Sec 1.4-1.6

Day 3

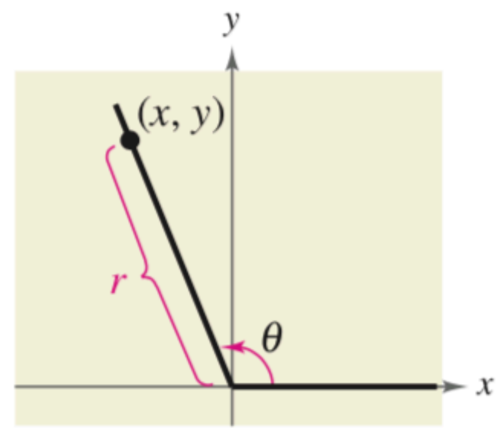
Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}, \quad x \neq 0 \qquad \cot \theta = \frac{x}{y}, \quad y \neq 0$$

$$\sec \theta = \frac{r}{x}, \quad x \neq 0 \qquad \csc \theta = \frac{r}{y}, \quad y \neq 0$$



Try as a Class

Let $(-3, 4)$ be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

Solution

Solution

Referring to Figure 1.36, you can see that $x = -3$, $y = 4$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$$

So, you have the following.

$$\sin\theta = \frac{y}{r} = \frac{4}{5} \qquad \cos\theta = \frac{x}{r} = -\frac{3}{5} \qquad \tan\theta = \frac{y}{x} = -\frac{4}{3}$$

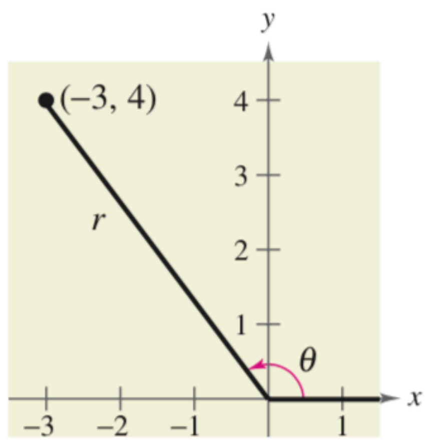


FIGURE 1.36

Try as a Class

Evaluating Trigonometric Functions

Given $\tan\,\theta = -\frac{5}{4}$ and $\cos\,\theta > 0$, find $\sin\,\theta$ and $\sec\,\theta$.

Solution

Solution

Note that θ lies in Quadrant IV because that is the only quadrant in which the tangent is negative and the cosine is positive. Moreover, using

$$\tan\theta = \frac{y}{x}$$

$$= -\frac{5}{4}$$

and the fact that y is negative in Quadrant IV, you can let $y = -5$ and $x = 4$. So, $r = \sqrt{16 + 25} = \sqrt{41}$ and you have

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{41}}$$

$$\approx -0.7809$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{41}}{4}$$

$$\approx 1.6008.$$

CHECKPoint Now try Exercise 23.

Try as a Class

Trigonometric Functions of Quadrant Angles

Evaluate the cosine and tangent functions at the four quadrant angles 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Solution

Solution

To begin, choose a point on the terminal side of each angle, as shown in Figure 1.38. For each of the four points, $r = 1$, and you have the following.

$$\cos 0 = \frac{x}{r} = \frac{1}{1} = 1 \qquad \tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \qquad (x, y) = (1, 0)$$

$$\cos \frac{\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan \frac{\pi}{2} = \frac{y}{x} = \frac{1}{0} \implies \text{undefined} \qquad (x, y) = (0, 1)$$

$$\cos \pi = \frac{x}{r} = \frac{-1}{1} = -1 \qquad \tan \pi = \frac{y}{x} = \frac{0}{-1} = 0 \qquad (x, y) = (-1, 0)$$

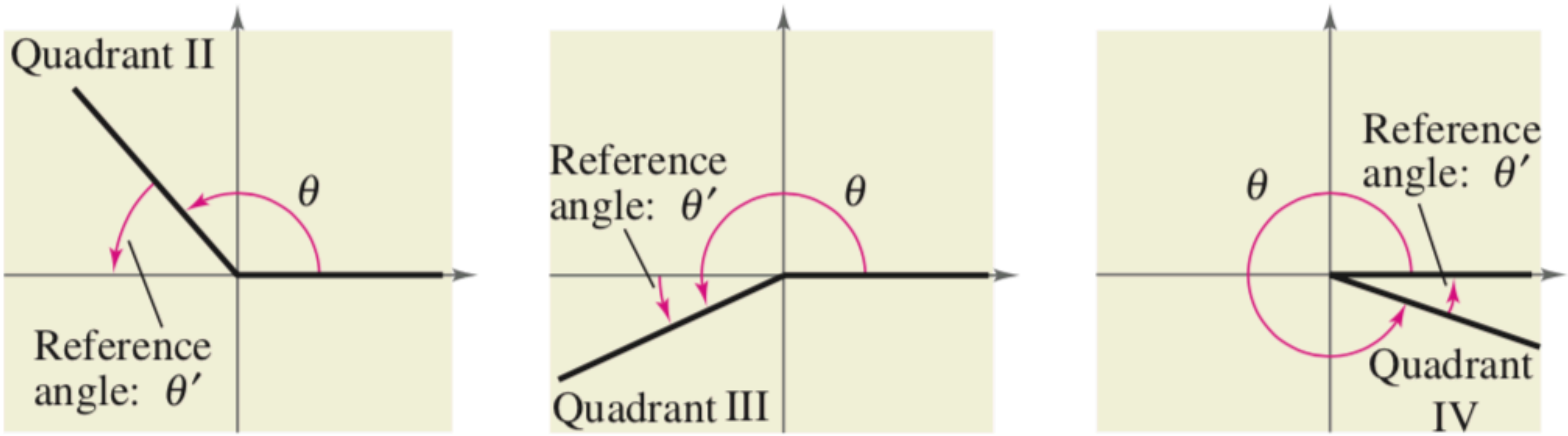
$$\cos \frac{3\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan \frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0} \implies \text{undefined} \qquad (x, y) = (0, -1)$$

CHECK*Point* Now try Exercise 37.

Definition of Reference Angle

Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Example



$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)

$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)

$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

FIGURE 1.39

Try as a Class

Finding Reference Angles

Find the reference angle θ'.

a. $\theta = 300°$ **b.** $\theta = 2.3$ **c.** $\theta = -135°$

Solution

Solution

a. Because 300° lies in Quadrant IV, the angle it makes with the *x*-axis is

$$\begin{aligned}\theta' &= 360^\circ - 300^\circ \\ &= 60^\circ. \qquad \text{Degrees}\end{aligned}$$

Figure 1.40 shows the angle $\theta = 300^\circ$ and its reference angle $\theta' = 60^\circ$.

b. Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\begin{aligned}\theta' &= \pi - 2.3 \\ &\approx 0.8416. \qquad \text{Radians}\end{aligned}$$

Figure 1.41 shows the angle $\theta = 2.3$ and its reference angle $\theta' = \pi - 2.3$.

c. First, determine that -135° is coterminal with 225°, which lies in Quadrant III. So, the reference angle is

$$\begin{aligned}\theta' &= 225^\circ - 180^\circ \\ &= 45^\circ. \qquad \text{Degrees}\end{aligned}$$

Figure 1.42 shows the angle $\theta = -135^\circ$ and its reference angle $\theta' = 45^\circ$.

CHECK*Point* Now try Exercise 45.

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value for the associated reference angle θ'.

2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

Try to recognize

Trigonometric Values of Common Angles

θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\ \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\ \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\ \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

Try as a Class

Evaluate each trigonometric function.

a. $\cos \dfrac{4\pi}{3}$ **b.** $\tan(-210°)$ **c.** $\csc \dfrac{11\pi}{4}$

Solution(Pt 1)

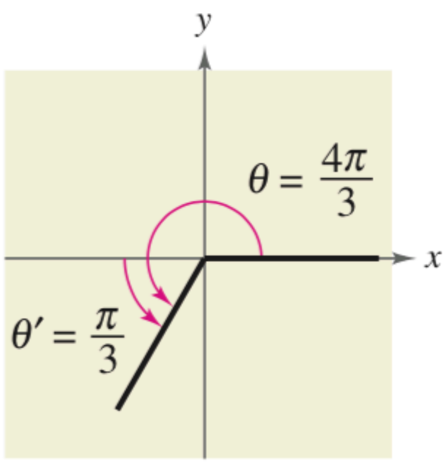


FIGURE 1.44

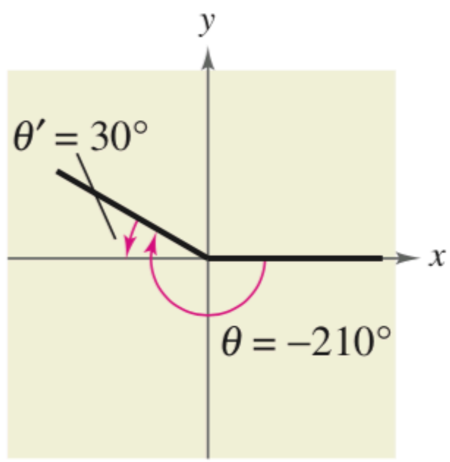


FIGURE 1.45

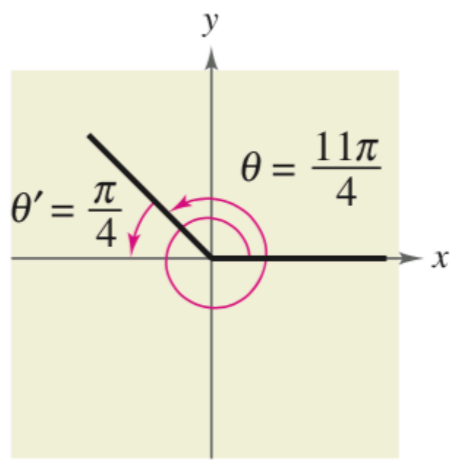


FIGURE 1.46

CHECK*Point* Now try Exercise 59.

