Calc I

Online Lecture Notes

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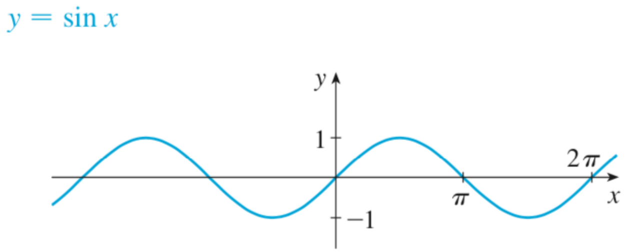
Inverse Trig Functions and their Derivatives

arcsin(x):

Recall, the domain of sin(x) is $\mathbb{R}$, its range is $[-1, 1]$, and has the graph



Clearly, sin(x) fails to be injective. So how can we define an inverse function for it?

Notice, if we restrict the domain of $\sin(x)$ to $[-\pi/2, \pi/2]$ we get that it is injective.

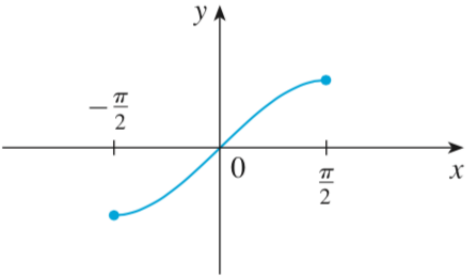


Figure: $\sin(x)$ restricted to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

It is with this restriction that we define $\sin^{-1}(x)$, also written as $\arcsin(x)$.

In other words, for $-1 \leq x \leq 1$ we have $-\frac{\pi}{2} \leq \sin^{-1}(x) \leq \frac{\pi}{2}$.

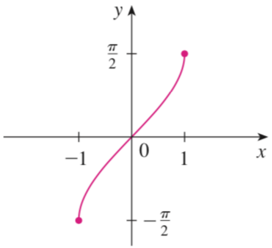


Figure: $\arcsin(x)$ or $\sin^{-1}(x)$

Now, since $\sin(x)$ is continuous and differentiable, we know $\arcsin(x)$.

And from Chapter 6 Section 1 we know

> *If f is a one-to-one differentiable function with inverse function f^{-1} and $f'\left(f^{-1}(a)\right) \neq 0$, then the inverse function is differentiable at a and*
>
> $$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Using this, let's find the derivative of $\arcsin(x)$.

Let $\arcsin(x) = y$, then $\sin(y) = x$; and, by implicit differentiation we get

$$\frac{d}{dx}\sin(y) = \frac{d}{dx}x$$

$$\implies \cos(y)\frac{dy}{dx} = 1$$

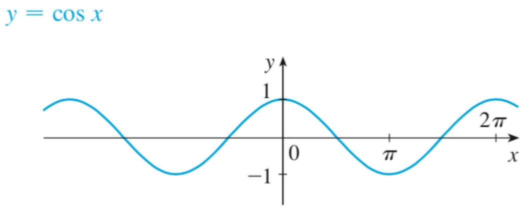
$$\implies \frac{dy}{dx} = \frac{1}{\cos(x)}$$

But, $\cos(y) = \sqrt{1 - \sin^2(y)}$ and we said $\sin(y) = x$ so we get,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} = \frac{d}{dx}\arcsin(x), \quad -1 \leq x \leq 1$$

arccos(x):

Recall, the domain of $\cos(x)$ is $\mathbb{R}$, its range is $[-1, 1]$, and has the graph



Clearly, $\cos(x)$ fails to be injective. So how can we define an inverse function for it?

Notice, if we restrict the domain of $\cos(x)$ to $[0, \pi]$ we get that it is injective.

