

Sec 1.2-1.3

Day 2

The Unit Circle

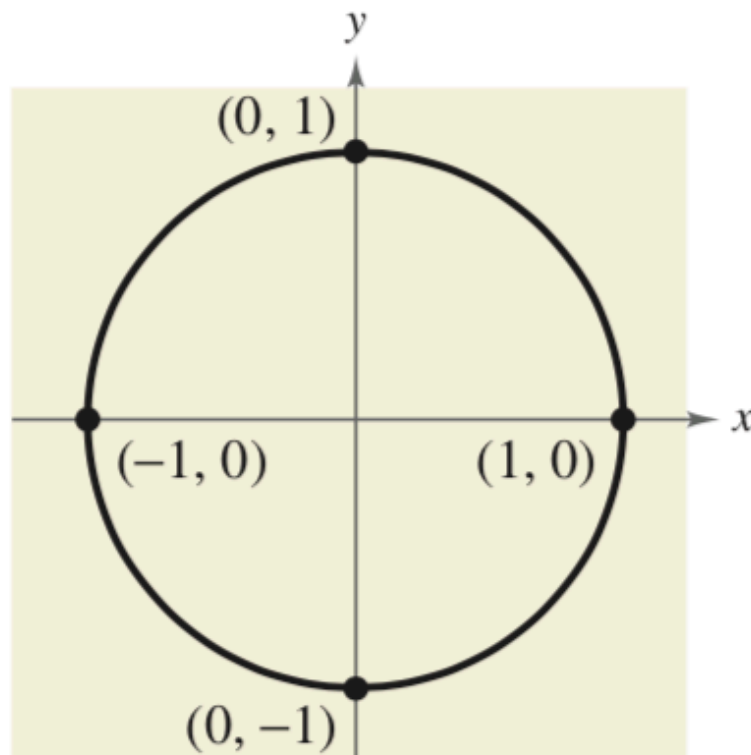


FIGURE 1.20

QOTW

- What's the equation of the unit circle?

Solution:

$$x^2+y^2=1$$

Definitions of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t .

$$\sin t = y$$

$$\cos t = x$$

$$\tan t = \frac{y}{x}, \quad x \neq 0$$

$$\csc t = \frac{1}{y}, \quad y \neq 0$$

$$\sec t = \frac{1}{x}, \quad x \neq 0$$

$$\cot t = \frac{x}{y}, \quad y \neq 0$$

(x,y) coordinates of 12 points on the unit circle

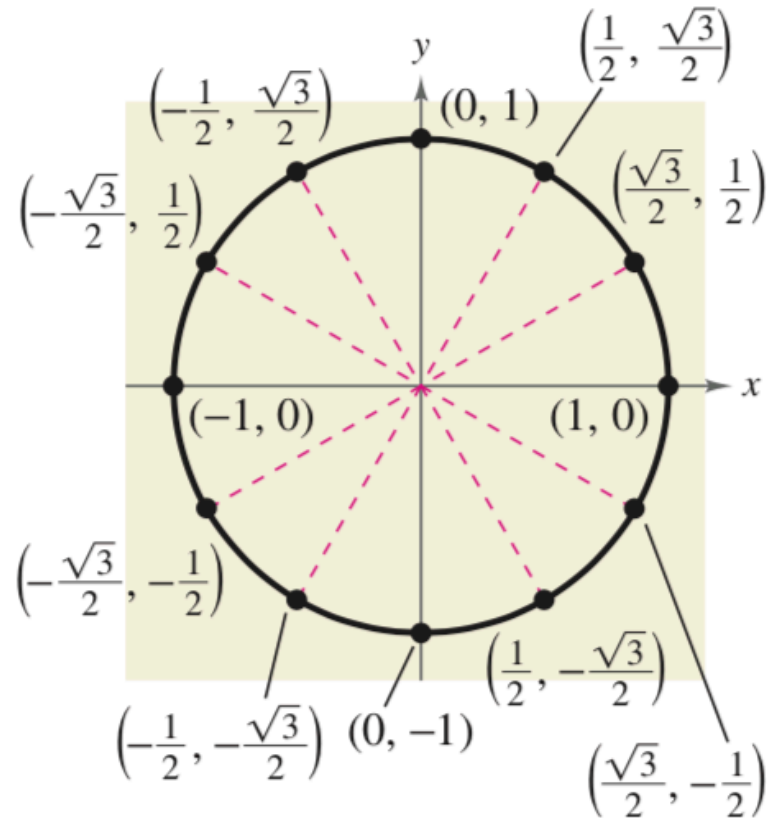


FIGURE 1.23

Try as a Class

Evaluate the six trigonometric functions at each real number.