Solution(Pt 2)

Solution

a. Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is

$$\theta' = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$$

as shown in Figure 1.44. Moreover, the cosine is negative in Quadrant III, so

$$\cos\frac{4\pi}{3} = (-)\cos\frac{\pi}{3}$$

$$= -\frac{1}{2}.$$

b. Because $-210° + 360° = 150°$, it follows that $-210°$ is coterminal with the second-quadrant angle $150°$. So, the reference angle is $\theta' = 180° - 150° = 30°$, as shown in Figure 1.45. Finally, because the tangent is negative in Quadrant II, you have

$$\tan(-210°) = (-)\tan 30°$$

$$= -\frac{\sqrt{3}}{3}.$$

c. Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$. So, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 1.46. Because the cosecant is positive in Quadrant II, you have

$$\csc\frac{11\pi}{4} = (+)\csc\frac{\pi}{4}$$

$$= \frac{1}{\sin(\pi/4)}$$

$$= \sqrt{2}.$$

Try as a Class

Let θ be an angle in Quadrant II such that $\sin\theta = \frac{1}{3}$. Find (a) $\cos\theta$ and (b) $\tan\theta$ by using trigonometric identities.

Solution

Solution

a. Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, you obtain

$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1 \qquad \text{Substitute } \tfrac{1}{3} \text{ for } \sin\theta.$$

$$\cos^2\theta = 1 - \frac{1}{9} = \frac{8}{9}.$$

Because $\cos\theta < 0$ in Quadrant II, you can use the negative root to obtain

$$\cos\theta = -\frac{\sqrt{8}}{\sqrt{9}}$$

$$= -\frac{2\sqrt{2}}{3}.$$

b. Using the trigonometric identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, you obtain

$$\tan\theta = \frac{1/3}{-2\sqrt{2}/3} \qquad \text{Substitute for } \sin\theta \text{ and } \cos\theta.$$

$$= -\frac{1}{2\sqrt{2}}$$

$$= -\frac{\sqrt{2}}{4}.$$

CHECK*Point* Now try Exercise 69.

Sec 1.5

Sin(x)

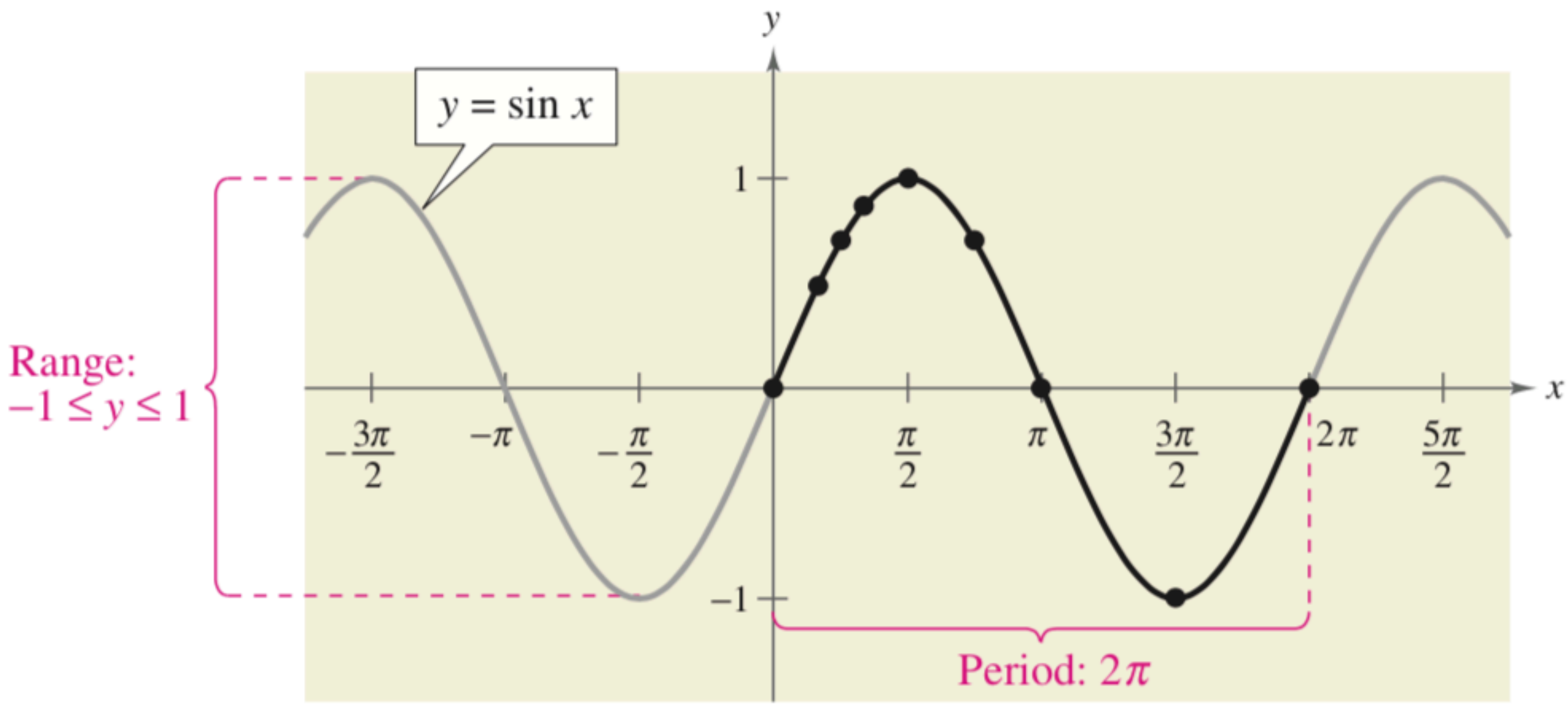


FIGURE 1.47

Cos(x)