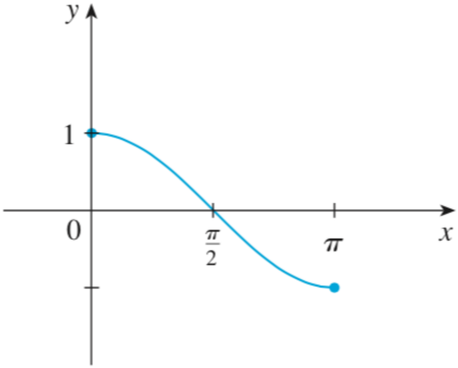


Figure: $\cos(x)$ restricted to $0 \leq x \leq \pi$

It is with this restriction that we define $\cos^{-1}(x)$, also written as $\arccos(x)$.

In other words, for $-1 \leq x \leq 1$ we have $0 \leq \cos^{-1}(x) \leq \pi$.

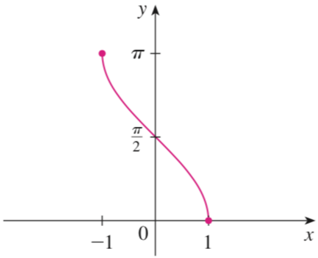


Figure: $\arccos(x)$ or $\cos^{-1}(x)$

Now, since $\cos(x)$ is continuous and differentiable, we know $\arccos(x)$.

And from Chapter 6 Section 1 we know

If f is a one-to-one differentiable function with inverse function f^{-1} and $f'\left(f^{-1}(a)\right) \neq 0$, then the inverse function is differentiable at a and

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Using this, let's find the derivative of $\arccos(x)$.

Let $\arccos(x) = y$, then $\cos(y) = x$; and, by implicit differentiation we get

$$\frac{d}{dx}\cos(y) = \frac{d}{dx}x$$

$$\implies -\sin(y)\frac{dy}{dx} = 1$$

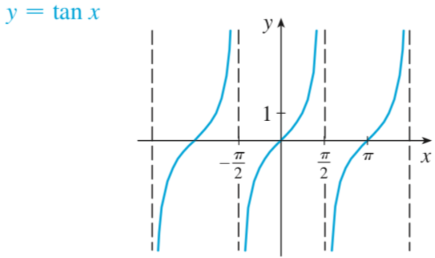
$$\implies \frac{dy}{dx} = -\frac{1}{\sin(x)}$$

But, $\sin(y) = \sqrt{1-\cos^2(y)}$ and we said $\cos(y) = x$ so we get,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}\arccos(x), \quad -1 \leq x \leq 1$$

arctan(x):

Recall, the domain of $\tan(x)$ is $\mathbb{R}$, its range is $\mathbb{R}$, and has the graph



Clearly, $\tan(x)$ fails to be injective. So how can we define an inverse function for it?

Notice, if we restrict the domain of tan(x) to $[-\pi/2, \pi/2]$ we get that it is injective.

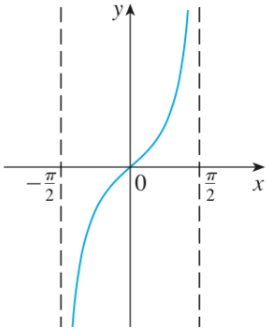


Figure: tan(x) restricted to $0 \leq x \leq \pi$

It is with this restriction that we define $\tan^{-1}(x)$, also written as $\arctan(x)$.

In other words, for $-\infty \leq x \leq \infty$ we have $-\pi/2 \leq \tan^{-1}(x) \leq \pi/2$.

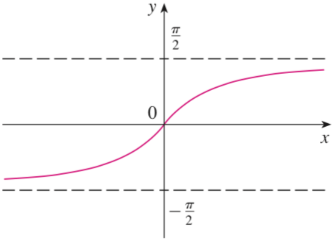


Figure: $\arctan(x)$ or $\tan^{-1}(x)$

Now, since $\tan(x)$ is continuous and differentiable, we know $\arctan(x)$.

