Calc I

Online Lecture Notes

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Expontential Functions and the Derivatives

Recall,

given n a positive integer and b any positive real number,

$$b^n = \underbrace{bb\cdots b}_{n-times},$$

$$b^{-n} = \underbrace{\frac{1}{b}\frac{1}{b}\cdots\frac{1}{b}}_{n-times},$$

and

$$b^{\frac{1}{n}} = \sqrt[n]{b}.$$

We call b the base and n or $\frac{1}{n}$ the exponent, together they

And exponential function $f(x)$ is a function such that,

$$f(x) = b^x.$$

Properties of Exponentials:

- $b^q b^p = b^{q+p}$
- $b^q b^{-p} = b^{q-p}$
- $(b^q)^p$
- $(ab)^p = a^p b^p$
- If $b > 1$, then $\lim_{x\to\infty} b^x = \infty$ and $\lim_{x\to-\infty} b^x = 0$
- If $1 \leq b \leq 0$, then $\lim_{x\to\infty} b^x = 0$ and $\lim_{x\to-\infty} b^x = \infty$

Limit Definition of the Derivative Applied to $f(x) = b^x$:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h}-b^x}{h}$$

$$= \lim_{h\to 0} \frac{b^x b^h - b^x}{h} = b^x \lim_{h\to 0} \frac{b^h - 1}{h}$$

$$= f(x) \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = f(x)f'(0)$$

We're going to come back to this later... But for now, a few examples are on the next slide.

Examples of Limit Definition of the Derivative Applied to $f(x) = b^x$:

- $\frac{d}{dx}2^x \approx 2^x(0.69)$
- $\frac{d}{dx}3^x \approx 3^x(1.10)$
- $\frac{d}{dx}4^x \approx 4^x(1.39)$

So, a two natual question emerge, ”is there a b such that for $f(x) = b^x$, $f'(0) = 1$? and $\frac{d}{dx}b^x = b^x$ skip to slide?” And, ”how can we find the exact values of $f'(0)$ skip to slide?”

Definition of the Number e:

$e \approx 2.718281828459045235 36$ is the number such that,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Notice if we set $f(x) = e^x$ and $x = 0$, then

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

So, the derivative of $f(x) = e^x$ is,

$$\frac{d}{dx} f(x) = e^x.$$

And when we apply chain rule we find the derivative of e^u is

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

or equivalently written,

$$\frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x)$$

Example:

- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}e^{2x+3} = e^{2x+3}(2)$
- $\frac{d}{dx}e^{\cos(x)} = e^{\cos(x)}(-\sin(x))$
- $\frac{d}{dx}\left(5x^2 + 3 + 4e^{3+x+2x^2+\cos(x)}\right) =$
  $10x + 0 + 4e^{3+x+2x^2+\cos(x)}\,(0 + 1 + 4x - \sin(x))$

Another Definition of the Number e:

Another common limit definition of e is,

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

And for e^x,

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Integral of e^x is

$$\int e^x dx = e^x + c$$

so when we have, $e^u \frac{du}{dx}$ we can apply subsitution($du = \frac{du}{dx} dx$) and get

$$\int e^u \frac{du}{dx} dx = \int e^u du = e^u + c$$

Example:

- $\int 4e^{4x}dx = e^{4x} + c \implies u = 4x$ and $du = 4dx$
- $\int \sec^2(x)e^{\tan(x)}dx = e^{\tan(x)} + c \implies u = \tan(x)$ and $du = \sec^2(x)dx$
- $\int xe^{\frac{x^2}{2}}dx = e^{\frac{x^2}{2}} + c \implies u = \frac{x^2}{2}$ and $du = xdx$
- $\int -13e^{x}dx = -13e^{x} + c \implies u = x$ and $du = dx$

Logarithmic Functions and the Derivatives

If

$$b^x = y,$$

then

$$\sqrt[x]{y} = b \quad \text{and} \quad \log_b(y) = x.$$

So,

$$\text{if } f(x) = b^x, \text{ then } f^{-1}(x) = \log_b(x).$$

Properties of Logarithms:

- $\log_b(b^x) = x$
- $b^{\log_b(x)} = x$ for every $x > 0$

And if $b > 1$, the function $f(x) = \log_b(x)$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbf{R}$. If $x, y > 0$and p is any real number, then

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
- $\log_b(x^p) = p \log_b(x)$

Limit Properties of Logarithms:

If $b > 0$, then

- $\lim_{x\to\infty} \log_b(x) = \infty$
- $\lim_{x\to 0^+} \log_b(x) = -\infty$

Examples:

- $2^5 = 32 \implies \log_2(32) = 5$
- $3^5 = 243$ and $3^2 = 9 \implies \log_3(\frac{243}{9})$
  $= \log_3(243) - \log_3(9) = 5 - 2 = 3$
- $\log_2(768) - \log_2(3) = \log_2(\frac{768}{3}) = \log_2(256) = 8$
- $\log_5(725) = \log_5\left((29)(25)\right)$
  $= \log_5(29) + \log_5(5^2) = \log_5(29) + 2$

Notation:

There are two commonly used logorithmic bases with their own unique notation:

- $b = 10, \quad \log_{10}(x) = \log(x)$
- $b = e, \quad \log_{e}(x) = \ln(x)$

By far, $b = e$ is the most commonly used base. So much so, that it wasn't enough to give it its own notation, we also gave it it's own name the, "natural logorithm."