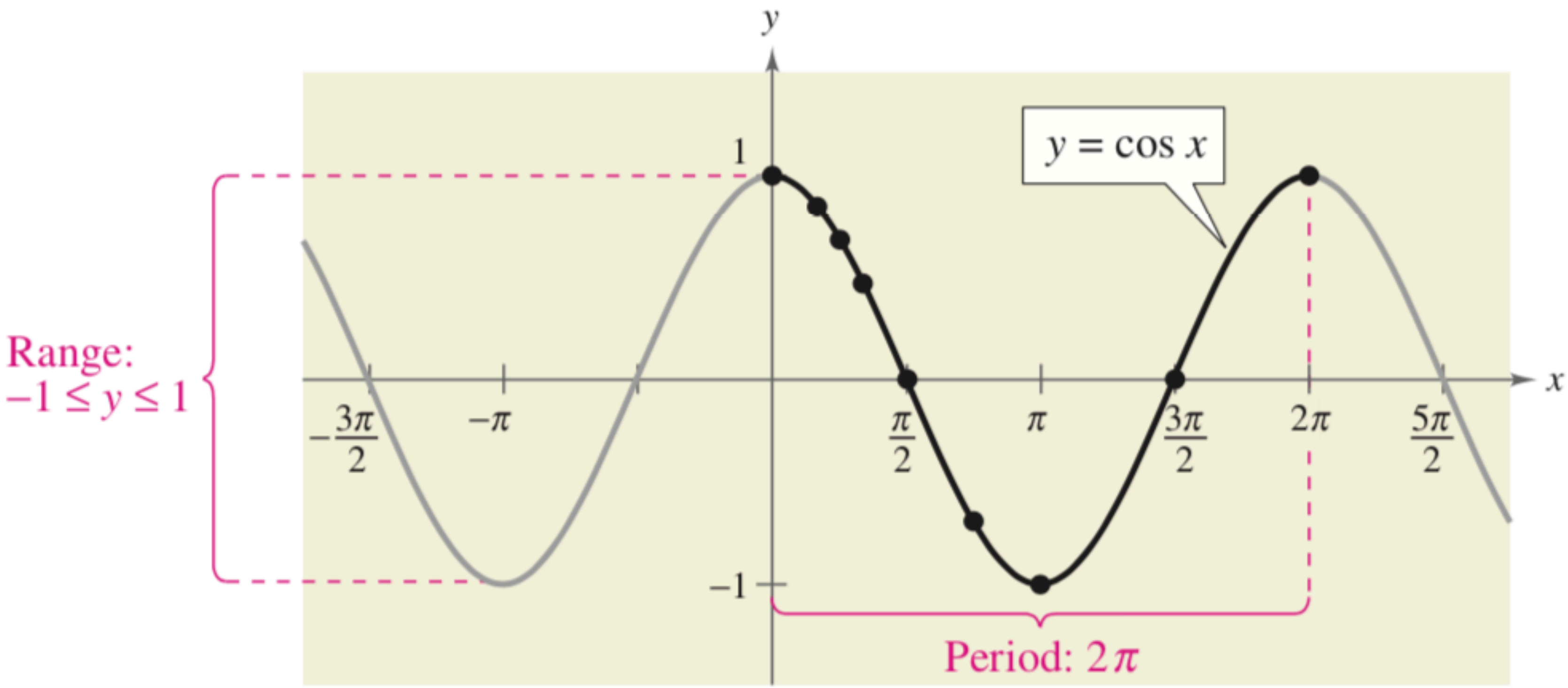


FIGURE 1.48

Anatomy of Sin and Cos

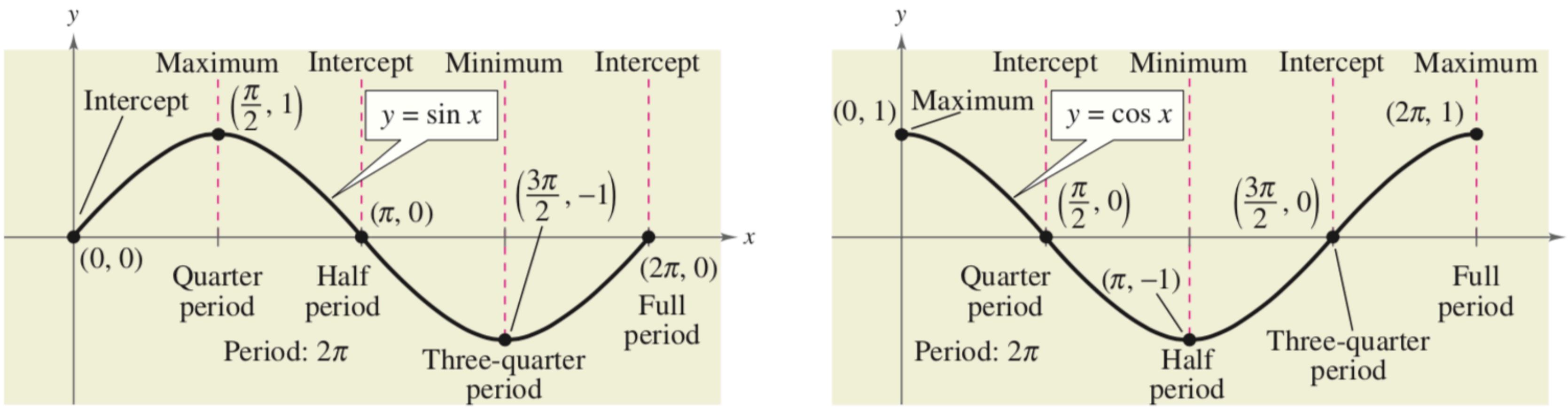


FIGURE 1.49

Try as a Class

Sketch the graph of $y = 2\sin x$ on the interval $[-\pi, 4\pi]$.

Solution

Solution

Note that

$$y = 2 \sin x = 2(\sin x)$$

indicates that the y-values for the key points will have twice the magnitude of those on the graph of $y = \sin x$. Divide the period 2π into four equal parts to get the key points for $y = 2 \sin x$.

Intercept	*Maximum*	*Intercept*	*Minimum*	*Intercept*
$(0, 0)$,	$\left(\frac{\pi}{2}, 2\right)$,	$(\pi, 0)$,	$\left(\frac{3\pi}{2}, -2\right)$, and	$(2\pi, 0)$

By connecting these key points with a smooth curve and extending the curve in both directions over the interval $[-\pi, 4\pi]$, you obtain the graph shown in Figure 1.50.

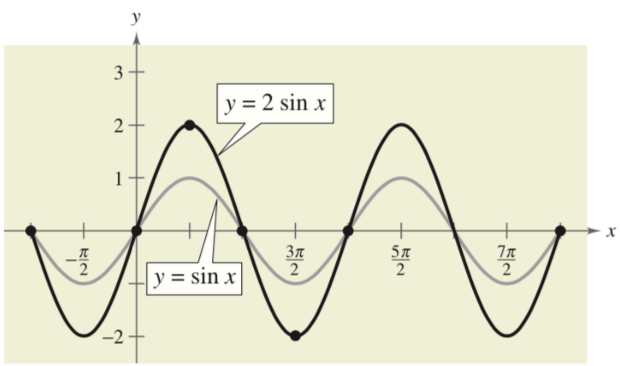


FIGURE 1.50

CHECK*Point* Now try Exercise 39.

Definition of Amplitude of Sine and Cosine Curves

The **amplitude** of $y = a \sin x$ and $y = a \cos x$ represents half the distance between the maximum and minimum values of the function and is given by

$$\text{Amplitude} = |a|.$$

Try as a Class

On the same coordinate axes, sketch the graph of each function.

a. $y = \frac{1}{2} \cos x$ **b.** $y = 3 \cos x$

Solution(Pt. 1)

Solution

a. Because the amplitude of $y = \frac{1}{2}\cos x$ is $\frac{1}{2}$, the maximum value is $\frac{1}{2}$ and the minimum value is $-\frac{1}{2}$. Divide one cycle, $0 \le x \le 2\pi$, into four equal parts to get the key points

Maximum	*Intercept*	*Minimum*	*Intercept*		*Maximum*
$\left(0, \frac{1}{2}\right)$,	$\left(\frac{\pi}{2}, 0\right)$,	$\left(\pi, -\frac{1}{2}\right)$,	$\left(\frac{3\pi}{2}, 0\right)$,	and	$\left(2\pi, \frac{1}{2}\right)$.

b. A similar analysis shows that the amplitude of $y = 3\cos x$ is 3, and the key points are

Maximum	*Intercept*	*Minimum*	*Intercept*		*Maximum*
$(0, 3)$,	$\left(\frac{\pi}{2}, 0\right)$,	$(\pi, -3)$,	$\left(\frac{3\pi}{2}, 0\right)$,	and	$(2\pi, 3)$.

The graphs of these two functions are shown in Figure 1.51. Notice that the graph of $y = \frac{1}{2}\cos x$ is a vertical *shrink* of the graph of $y = \cos x$ and the graph of $y = 3\cos x$ is a vertical *stretch* of the graph of $y = \cos x$.

CHECK*Point* Now try Exercise 41.

Solution(Pt. 2)

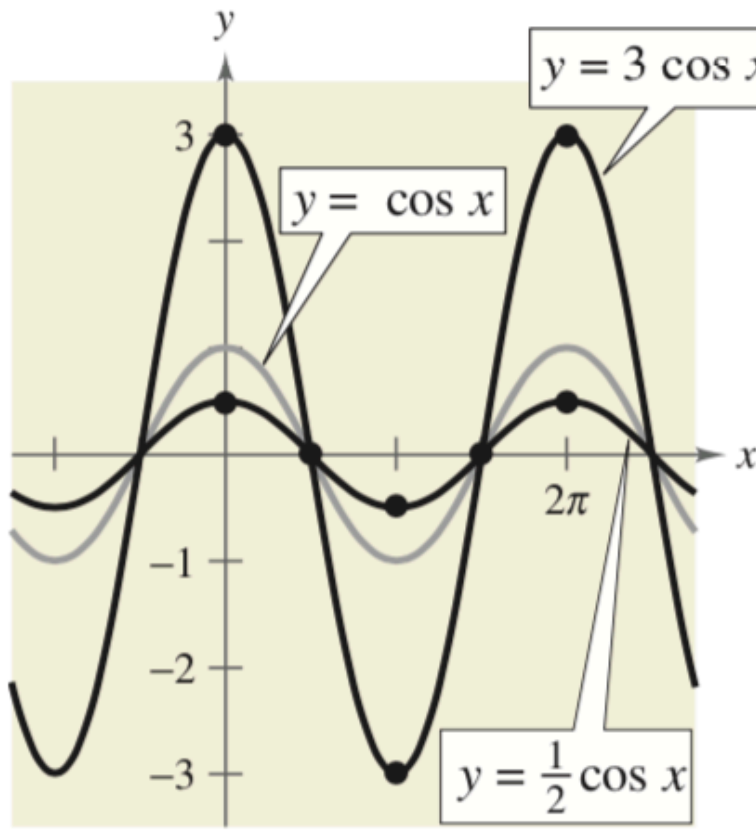


FIGURE 1.51

Period of Sine and Cosine Functions

Let b be a positive real number. The **period** of $y = a \sin bx$ and $y = a \cos bx$ is given by

$$\text{Period} = \frac{2\pi}{b}.$$

Try as a Class

Sketch the graph of $y = \sin\frac{x}{2}$.

Solution

Solution

The amplitude is 1. Moreover, because $b = \frac{1}{2}$, the period is

$$\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi. \qquad \text{Substitute for } b.$$

Now, divide the period-interval $[0, 4\pi]$ into four equal parts with the values π, 2π, and 3π to obtain the key points on the graph.