a. $t = \frac{\pi}{6}$ **b.** $t = \frac{5\pi}{4}$ **c.** $t = 0$ **d.** $t = \pi$

Solutions

Solution

For each t -value, begin by finding the corresponding point (x, y) on the unit circle. Then use the definitions of trigonometric functions listed on page 147.

a. $t = \frac{\pi}{6}$ corresponds to the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

$$\sin \frac{\pi}{6} = y = \frac{1}{2}$$

$$\csc \frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\cos \frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$$

$$\sec \frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan \frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot \frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

b. $t = \frac{5\pi}{4}$ corresponds to the point $(x, y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$$\sin \frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2}$$

$$\csc \frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cos \frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2}$$

$$\sec \frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\tan \frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

$$\cot \frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

c. $t = 0$ corresponds to the point $(x, y) = (1, 0)$.

$$\sin 0 = y = 0$$

$$\csc 0 = \frac{1}{y} \text{ is undefined.}$$

$$\cos 0 = x = 1$$

$$\sec 0 = \frac{1}{x} = \frac{1}{1} = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$$

$$\cot 0 = \frac{x}{y} \text{ is undefined.}$$

Example

Evaluate the six trigonometric functions at $t = -\frac{\pi}{3}$.

Solution

Moving *clockwise* around the unit circle, it follows that $t = -\pi/3$ corresponds to the point $(x, y) = (1/2, -\sqrt{3}/2)$.

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\csc\left(-\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\sec\left(-\frac{\pi}{3}\right) = 2$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

$$\cot\left(-\frac{\pi}{3}\right) = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Definition of Periodic Function

A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f . The smallest number c for which f is periodic is called the **period** of f .

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$$\cos(-t) = \cos t \quad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$$\sin(-t) = -\sin t \quad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \quad \cot(-t) = -\cot t$$

Using the Period to Evaluate the Sine and Cosine

a. Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, you have $\sin \frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}$.

b. Because $-\frac{7\pi}{2} = -4\pi + \frac{\pi}{2}$, you have

$$\cos\left(-\frac{7\pi}{2}\right) = \cos\left(-4\pi + \frac{\pi}{2}\right) = \cos \frac{\pi}{2} = 0.$$

c. For $\sin t = \frac{4}{5}$, $\sin(-t) = -\frac{4}{5}$ because the sine function is odd.

Take a moment and try to relate this
back to a circle...

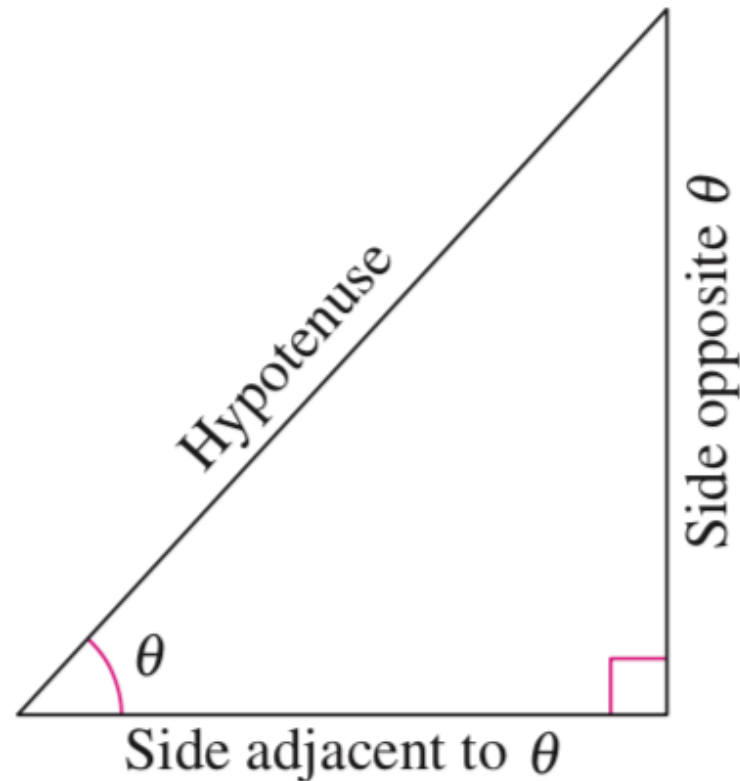


FIGURE 1.26

Right Triangle Definitions of Trigonometric Functions

Let θ be an *acute* angle of a right triangle. The six trigonometric functions of the angle θ are defined as follows. (Note that the functions in the second row are the *reciprocals* of the corresponding functions in the first row.)

