Calc I

Online Lecture Notes

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Inverse Functions

Definition: Injective(One-to-One)

A function $f(x)$ is said to be injective if,

$$\text{For each } a \neq b, \text{ then } f(a) \neq f(b).$$

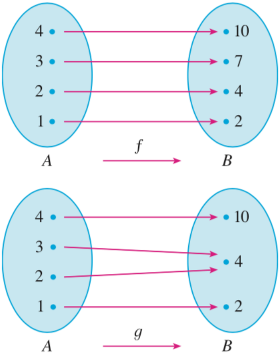

FIGURE 1
f is one-to-one; g is not.

Horizontal Line Test

A function is injective if and only if there exist no horizontal lines which pass through the function more than once.

Consider the functions

- $f(x) = x^2$
- $g(x) = x^3$
- $h(x) = \cos(x)$
- $j(x) = x + 4$

of these, which are injective?

Definition: Inverse Function

Given an injective function $f(x) = y$, its inverse function $f^{-1}(y) = x$ is a function such that

$$f^{-1}(y) = x \iff f(x) = y$$

so

$$f^{-1}\left(f(x)\right) = x \quad and \quad f\left(f^{-1}(y)\right) = y$$

Now, it's important to know where these x and y are coming from.

That way we can understand a bit better what's happening.

The function f takes it's **independent variables**, x, to be from the set A, called it's **domain**; and, *sends/maps* them to their **dependent variable(s)**, y, in the set B, called it's **range**.

The function f^{-1} takes it's **independent variables**, y, to be from the set B, called it's **domain**; and, *sends* them to their **dependent variable(s)**, x, in the set A, called it's **range**.

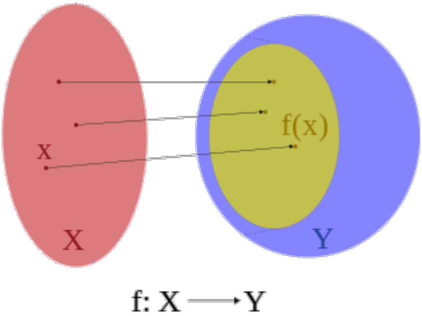

Figure: Here we have a depiction of the function f sending elements of its **domain**(in red), its set of independent variables, to elements in its **range**(in yellow), its set of dependent variabels. The larger blue set is called the ”co-domain” of f.

So,

- the domain of f^{-1} = the range of f
- the range of f^{-1} = the domain of f

To find the inverse of a function:

1. Write $f(x) = y$
2. Solve for x(if possible)
3. Swap x and y.
4. Write $f^{-1}(x) = y$

Let's try two quick examples-

$f(x) = x^3$:

1. $x^3 = y$
2. $x = \sqrt[3]{y}$
3. $y = \sqrt[3]{x}$ (swapping x and y)
4. $f^{-1}(x) = \sqrt[3]{x}$

$g(x) = x^2$:

1. $x^2 = y$
2. $x = \pm\sqrt{y}$
3. $y = \pm\sqrt{x}$ (swapping x and y)
4. $g^{-1}(x) = \pm\sqrt{x}$

But, $g^{-1}(x) = \pm\sqrt{x}$ fails the vertical line test, so it can't be a function. And $g^{-1}(x) = \sqrt{x}$ only gives positive values back, so its range $\neq$ the domain of f. So, $g(x)$ doesn't have an inverse function!

What do we do here? How do we proceed? And, was there an easier way to show/see that?

In the case of $g(x) = x^2$, had we noticed if $a \neq b$, then $g(a) = g(b)$ only when $a = -b$, we could have said ”it fails to be injective so it does not have an inverse.”

Another way, would have to apply an obersvartion about the graphs of functions and their inverses.

The graph of an inverse function $f^{-1}(x)$ is the reflection of the original function $f(x)$'s graph about the line $y = x$.

When we apply that observation to $g(x) = x^2$ we find that the graph of $g^{-1}(x)$ would fail to pass the vertical line test and only give solutions for values 0 or grater. So, g^{-1} can't exist.

BUT! If we had decided to restrict $g(x)$ to $0 \leq x$, then we could say $g^{-1}(x)$ exist!

Theorem:

If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Well if that's the, what can we say about the derivatives of f^{-1}, if they exist?

Let's find the derivative of the inverse function of $f(x) = \sqrt{4-x}$ at $f^{-1}(3)$

Finding $f^{-1}(x)$:

1. $y = \sqrt{4-x}$
2. $x = 4 - y^2$
3. $y = 4 - x^2$ (swapping x and y)
4. $f^{-1}(x) = 4 - x^2, x \geq 0$

So, apply the derivative we get $\frac{d}{dx} f^{-1}(x) = \frac{d}{dx} 4 - x^2 = -2x, x > 0$.