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$(0, 0)$,	$(\pi, 1)$,	$(2\pi, 0)$,	$(3\pi, -1)$,	and	$(4\pi, 0)$

The graph is shown in Figure 1.53.

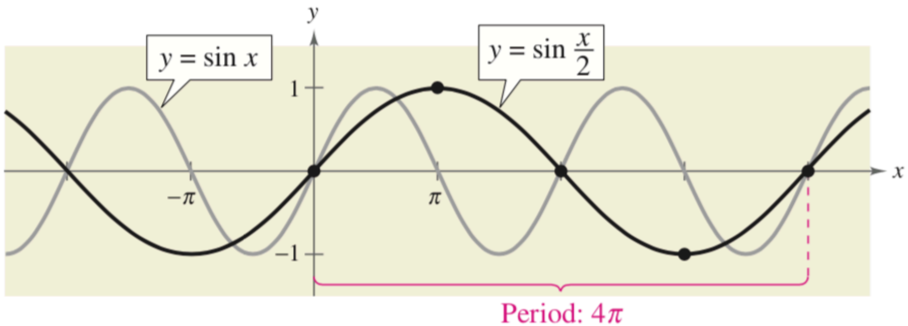


FIGURE 1.53

CHECKPoint Now try Exercise 43.

Graphs of Sine and Cosine Functions

The graphs of $y = a \sin(bx - c)$ and $y = a \cos(bx - c)$ have the following characteristics. (Assume $b > 0$.)

$$\text{Amplitude} = |a| \qquad \text{Period} = \frac{2\pi}{b}$$

The left and right endpoints of a one-cycle interval can be determined by solving the equations $bx - c = 0$ and $bx - c = 2\pi$.

Try as a Class

Analyze the graph of $y = \frac{1}{2}\sin\left(x - \frac{\pi}{3}\right)$.

Solution

Algebraic Solution

The amplitude is $\frac{1}{2}$ and the period is 2π. By solving the equations

$$x - \frac{\pi}{3} = 0 \quad \Rightarrow \quad x = \frac{\pi}{3}$$

and

$$x - \frac{\pi}{3} = 2\pi \quad \Rightarrow \quad x = \frac{7\pi}{3}$$

you see that the interval $[\pi/3, 7\pi/3]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the key points

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$\left(\frac{\pi}{3}, 0\right)$,	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$,	$\left(\frac{4\pi}{3}, 0\right)$,	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$,	and	$\left(\frac{7\pi}{3}, 0\right)$.

CHECK*Point* Now try Exercise 49.

Try as a Class

Sketch the graph of

$$y = -3\cos(2\pi x + 4\pi).$$

Solution(Pt. 1)

Solution

The amplitude is 3 and the period is $2\pi/2\pi = 1$. By solving the equations

$$\begin{aligned} 2\pi x + 4\pi &= 0 \\ 2\pi x &= -4\pi \\ x &= -2 \end{aligned}$$

and

$$\begin{aligned} 2\pi x + 4\pi &= 2\pi \\ 2\pi x &= -2\pi \\ x &= -1 \end{aligned}$$

you see that the interval $[-2, -1]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the key points

Minimum	*Intercept*	*Maximum*	*Intercept*		*Minimum*
$(-2, -3)$,	$\left(-\frac{7}{4}, 0\right)$,	$\left(-\frac{3}{2}, 3\right)$,	$\left(-\frac{5}{4}, 0\right)$,	and	$(-1, -3)$.

The graph is shown in Figure 1.55.

CHECK*Point* Now try Exercise 51.

Solution(Pt. 2)

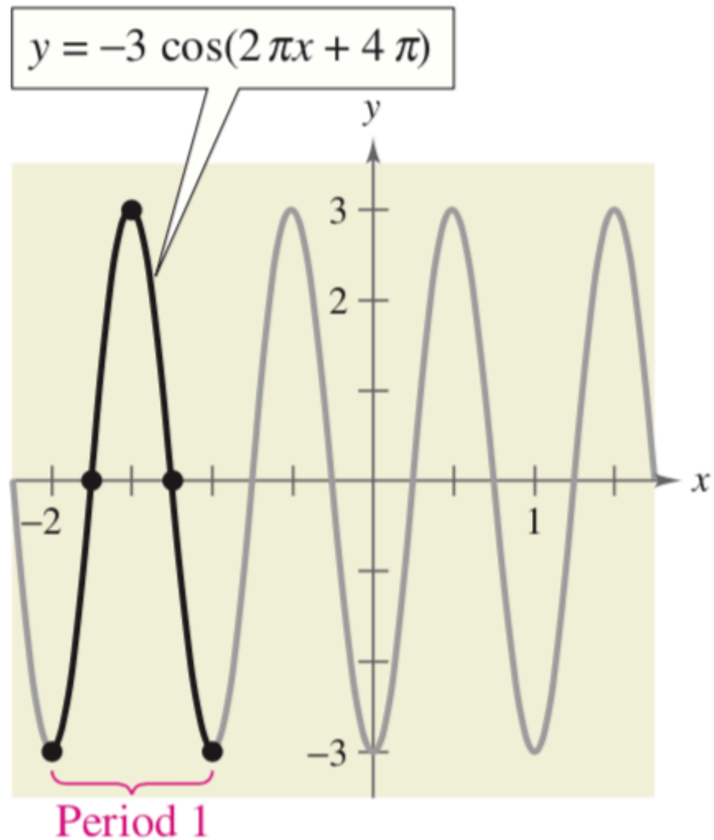


FIGURE 1.55

Try as a Class

Sketch the graph of

$$y = 2 + 3\cos 2x.$$

Solution(Pt. 1)

Solution

The amplitude is 3 and the period is π. The key points over the interval $[0, \pi]$ are

$$(0, 5), \quad \left(\frac{\pi}{4}, 2\right), \quad \left(\frac{\pi}{2}, -1\right), \quad \left(\frac{3\pi}{4}, 2\right), \quad \text{and} \quad (\pi, 5).$$

The graph is shown in Figure 1.56. Compared with the graph of $f(x) = 3 \cos 2x$, the graph of $y = 2 + 3 \cos 2x$ is shifted upward two units.

CHECKPoint Now try Exercise 57.

Solution(Pt. 2)

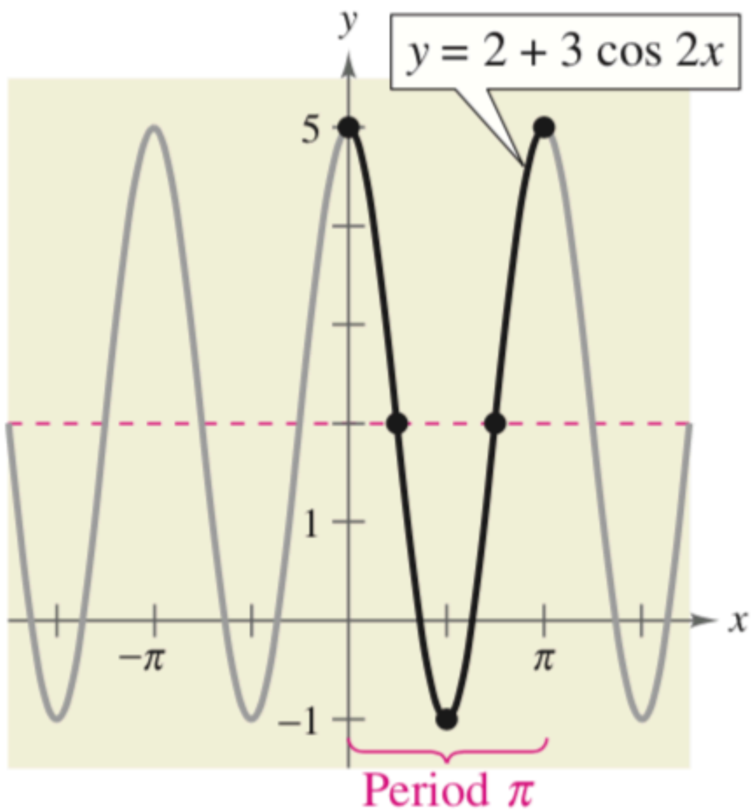


FIGURE 1.56

Try as a Class

Throughout the day, the depth of water at the end of a dock in Bar Harbor, Maine varies with the tides. The table shows the depths (in feet) at various times during the morning. (Source: Nautical Software, Inc.)

a. Use a trigonometric function to model the data.

b. Find the depths at 9 A.M. and 3 P.M.

c. A boat needs at least 10 feet of water to moor at the dock. During what times in the afternoon can it safely dock?