Notice, $b^x = e^{\ln(b)x}$ so,

$$\frac{d}{dx}b^x = \frac{d}{dx}e^{\ln(b)x} = \ln(b)e^{\ln(b)x} = \ln(b)b^x$$

This also means,

$$\int b^x dx = \frac{b^x}{\ln(b)} + c$$

Examples:

- $\ln(5x+3) = 9 \implies e^{\ln(5x+3)} = e^9$

$$= 5x + 3 = e^9$$

$$\implies 5x = e^9 - 3$$

$$\implies x = \frac{e^9 - 3}{5}$$

Examples:

- $e^{6-4x} = 9 \implies \ln\left(e^{6-4x}\right) = \ln(9)$

$$= 6 - 4x = \ln(9)$$

$$\implies -4x = \ln(9) - 6$$

$$\implies x = -\frac{\ln(9) - 6}{4} = \frac{6 - \ln(9)}{4}$$

What if we have our answer in terms of a base other than what we wanted? So, let's say we have $\ln(x)$ but want $\log_b(x)$. Well,

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Derivative of Logorithmic Functions:

- $\frac{d}{dx}\ln(x) = \frac{1}{x}$
- $\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}$ or equivalently written $\frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)}g'(x)$

Notice, from that second formula,

$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}$$

so

$$\int \frac{1}{x}dx = \ln(|x|) + c$$

Integral Example:

Exapanding, now, on $\int \frac{1}{x} dx = \ln(|x|) + c$. Let's say we want to find $\int \tan(x) dx$. How might we do that?

First, let's observe $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and

$$\int \tan(x) dx = \int \frac{\sin(x)}{\cos(x)} dx.$$

So, by setting $u = \cos(x)$ we get $\frac{du}{dx} = -\sin(x) \implies du = -\sin(x)dx$ and

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = \int \frac{-1}{u}du = -\ln(|u|)+c = -\ln(|\cos(x)|)+c$$

But $-\ln(x) = \ln\left(x^{-1}\right) = \ln\left(\frac{1}{x}\right)$ and $\frac{1}{\cos(x)} = \sec(x)$, so

$$\int \tan(x)dx = -\ln(|\cos(x)|) + c = \ln(|\sec(x)|) + c$$

Appendex

Derivative Identities:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\tan(x) = \sec(x)$

Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

   where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a "family" of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, "the integral('$\int$') of f(x) with respect to x('dx')."

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
- $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n}x^{n+1}$
- $\int \frac{1}{x}dx = ln(x) + C$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, "the integral('$\int$') from a to b of f(x) with respect to x('dx')."

Properties of the Definite Integral:

- $\int_a^b c\, dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a,b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Area between Two Curves:

The area between two curves is,

$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k^*)-g(x_k^*)|\Delta x=\int_a^b |f(x)-g(x)|dx$$

Volume of a Solid:

To find the volume of a solid, we calculate

$$V = \int_a^b A(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x.$$

A note: the function $A(x)$,the cross-sectional area of our solid at some x value, is going to depend on the solid.

Volume of a Solid of Revolution

The Disk Method:

When restricted to the volume of a solid of revolution(a solid formed by revolving a region, under a curve, about an axis), we calculate

$$V = \int_a^b \pi f(x)^2 dx.$$

A note: here $A(x) = \pi f(x)^2$, the cross-sectional area of a disk at x formed by revolving our function about the axis.

Volume of a Solid of Revolution

The Washer Method:

To the find the volume between two solids of revolution, we evaluate

$$V = \int_a^b \pi \left| f(x)^2 - g(x)^2 \right| dx.$$

A note: here $A_1(x) = \pi f(x)^2$ and $A_2(x) = \pi g(x)^2$, are the cross-sectional area of disks at x formed by revolving our functions about the axis. Notice too, that since π is positive we just factored it out side of the absolute value.

Work

Given a force F, which can be written as a function $f(x)$ of distance/position, being applied to an object over some distance, let's say $x = a$ to $x = b$; then, the work done by the thing applying the force is

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)dx = \int_a^b F\, dx.$$

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists at least one number c in $[a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c) = f_{ave.\ on\ [a,b]}$$

or

$$\int_a^b f(x)dx = (b-a)f(c)$$

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 5

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 8

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 11 through 13

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 29 and 30