And from Chapter 6 Section 1 we know

> *If f is a one-to-one differentiable function with inverse function f^{-1} and $f'\left(f^{-1}(a)\right) \neq 0$, then the inverse function is differentiable at a and*
>
> $$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Using this, let's find the derivative of $\arctan(x)$.

Let $\arctan(x) = y$, then $\tan(y) = x$; and, by implicit differentiation we get

$$\frac{d}{dx}\tan(y) = \frac{d}{dx}x$$

$$\implies \sec^2(y)\frac{dy}{dx} = 1$$

$$\implies \frac{dy}{dx} = \frac{1}{\sec^2(x)}$$

But, $\sec^2(y) = 1 + \tan^2(y)$ and we said $\tan(y) = x$ so we get,

$$\frac{dy}{dx} = \frac{1}{1+x^2} = \frac{d}{dx}\arctan(x).$$

List of Inverse Trig Functions and their Derivatives:

- $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$

Example:

Use the list of trig functions and their derivatives to find:

$$\int \frac{1}{x^2 + a^2} dx$$

Notice, if we multiply by $1 = \frac{1/a^2}{1/a^2}$ we get

$$\int \frac{1}{x^2 + a^2} dx = \int \frac{1}{a^2} \frac{1}{(x/a)^2 + 1} dx = .$$

So if we apply the subsitution $u = x/a$, which yields $\frac{du}{dx} = \frac{1}{a} \implies dx = a\,du$, we get

$$\int \frac{1}{a^2} \frac{a\,du}{u^2 + 1} = \frac{1}{a} \int \frac{du}{u^2 + 1}.$$

From our list we know, $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$ so we have

$$\frac{1}{a}\int \frac{du}{u^2+1} = \frac{1}{a}\arctan(u) + c.$$

And by subsituting $u = x/a$ back in, we find

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c \quad \blacksquare$$

Indeterminate Forms and l′Hospital′s Rule

The indeterminate forms of a limit are:

- $\frac{0}{0}$
- $\frac{\pm\infty}{\pm\infty}$
- $\infty - \infty$
- $0(\infty)$
- 0^0
- 1^∞
- ∞^0

To handle these forms, we make use of a tool called l'Hospital's Rule.

l′Hospital′s Rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0$$

or that $\lim_{x \to a} f(x) = \pm\infty$

$$\lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

It is important to note, inorder to apply l'Hospital's Rule the stated conditions HAVE TO be met.

> *" Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*
>
> $$\lim_{x\to a} f(x) = 0 \quad \textit{and} \quad \lim_{x\to a} g(x) = 0$$
>
> *or that* $\lim_{x\to a} f(x) = \pm\infty$
>
> $$\lim_{x\to a} g(x) = \pm\infty \text{ "}$$

It's also important to note, l'Hospital's Rule can be applied to one sided limits.

Example(Form $\lim_{x\to a}\frac{f(x)}{g(x)}$):

Find

$$\lim_{x\to 1}\frac{\ln(x)}{x-1}.$$

Clearly,

$$\frac{\lim_{x\to 1}\ln(x)}{\lim_{x\to 1}(x-1)}=\frac{0}{0}$$

But by applying l'Hospital's Rule we get,

$$\lim_{x\to 1}\frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}(x-1)}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=\lim_{x\to 1}\frac{1}{x}=1$$

Example(Form $\lim_{x\to a} \frac{f(x)}{g(x)}$):

Find

$$\lim_{x\to\infty} \frac{e^x}{x-1}.$$

Clearly,

$$\frac{\lim_{x\to\infty} e^x}{\lim_{x\to\infty}(x-1)} = \frac{\infty}{\infty}$$

But by applying l'Hospital's Rule we get,

$$\lim_{x\to\infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}(x-1)} = \lim_{x\to\infty} \frac{e^x}{1} = \lim_{x\to\infty} e^x = \infty$$

Example(Form $\lim_{x\to a}\frac{f(x)}{g(x)}$):