Time, t	Depth, y
Midnight	3.4
2 A.M.	8.7
4 A.M.	11.3
6 A.M.	9.1
8 A.M.	3.8
10 A.M.	0.1
Noon	1.2

Solution

a. Begin by graphing the data, as shown in Figure 1.57. You can use either a sine or a cosine model. Suppose you use a cosine model of the form

$$y = a\cos(bt - c) + d.$$

The difference between the maximum height and the minimum height of the graph is twice the amplitude of the function. So, the amplitude is

$$a = \frac{1}{2}[(\text{maximum depth}) - (\text{minimum depth})] = \frac{1}{2}(11.3 - 0.1) = 5.6.$$

The cosine function completes one half of a cycle between the times at which the maximum and minimum depths occur. So, the period is

$$p = 2[(\text{time of min. depth}) - (\text{time of max. depth})] = 2(10 - 4) = 12$$

which implies that $b = 2\pi/p \approx 0.524$. Because high tide occurs 4 hours after midnight, consider the left endpoint to be $c/b = 4$, so $c \approx 2.094$. Moreover, because the average depth is $\frac{1}{2}(11.3 + 0.1) = 5.7$, it follows that $d = 5.7$. So, you can model the depth with the function given by

$$y = 5.6\cos(0.524t - 2.094) + 5.7.$$

b. The depths at 9 A.M. and 3 P.M. are as follows.

$$y = 5.6\cos(0.524 \cdot 9 - 2.094) + 5.7$$

$$\approx 0.84 \text{ foot} \qquad \text{9 A.M.}$$

$$y = 5.6\cos(0.524 \cdot 15 - 2.094) + 5.7$$

$$\approx 10.57 \text{ feet} \qquad \text{3 P.M.}$$

c. To find out when the depth y is at least 10 feet, you can graph the model with the line $y = 10$ using a graphing utility, as shown in Figure 1.58. Using the *intersect* feature, you can determine that the depth is at least 10 feet between 2:42 P.M. ($t \approx 14.7$) and 5:18 P.M. ($t \approx 17.3$).

CHECK*Point* Now try Exercise 91.

Solution(Pt. 2)

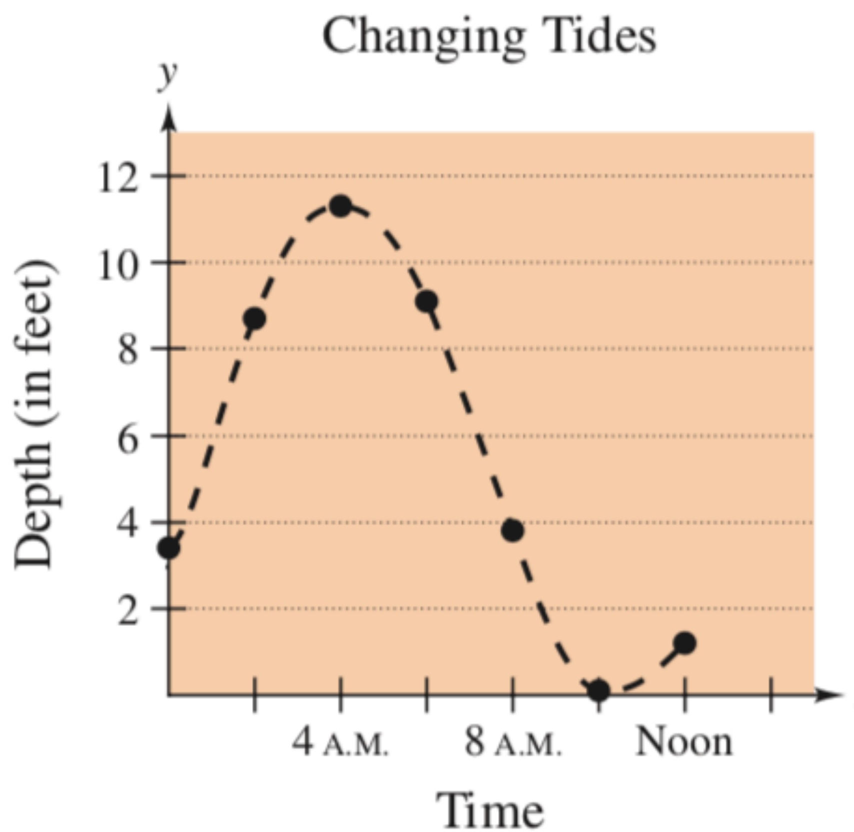


FIGURE 1.57

Sec 1.6

But what about tan(x)?????

x	$-\frac{\pi}{2}$	-1.57	-1.5	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	1.5	1.57	$\frac{\pi}{2}$
$\tan x$	Undef.	-1255.8	-14.1	-1	0	1	14.1	1255.8	Undef.

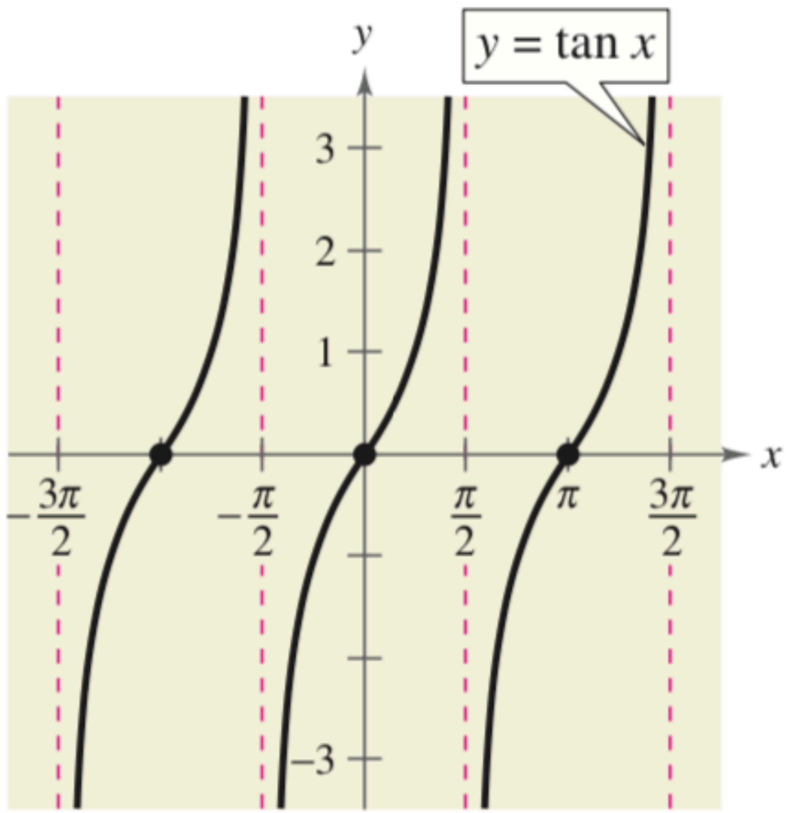


FIGURE 1.59

PERIOD: π

DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$

RANGE: $(-\infty, \infty)$

VERTICAL ASYMPTOTES: $x = \frac{\pi}{2} + n\pi$

SYMMETRY: ORIGIN

Try as a Class

Sketch the graph of $y = \tan(x/2)$.

Solution(Pt. 1)

Solution

By solving the equations

$$\frac{x}{2} = -\frac{\pi}{2} \qquad \text{and} \qquad \frac{x}{2} = \frac{\pi}{2}$$

$$x = -\pi \qquad\qquad\qquad x = \pi$$

you can see that two consecutive vertical asymptotes occur at $x = -\pi$ and $x = \pi$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 1.60.

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$\tan\frac{x}{2}$	Undef.	-1	0	1	Undef.

CHECK*Point* Now try Exercise 15.

Solution(Pt. 2)

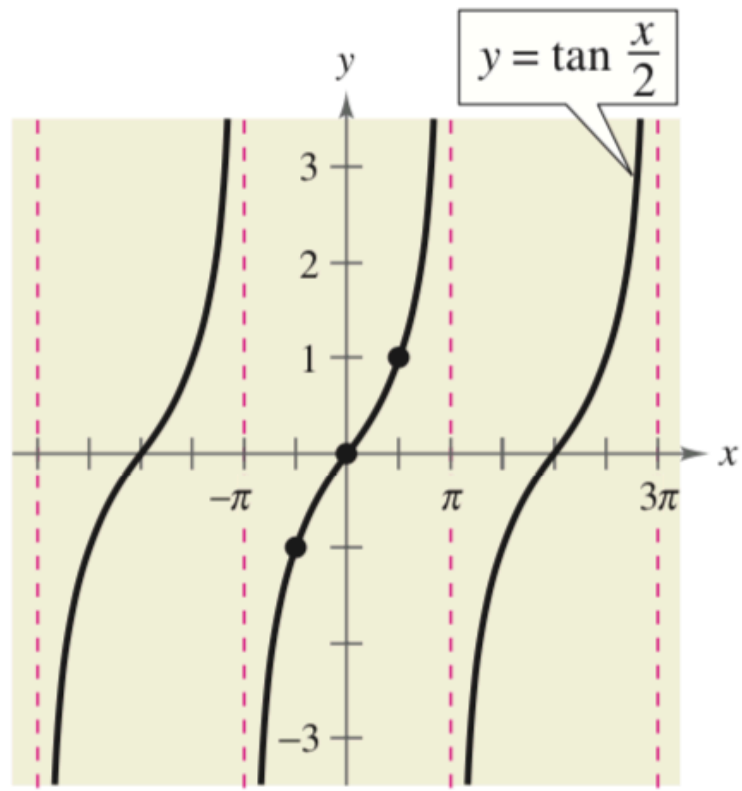


FIGURE 1.60

Try as a Class

Sketch the graph of $y = -3 \tan 2x$.

