + 2i| \\
&= \sqrt{6^2 + 2^2} \\
&= \sqrt{40} \\
&= 2\sqrt{10}
\end{aligned}
$$

and the angle θ is

$$\tan\theta = \frac{b}{a} = \frac{2}{6} = \frac{1}{3}.$$

Because $z = 6 + 2i$ is in Quadrant I, you can conclude that

$$\theta = \arctan\frac{1}{3} \approx 0.32175 \text{ radian} \approx 18.4°.$$

So, the trigonometric form of z is

$$
\begin{aligned}
z &= r(\cos\theta + i\sin\theta) \\
&= 2\sqrt{10}\left[\cos\left(\arctan\frac{1}{3}\right) + i\sin\left(\arctan\frac{1}{3}\right)\right] \\
&\approx 2\sqrt{10}(\cos 18.4° + i\sin 18.4°).
\end{aligned}
$$

This result is illustrated graphically in Figure 4.9.

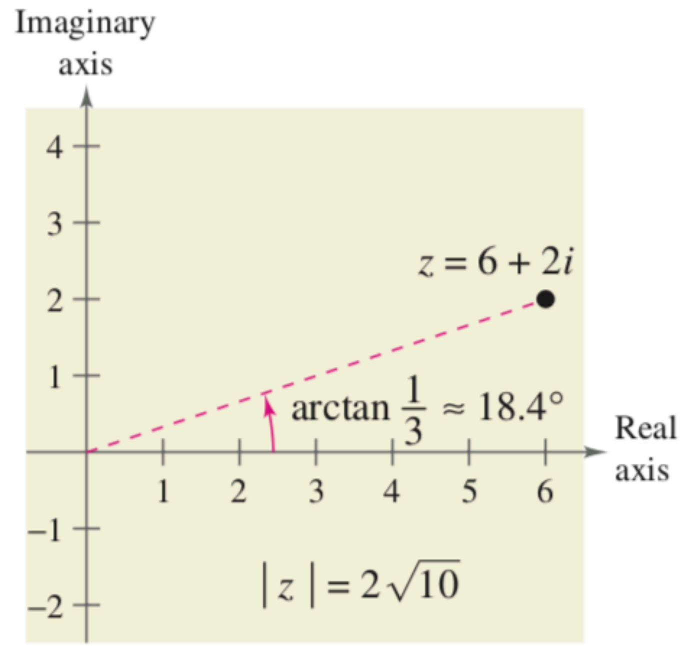


FIGURE 4.9

Try as a Class

Writing a Complex Number in Standard Form

Write the complex number in standard form $a + bi$.

$$z = \sqrt{8}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

Solution

Solution

Because $\cos(-\pi/3) = \frac{1}{2}$ and $\sin(-\pi/3) = -\sqrt{3}/2$, you can write

$$\begin{aligned} z &= \sqrt{8}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] \\ &= 2\sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= \sqrt{2} - \sqrt{6}i. \end{aligned}$$

Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$ Product

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \quad z_2 \neq 0$$ Quotient

Try as a Class

Dividing Complex Numbers

Find the quotient z_1/z_2 of the complex numbers.

$$z_1 = 24(\cos 300° + i \sin 300°) \qquad z_2 = 8(\cos 75° + i \sin 75°)$$

Solution

Solution

$$\frac{z_1}{z_2} = \frac{24(\cos 300^\circ + i \sin 300^\circ)}{8(\cos 75^\circ + i \sin 75^\circ)}$$

$$= \frac{24}{8}[\cos(300^\circ - 75^\circ) + i \sin(300^\circ - 75^\circ)]$$

$$= 3(\cos 225^\circ + i \sin 225^\circ)$$

$$= 3\left[\left(-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$= -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$

Try as a Class

Multiplying Complex Numbers

Find the product $z_1 z_2$ of the complex numbers.

$$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \qquad z_2 = 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

Solution

Solution

$$
\begin{aligned}
z_1z_2 &= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\cdot 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)\\
&= 16\left[\cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i\sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right] && \text{Multiply moduli and add arguments.}\\
&= 16\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)\\
&= 16\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\\
&= 16[0 + i(1)]\\
&= 16i
\end{aligned}
$$