Find

$$\lim_{x\to 0}\frac{x}{\sin(x)}.$$

Clearly,

$$\frac{\lim_{x\to 0} x}{\lim_{x\to 0}\sin(x)} = \frac{0}{0}$$

But by applying l'Hospital's Rule we get,

$$\lim_{x\to 0}\frac{\frac{d}{dx}(x)}{\frac{d}{dx}\sin(x)} = \lim_{x\to 0}\frac{1}{\cos(x)} = \lim_{x\to 0}\sec(x) = 1$$

Example(Form $\lim_{x\to a} f(x)g(x)$):

Find

$$\lim_{x\to 0^+} x\ln(x).$$

Clearly,

$$\left(\lim_{x\to 0^+} x\right)\left(\lim_{x\to 0^+} \ln(x)\right) = 0(-\infty)$$

But $x\ln(x) = \frac{\ln(x)}{\frac{1}{x}}$ and by applying l'Hospital's Rule we get,

$$\lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

Example(Form $\lim_{x\to a} f(x)g(x)$):

Find

$$\lim_{x\to 0^+} \sin(x)\ln(x).$$

Clearly,

$$\left(\lim_{x\to 0^+} \sin(x)\right)\left(\lim_{x\to 0^+} \ln(x)\right) = 0(-\infty)$$

But $\sin(x)\ln(x) = \frac{\sin(x)}{\frac{1}{\ln(x)}}$ and by applying l'Hospital's Rule we get,

$$\lim_{x\to 0^+} \frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}\ln(x)} = \lim_{x\to 0^+} \frac{\cos(x)}{\frac{1}{x}} = \frac{\lim_{x\to 0^+}\cos(x)}{\lim_{x\to 0^+}\frac{1}{x}} = 0$$

Example(Form $\lim_{x\to a}[f(x) - g(x)]$):

Find

$$\lim_{x\to\pi/2^-}\left[\sec(x) - tan(x)\right].$$

Clearly,

$$\lim_{x\to\pi/2^-}\left[\sec(x) - tan(x)\right] = \infty - \infty$$

But $\sec(x) = \frac{1}{\cos(x)}$ and $\tan(x) = \frac{\sin(x)}{\cos(x)}$ so by applying l'Hospital's Rule we get,

$$\lim_{x\to\pi/2^-}\left[\sec(x) - tan(x)\right] = \lim_{x\to\pi/2^-}\left[\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right] = \lim_{x\to\pi/2^-}\frac{1-\sin(x)}{\cos(x)}$$

Example Continued:

$$\lim_{x\to\pi/2^-}\frac{1-\sin(x)}{\cos(x)}=\lim_{x\to\pi/2^-}\frac{\frac{d}{dx}(1-\sin(x))}{\frac{d}{dx}\cos(x)}=$$

$$\lim_{x\to\pi/2^-}\frac{-\cos(x)}{-\sin(x)}=\lim_{x\to\pi/2^-}\frac{\cos(x)}{\sin(x)}=\frac{0}{1}=0$$

Now, in the case of $\lim_{x\to a} f(x)^{g(x)}$ we can take two different approaches. We can either use

$$y = f(x)^{g(x)} \implies \ln(y) = g(x)\ln(f(x))$$

or we can use

$$y = f(x)^{g(x)} = e^{\ln(f(x))g(x)}$$

Example(Form $\lim_{x\to a} f(x)^{g(x)}$):

Find

$$\lim_{x\to 0} \cos(x)^{1/x}.$$

Let $y = \cos(x)^{1/x}$, then $\ln(y) = \frac{\ln(\cos(x))}{x}$ and

$$\lim_{x\to 0} \frac{\ln(\cos(x))}{x} = \frac{0}{0}$$

But by applying l'Hospital's Rule we get,

$$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} \frac{\frac{d}{dx}\ln(\cos(x))}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{\frac{-\sin(x)}{\cos(x)}}{1} = \lim_{x\to 0} -\tan(x) = 0$$

Example Continued:

So, since $\lim_{x \to 0} \ln(y) = 0$ we get that

$$\lim_{x \to 0} y = \lim_{x \to 0} e^{\ln(y)} = e^0 = 1$$

Example(Form $\lim_{x\to a} f(x)^{g(x)}$):

Find

$$\lim_{x\to 0^+} x^x.$$

Since $x^x = e^{x\ln(x)}$ let's find $\lim_{x\to 0^+} x\ln(x)$ and plug it back in.