Solution(Pt. 1)

Solution

By solving the equations

$$2x = -\frac{\pi}{2} \qquad \text{and} \qquad 2x = \frac{\pi}{2}$$

$$x = -\frac{\pi}{4} \qquad\qquad x = \frac{\pi}{4}$$

you can see that two consecutive vertical asymptotes occur at $x = -\pi/4$ and $x = \pi/4$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 1.61.

x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$-3 \tan 2x$	Undef.	3	0	-3	Undef.

By comparing the graphs in Examples 1 and 2, you can see that the graph of $y = a \tan(bx - c)$ increases between consecutive vertical asymptotes when $a > 0$, and decreases between consecutive vertical asymptotes when $a < 0$. In other words, the graph for $a < 0$ is a reflection in the x-axis of the graph for $a > 0$.

CHECKPoint Now try Exercise 17.

Solution(Pt. 2)

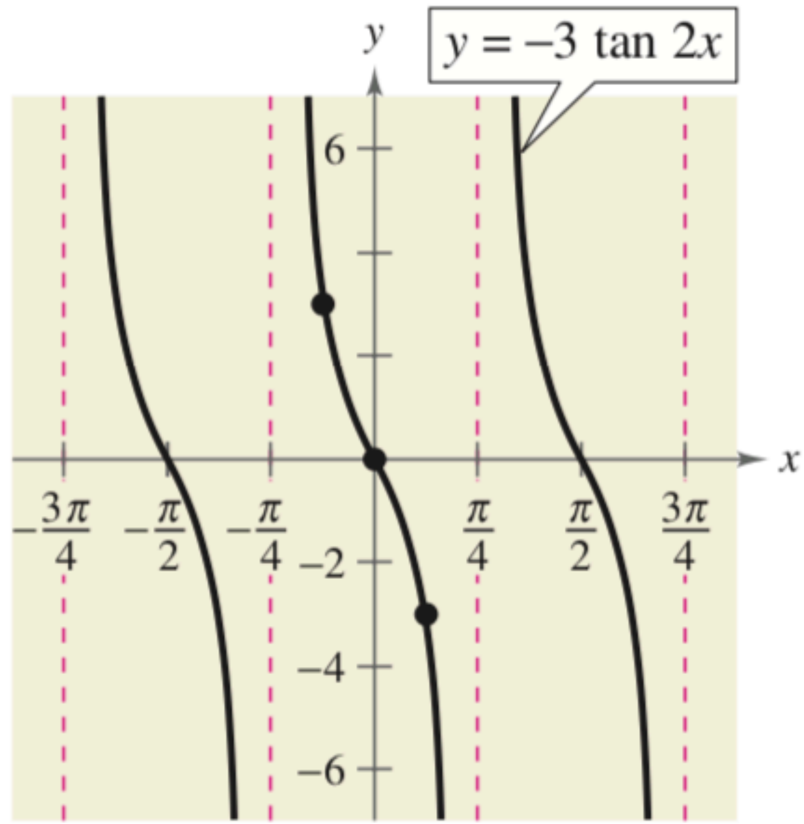


FIGURE 1.61

And cot(x)…?

$$y = \cot x = \frac{\cos x}{\sin x}$$

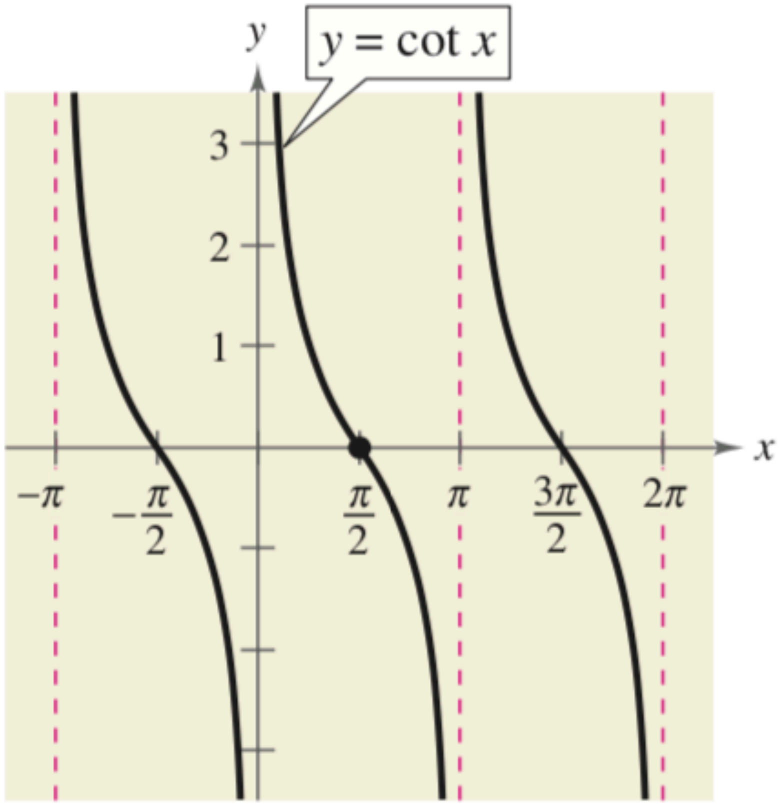


FIGURE 1.62

PERIOD: π

DOMAIN: ALL $x \neq n\pi$

RANGE: $(-\infty, \infty)$

VERTICAL ASYMPTOTES: $x = n\pi$

SYMMETRY: ORIGIN

Try as a Class

Sketch the graph of $y = 2\cot\dfrac{x}{3}$.

Solution(Pt. 1)

Solution

By solving the equations

$$\frac{x}{3} = 0 \qquad \text{and} \qquad \frac{x}{3} = \pi$$

$$x = 0 \qquad\qquad x = 3\pi$$

you can see that two consecutive vertical asymptotes occur at $x = 0$ and $x = 3\pi$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 1.63. Note that the period is 3π, the distance between consecutive asymptotes.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$2 \cot \frac{x}{3}$	Undef.	2	0	-2	Undef.

CHECK*Point* Now try Exercise 27.

Solution(Pt. 2)

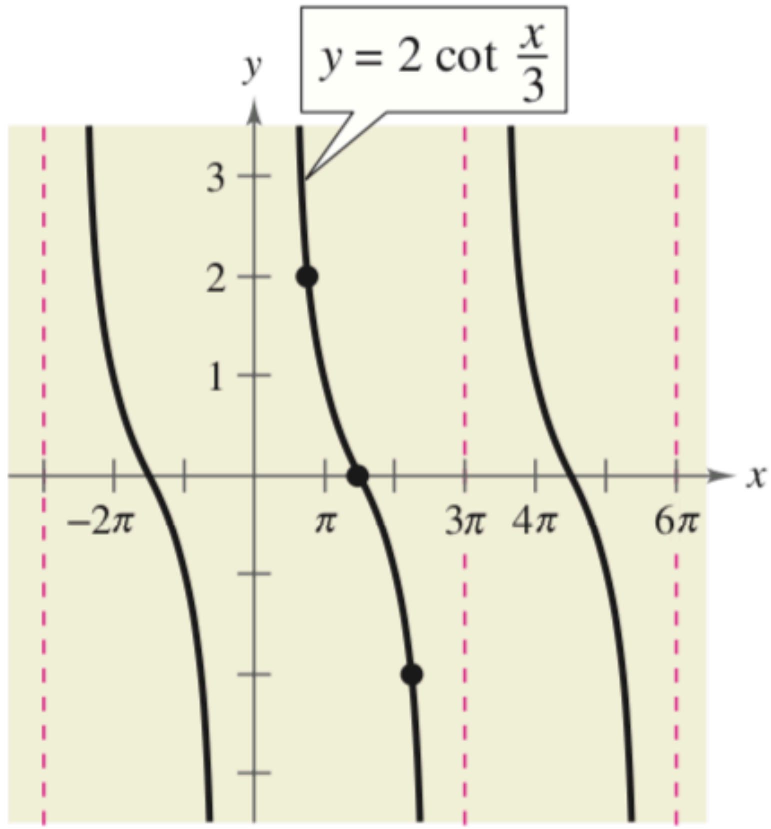
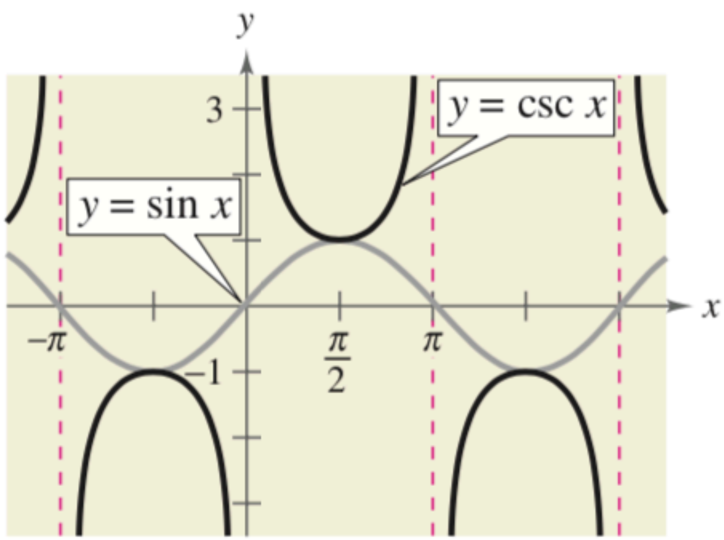