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos \theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec \theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

The abbreviations opp, adj, and hyp represent the lengths of the three sides of a right triangle.

opp = the length of the side *opposite* θ

adj = the length of the side *adjacent* to θ

hyp = the length of the *hypotenuse*

Try as a Class

Use the triangle in Figure 1.27 to find the values of the six trigonometric functions of θ .

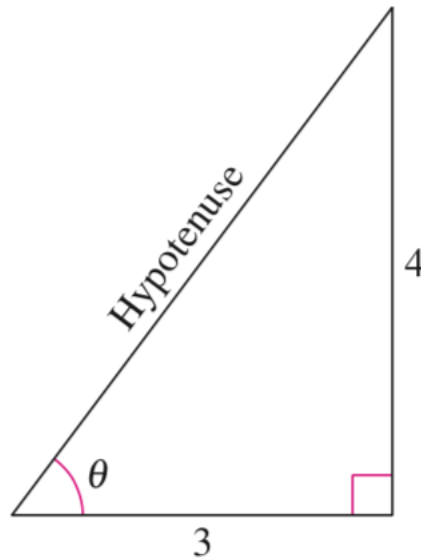


FIGURE 1.27

Solution

Solution

By the Pythagorean Theorem, $(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$, it follows that

$$\begin{aligned}\text{hyp} &= \sqrt{4^2 + 3^2} \\ &= \sqrt{25} \\ &= 5.\end{aligned}$$

So, the six trigonometric functions of θ are

$$\begin{aligned}\sin \theta &= \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} = \frac{5}{4} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} = \frac{5}{3} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} = \frac{4}{3} & \cot \theta &= \frac{\text{adj}}{\text{opp}} = \frac{3}{4}.\end{aligned}$$

Try as a Class

Find the values of $\sin 45^\circ$, $\cos 45^\circ$, and $\tan 45^\circ$.

Solution

Solution

Construct a right triangle having 45° as one of its acute angles, as shown in Figure 1.28. Choose the length of the adjacent side to be 1. From geometry, you know that the other acute angle is also 45° . So, the triangle is isosceles and the length of the opposite side is also 1. Using the Pythagorean Theorem, you find the length of the hypotenuse to be $\sqrt{2}$.

$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

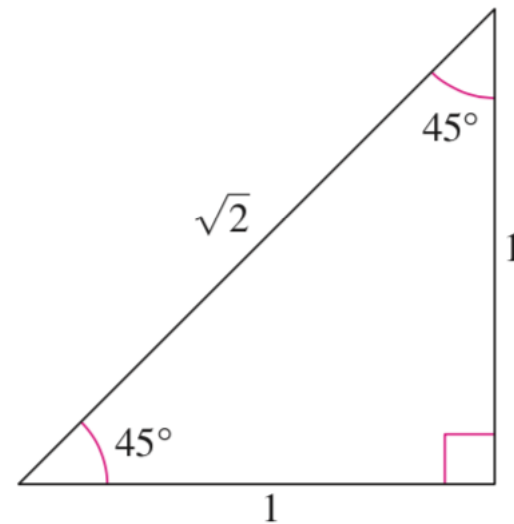


FIGURE 1.28

Try as a Class

Use the equilateral triangle shown in Figure 1.29 to find the values of $\sin 60^\circ$, $\cos 60^\circ$, $\sin 30^\circ$, and $\cos 30^\circ$.

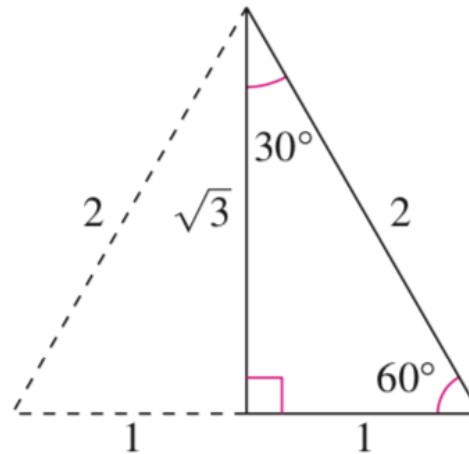


FIGURE 1.29

Solution

Solution

Use the Pythagorean Theorem and the equilateral triangle in Figure 1.29 to verify the lengths of the sides shown in the figure. For $\theta = 60^\circ$, you have $\text{adj} = 1$, $\text{opp} = \sqrt{3}$, and $\text{hyp} = 2$. So,

$$\sin 60^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}.$$

For $\theta = 30^\circ$, $\text{adj} = \sqrt{3}$, $\text{opp} = 1$, and $\text{hyp} = 2$. So,

$$\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \quad \text{and} \quad \cos 30^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}.$$