We already found $\lim_{x\to 0^+} x\ln(x) = 0$ in an earlier example. So,

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)} = e^0 = 1$$

Notice, both methods reduce to being the exact same. Solving the limit applied to $g(x)\ln(f(x))$ and then applying that solution to the exponential.

Cauchy's Mean Value Theorem:

Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b)

Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Hyperbolic Functions

For the sake of brevity, the following slides on hyperbolic functions will only be lists of functions, identities, and inverses. I will leave the derivation of their derivatives to y'all.

Hyperbolic Functions

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$ (hyperbolic sine)
- $\cosh(x) = \frac{e^x + e^{-x}}{2}$ (hyperbolic cosine)
- $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ (hyperbolic tangent)
- $\operatorname{csch}(x) = \frac{1}{\sinh(x)}$
- $\operatorname{sech}(x) = \frac{1}{\cosh(x)}$
- $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$

Notice the similarities between the hyperbolic functions and our traditional trig functions.

Inverse Hyperbolic Functions

- $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$
- $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$
- $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Hyperbolic Identities

- $\sinh(-x) = -\sinh(x)$
- $\cosh(-x) = \cosh(x)$
- $\cosh^2(x) - \sinh^2(x) = 1$
- $1 - \tanh^2(x) = \operatorname{sech}^2(x)$
- $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
- $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

Appendex

Derivative Identities:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\tan(x) = \sec(x)$

Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

   where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a ”family” of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, "the integral('$\int$') of f(x) with respect to x('dx')."

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
- $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n}x^{n+1}$
- $\int \frac{1}{x}dx = ln(x) + C$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, ”the integral('$\int$') from a to b of f(x) with respect to x('dx').”

Properties of the Definite Integral:

- $\int_a^b c\,dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a,b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Area between Two Curves:

The area between two curves is,

$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k^*)-g(x_k^*)|\Delta x=\int_a^b|f(x)-g(x)|dx$$

Volume of a Solid:

To find the volume of a solid, we calculate

$$V = \int_a^b A(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x.$$

A note: the function $A(x)$,the cross-sectional area of our solid at some x value, is going to depend on the solid.

Volume of a Solid of Revolution

The Disk Method:

When restricted to the volume of a solid of revolution(a solid formed by revolving a region, under a curve, about an axis), we calculate

$$V = \int_a^b \pi f(x)^2 dx.$$

A note: here $A(x) = \pi f(x)^2$, the cross-sectional area of a disk at x formed by revolving our function about the axis.

Volume of a Solid of Revolution

The Washer Method:

To the find the volume between two solids of revolution, we evaluate

$$V = \int_a^b \pi \left| f(x)^2 - g(x)^2 \right| dx.$$

A note: here $A_1(x) = \pi f(x)^2$ and $A_2(x) = \pi g(x)^2$, are the cross-sectional area of disks at x formed by revolving our functions about the axis. Notice too, that since π is positive we just factored it out side of the absolute value.

Work

Given a force F, which can be written as a function $f(x)$ of distance/position, being applied to an object over some distance, let's say $x = a$ to $x = b$; then, the work done by the thing applying the force is

$$W = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)dx = \int_a^b F\, dx.$$

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$*, then there exists at least one number c in* $[a, b]$ *such that*

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c) = f_{ave.\ on\ [a,b]}$$

or

$$\int_a^b f(x)dx = (b-a)f(c)$$

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 5

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 8

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 11 through 13

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 29 and 30