FIGURE 1.63

And the other functions…?

$$\csc x = \frac{1}{\sin x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}.$$



PERIOD: 2π

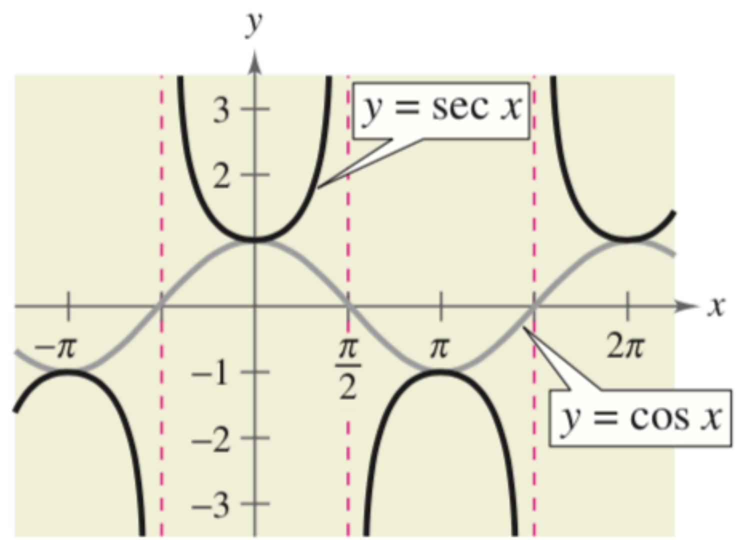
DOMAIN: ALL $x \neq n\pi$

RANGE: $(-\infty, -1] \cup [1, \infty)$

VERTICAL ASYMPTOTES: $x = n\pi$

SYMMETRY: ORIGIN

FIGURE 1.64



PERIOD: 2π

DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$

RANGE: $(-\infty, -1] \cup [1, \infty)$

VERTICAL ASYMPTOTES: $x = \frac{\pi}{2} + n\pi$

SYMMETRY: y-AXIS

Try as a Class

Sketch the graph of $y = 2\csc\left(x + \dfrac{\pi}{4}\right)$.

Solution(Pt. 1)

Solution

Begin by sketching the graph of

$$y = 2\sin\left(x + \frac{\pi}{4}\right).$$

For this function, the amplitude is 2 and the period is 2π. By solving the equations

$$x + \frac{\pi}{4} = 0 \qquad \text{and} \qquad x + \frac{\pi}{4} = 2\pi$$

$$x = -\frac{\pi}{4} \qquad\qquad\qquad x = \frac{7\pi}{4}$$

you can see that one cycle of the sine function corresponds to the interval from $x = -\pi/4$ to $x = 7\pi/4$. The graph of this sine function is represented by the gray curve in Figure 1.66. Because the sine function is zero at the midpoint and endpoints of this interval, the corresponding cosecant function

$$\begin{aligned} y &= 2\csc\left(x + \frac{\pi}{4}\right) \\ &= 2\left(\frac{1}{\sin[x + (\pi/4)]}\right) \end{aligned}$$

has vertical asymptotes at $x = -\pi/4, x = 3\pi/4, x = 7\pi/4$, etc. The graph of the cosecant function is represented by the black curve in Figure 1.66.

CHECK*Point* Now try Exercise 33.

Solution(Pt. 2)

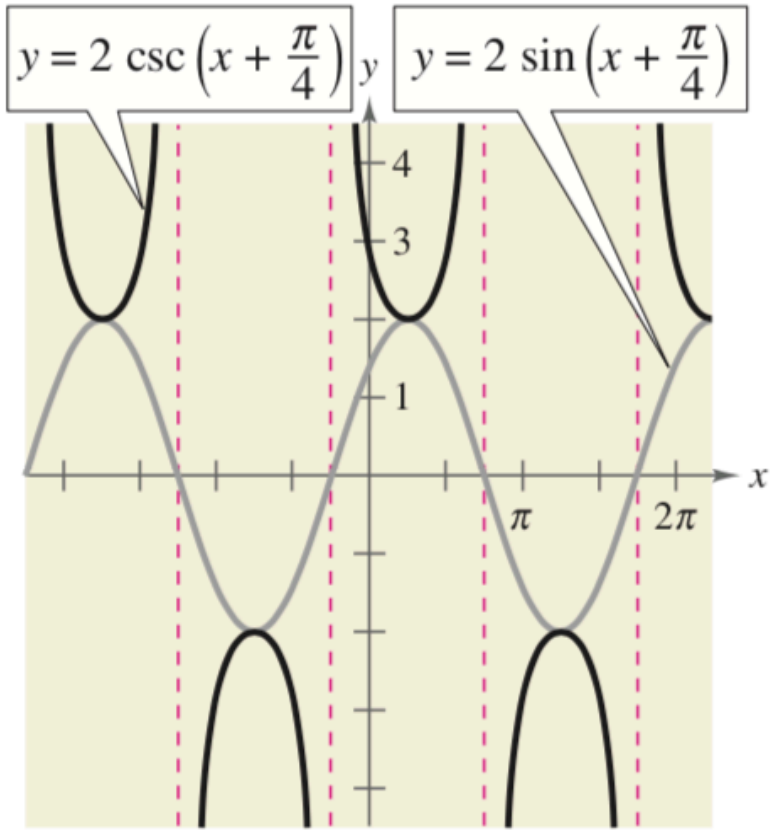


FIGURE 1.66

Try as a Class

Sketch the graph of $y = \sec 2x$.

Solution(Pt. 1)

Solution

Begin by sketching the graph of $y = \cos 2x$, as indicated by the gray curve in Figure 1.67. Then, form the graph of $y = \sec 2x$ as the black curve in the figure. Note that the x-intercepts of $y = \cos 2x$

$$\left(-\frac{\pi}{4}, 0\right), \qquad \left(\frac{\pi}{4}, 0\right), \qquad \left(\frac{3\pi}{4}, 0\right), \ldots$$

correspond to the vertical asymptotes

$$x = -\frac{\pi}{4}, \qquad x = \frac{\pi}{4}, \qquad x = \frac{3\pi}{4}, \ldots$$

of the graph of $y = \sec 2x$. Moreover, notice that the period of $y = \cos 2x$ and $y = \sec 2x$ is π.

CHECK*Point* Now try Exercise 35.

Solution(Pt. 2)

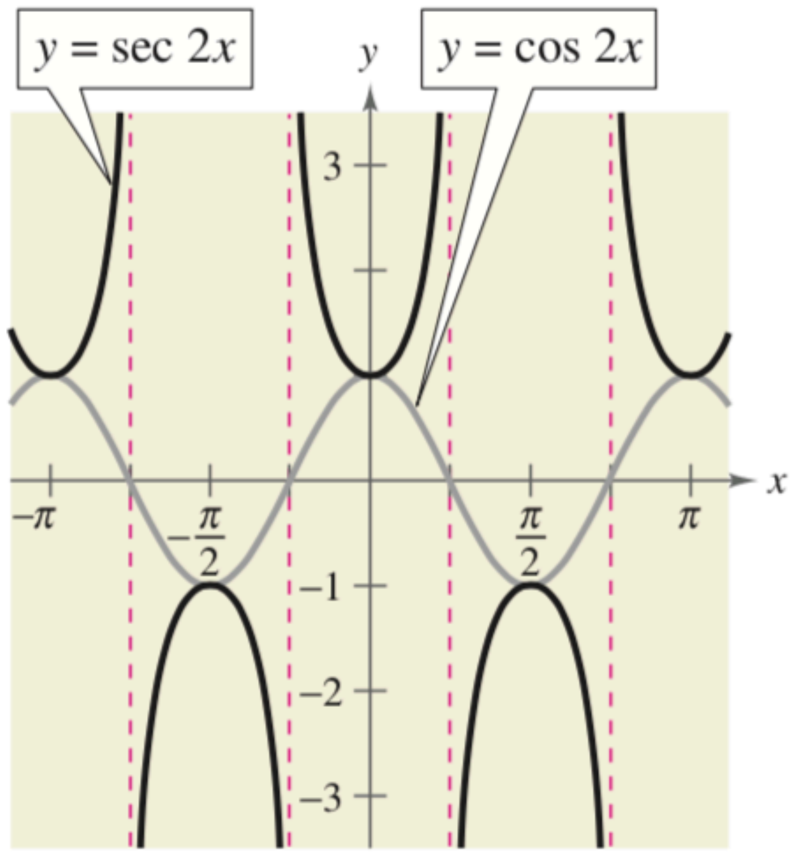


FIGURE 1.67

…now… for the fun stuff!