PRO TIP

Sines, Cosines, and Tangents of Special Angles

$$\sin 30^\circ = \sin \frac{\pi}{6} = \frac{1}{2} \quad \cos 30^\circ = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \tan 30^\circ = \tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\sin 45^\circ = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \cos 45^\circ = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \tan 45^\circ = \tan \frac{\pi}{4} = 1$$

$$\sin 60^\circ = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \cos 60^\circ = \cos \frac{\pi}{3} = \frac{1}{2} \quad \tan 60^\circ = \tan \frac{\pi}{3} = \sqrt{3}$$

KNOW THESE!

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

Quotient Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

Try as a Class

Let θ be an acute angle such that $\sin \theta = 0.6$. Find the values of (a) $\cos \theta$ and (b) $\tan \theta$ using trigonometric identities.

Solution(Pt 1)

Solution

- a. To find the value of $\cos \theta$, use the Pythagorean identity

$$\sin^2 \theta + \cos^2 \theta = 1.$$

So, you have

$$(0.6)^2 + \cos^2 \theta = 1$$

Substitute 0.6 for $\sin \theta$.

$$\cos^2 \theta = 1 - (0.6)^2 = 0.64$$

Subtract $(0.6)^2$ from each side.

$$\cos \theta = \sqrt{0.64} = 0.8.$$

Extract the positive square root.

- b. Now, knowing the sine and cosine of θ , you can find the tangent of θ to be

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$= \frac{0.6}{0.8}$$

$$= 0.75.$$

Use the definitions of $\cos \theta$ and $\tan \theta$, and the triangle shown in Figure 1.30, to check these results.

Solution(Pt 2)

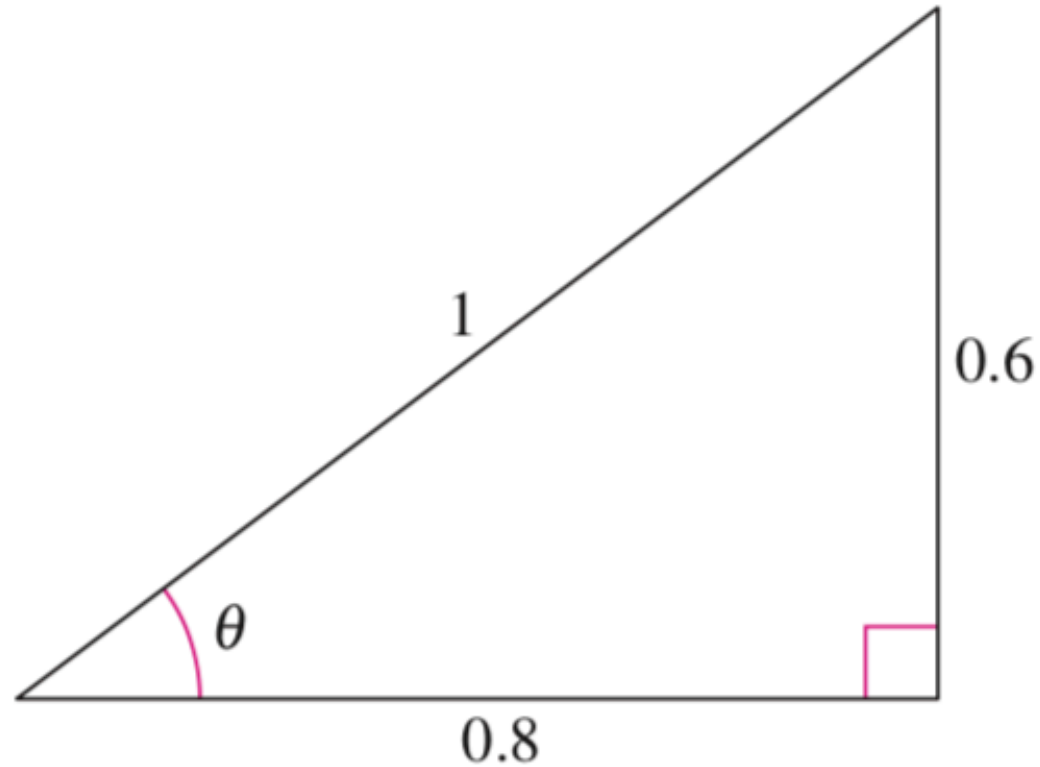


FIGURE 1.30

Try as a Class

Let θ be an acute angle such that $\tan \theta = 3$. Find the values of (a) $\cot \theta$ and (b) $\sec \theta$ using trigonometric identities.