Damped Trigonometric Graphs

A *product* of two functions can be graphed using properties of the individual functions. For instance, consider the function

$$f(x) = x \sin x$$

Example

$$-|x| \le x \sin x \le |x|$$

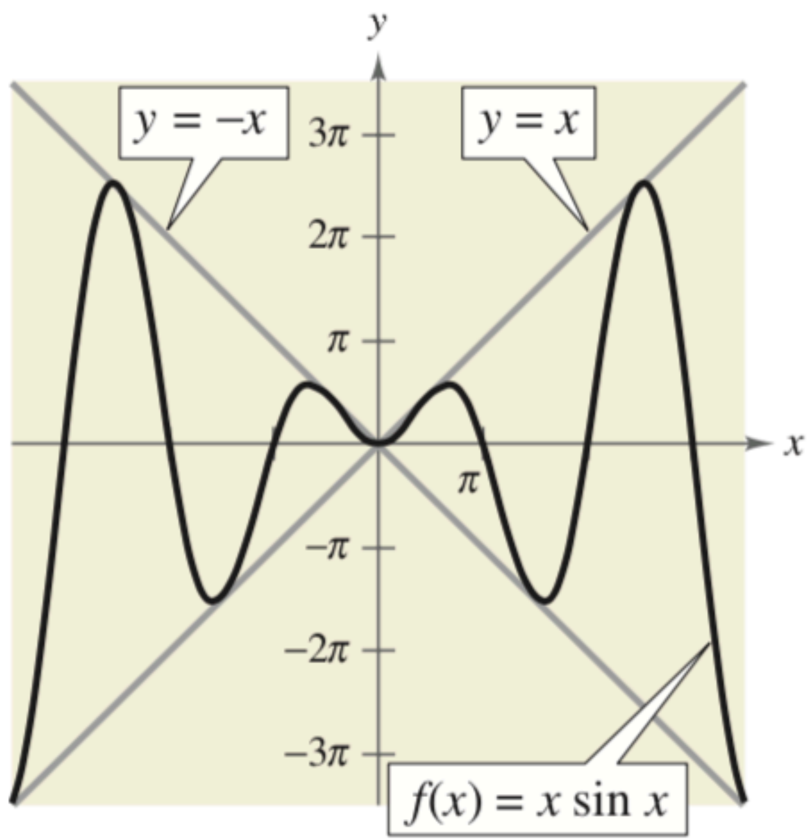


FIGURE 1.68

Try as a Class

Sketch the graph of

$$f(x) = x^2 \sin 3x.$$

Solution(Pt. 1)

Solution

Consider $f(x)$ as the product of the two functions

$$y = x^2 \qquad \text{and} \qquad y = \sin 3x$$

each of which has the set of real numbers as its domain. For any real number x, you know that $x^2 \geq 0$ and $|\sin 3x| \leq 1$. So, $x^2\,|\sin 3x| \leq x^2$, which means that

$$-x^2 \leq x^2 \sin 3x \leq x^2.$$

Furthermore, because

$$f(x) = x^2 \sin 3x = \pm x^2 \quad \text{at} \quad x = \frac{\pi}{6} + \frac{n\pi}{3}$$

and

$$f(x) = x^2 \sin 3x = 0 \quad \text{at} \quad x = \frac{n\pi}{3}$$

the graph of f touches the curves $y = -x^2$ and $y = x^2$ at $x = \pi/6 + n\pi/3$ and has intercepts at $x = n\pi/3$. A sketch is shown in Figure 1.69.

CHECKPoint Now try Exercise 81.

Solution(Pt. 2)

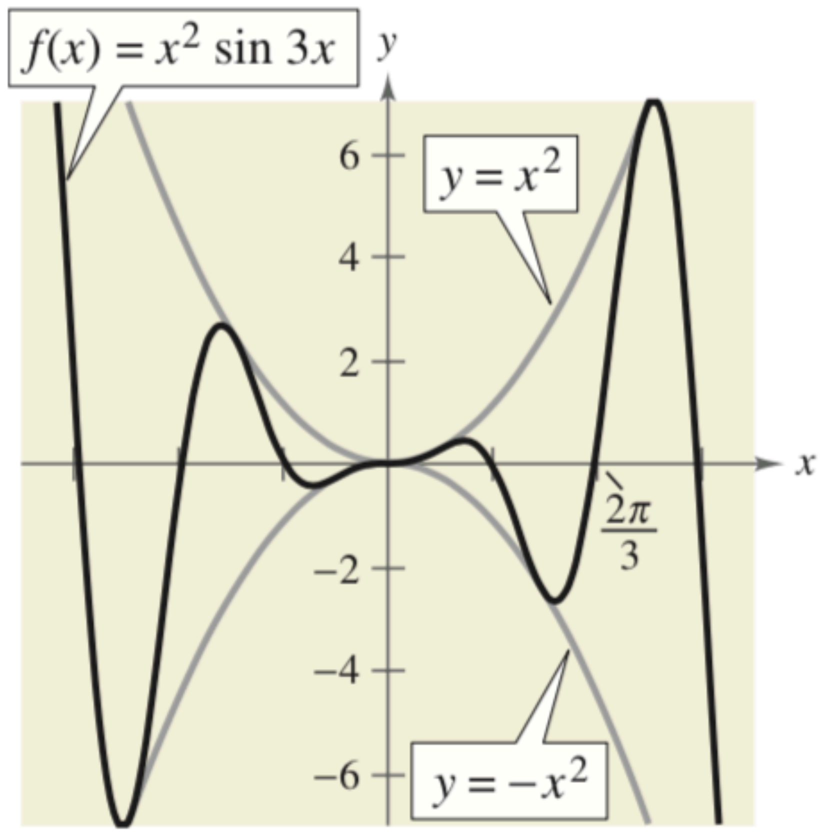
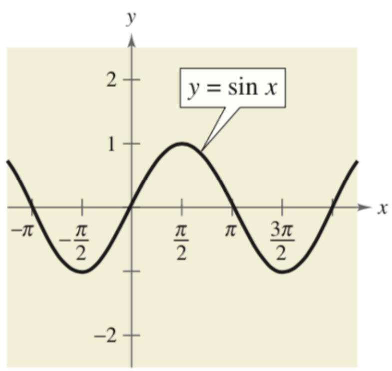
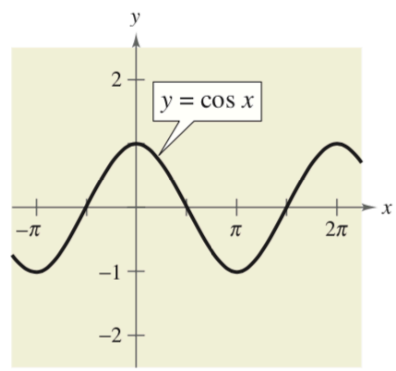


FIGURE 1.69

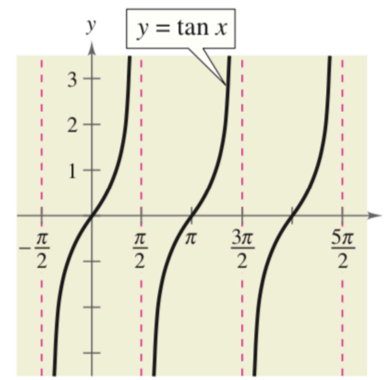
Review of the the graph of basic trig. functions



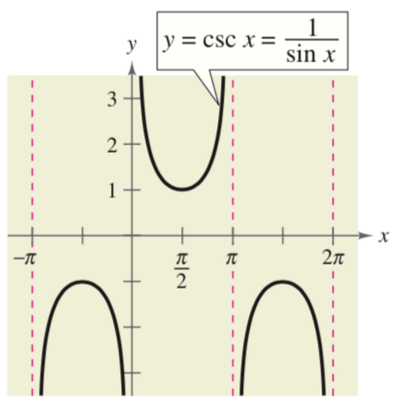
DOMAIN: $(-\infty, \infty)$
RANGE: $[-1, 1]$
PERIOD: 2π



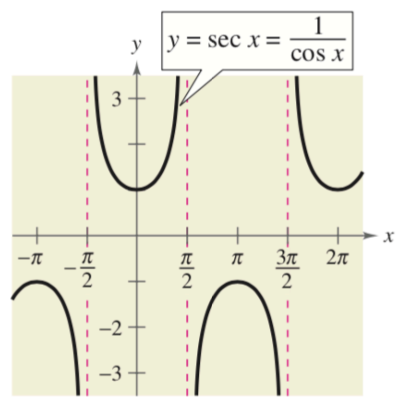
DOMAIN: $(-\infty, \infty)$
RANGE: $[-1, 1]$
PERIOD: 2π



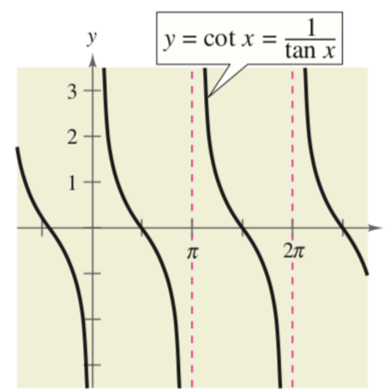
DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π

FIGURE 1.70