Solution

Solution

a. $\cot \theta = \frac{1}{\tan \theta}$

$$\cot \theta = \frac{1}{3}$$

b. $\sec^2 \theta = 1 + \tan^2 \theta$

$$\sec^2 \theta = 1 + 3^2$$

$$\sec^2 \theta = 10$$

$$\sec \theta = \sqrt{10}$$

Reciprocal identity

Pythagorean identity

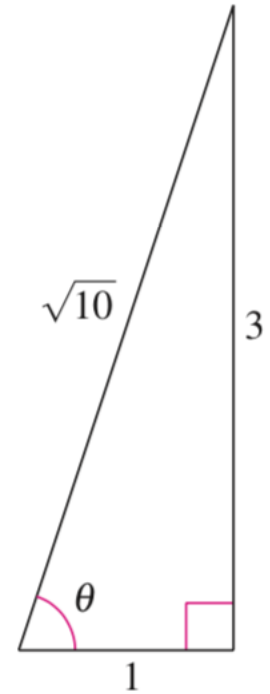


FIGURE 1.31

Use the definitions of $\cot \theta$ and $\sec \theta$, and the triangle shown in Figure 1.31, to check these results.

Angle of...

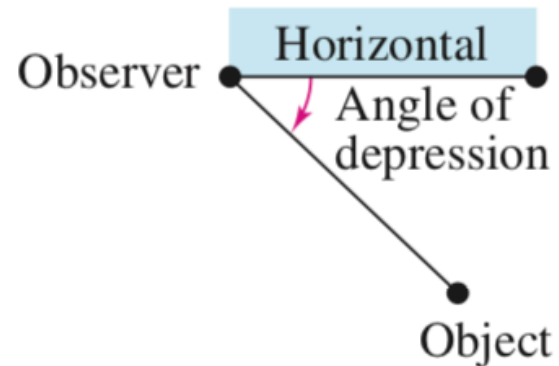
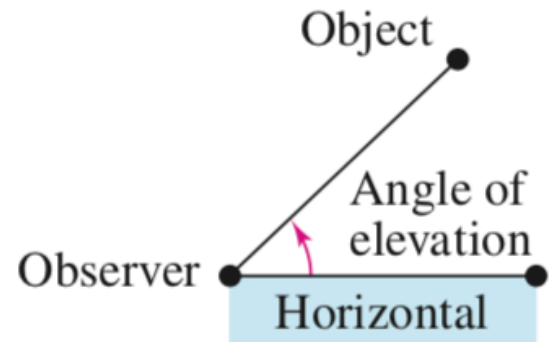


FIGURE 1.32

Try as a Class

A surveyor is standing 115 feet from the base of the Washington Monument, as shown in Figure 1.33. The surveyor measures the angle of elevation to the top of the monument as 78.3° . How tall is the Washington Monument?

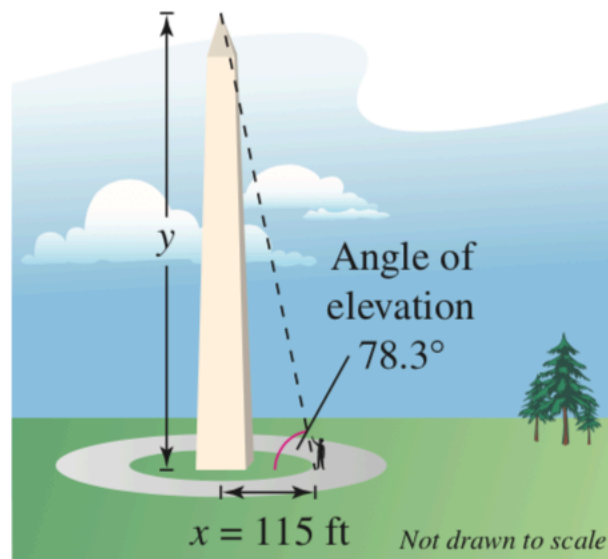


FIGURE 1.33

Solution

Solution

From Figure 1.33, you can see that

$$\tan 78.3^\circ = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

where $x = 115$ and y is the height of the monument. So, the height of the Washington Monument is

$$y = x \tan 78.3^\circ \approx 115(4.82882) \approx 555 \text{ feet.}$$

Example

A historic lighthouse is 200 yards from a bike path along the edge of a lake. A walkway to the lighthouse is 400 yards long. Find the acute angle θ between the bike path and the walkway, as illustrated in Figure 1.34.

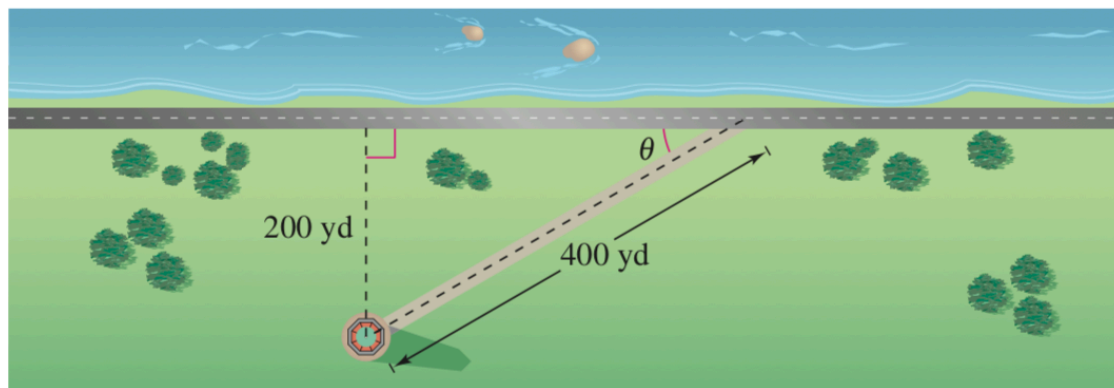


FIGURE 1.34

Solution

From Figure 1.34, you can see that the sine of the angle θ is

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{200}{400} = \frac{1}{2}.$$

Now you should recognize that $\theta = 30^\circ$.

Try as a Class

Find the length c of the skateboard ramp shown in Figure 1.35.

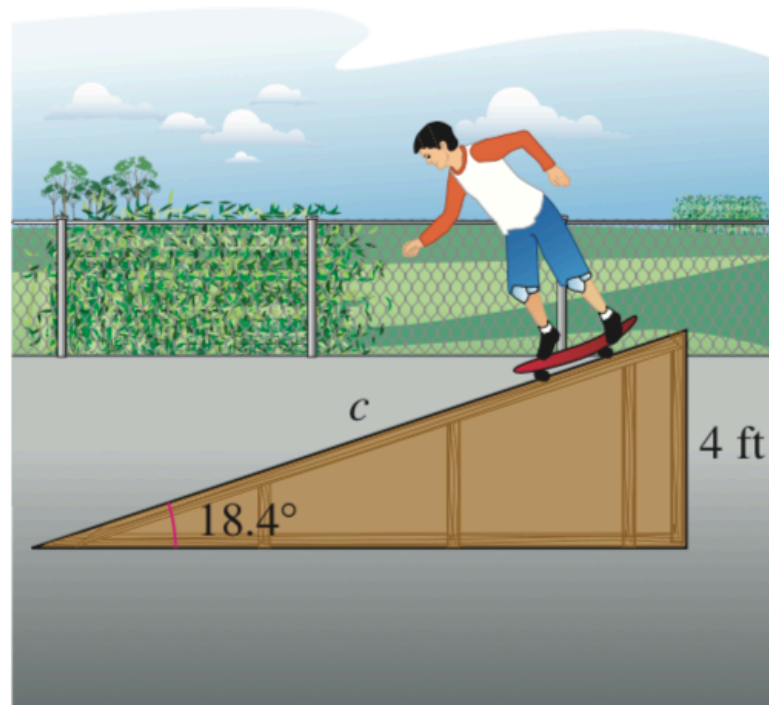


FIGURE 1.35

Solution

Solution

From Figure 1.35, you can see that

$$\begin{aligned}\sin 18.4^\circ &= \frac{\text{opp}}{\text{hyp}} \\ &= \frac{4}{c}.\end{aligned}$$

So, the length of the skateboard ramp is

$$\begin{aligned}c &= \frac{4}{\sin 18.4^\circ} \\ &\approx \frac{4}{0.3156} \\ &\approx 12.7 \text{ feet.}\end{aligned}$$