So, $\frac{d}{dx} f^{-1}(3) = -6$

That was long though. If there another way we might be able to find them?

Theorem:

If f is a one-to-one differentiable function with inverse function f^{-1} and $f'\left(f^{-1}(a)\right) \neq 0$, then the inverse function is differentiable at a and

$$\frac{d}{dx}f^{-1}(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Revisiting the derivative of the inverse function of $f(x) = \sqrt{4-x}$ at $f^{-1}(3)$

We know, $f'(x) = -\frac{1}{2\sqrt{4-x}}$. And, $f(-5) = 3$ so $f^{-1}(3) = -5$ and applying the theorem from the last slide we get

$$\frac{d}{dx} f^{-1}(3) = \frac{1}{f'\left(f^{-1}(3)\right)} = \frac{1}{f'(-5)} = -2\sqrt{4+5} = -6$$

A bit less work.

Example:

Find the derivative of f^{-1} at $f^{-1}(\frac{39\pi}{2}+1)$ provided $f(x) = 3x + \sin^2(x)$.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(3x + \sin^2(x)\right) = \frac{d}{dx}(3x) + \frac{d}{dx}\left(\sin^2(x)\right)$$

Applying the "half-angle formula" $\sin^2(x) = \frac{1-\cos(2x)}{2}$ we get,

$$\frac{d}{dx}f(x) = (3) + \frac{1}{2}\frac{d}{dx}(1 - \cos(2x)) = 3 + \frac{1}{2}(0 + 2\sin(2x)) = 3 + \sin(2x)$$

Notice, because $-1 \leq \sin(2x) \leq 1$ we have $0 < f'(x)$ so our function is always increasing. Hence it must be injective, and we can apply $\frac{d}{dx} f^{-1}(a) = \frac{1}{f'(f^{-1}(a))}$.

Now, as $f(\frac{13\pi}{2}) = \frac{39\pi}{2} + 1$ we get,

$$\frac{d}{dx} f^{-1}(\frac{39\pi}{2} + 1) = \frac{1}{f'\left(f^{-1}(\frac{39\pi}{2} + 1)\right)} = \frac{1}{f'\left(\frac{13\pi}{2}\right)} = \frac{1}{3 + \sin(13\pi)} = \frac{1}{3}$$

Appendex

Derivative Identities:

- $\frac{d}{dx} x^n = nx^{n-1}$
- $\frac{d}{dx} \cos(x) = -\sin(x)$
- $\frac{d}{dx} \sin(x) = \cos(x)$
- $\frac{d}{dx} \tan(x) = \sec(x)$

Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

   where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a "family" of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, "the integral('$\int$') of f(x) with respect to x('dx')."

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
- $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n}x^{n+1}$
- $\int \frac{1}{x}dx = ln(x) + C$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, "the integral('$\int$') from a to b of f(x) with respect to x('dx')."

Properties of the Definite Integral:

- $\int_a^b c\,dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a, b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Area between Two Curves:

The area between two curves is,

$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k^*)-g(x_k^*)|\Delta x=\int_a^b|f(x)-g(x)|dx$$

Volume of a Solid:

To find the volume of a solid, we calculate

$$V = \int_a^b A(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x.$$

A note: the function $A(x)$,the cross-sectional area of our solid at some x value, is going to depend on the solid.

Volume of a Solid of Revolution

The Disk Method:

When restricted to the volume of a solid of revolution(a solid formed by revolving a region, under a curve, about an axis), we calculate

$$V = \int_a^b \pi f(x)^2 dx.$$

A note: here $A(x) = \pi f(x)^2$, the cross-sectional area of a disk at x formed by revolving our function about the axis.

Volume of a Solid of Revolution

The Washer Method:

To the find the volume between two solids of revolution, we evaluate

$$V = \int_a^b \pi \left| f(x)^2 - g(x)^2 \right| dx.$$

A note: here $A_1(x) = \pi f(x)^2$ and $A_2(x) = \pi g(x)^2$, are the cross-sectional area of disks at x formed by revolving our functions about the axis. Notice too, that since π is positive we just factored it out side of the absolute value.

Work

Given a force F, which can be written as a function $f(x)$ of distance/position, being applied to an object over some distance, let's say $x = a$ to $x = b$; then, the work done by the thing applying the force is

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)dx = \int_a^b F\, dx.$$

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$*, then there exists at least one number c in* $[a, b]$ *such that*

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c) = f_{ave.\ on\ [a,b]}$$

or

$$\int_a^b f(x)dx = (b-a)f(c)$$

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 5

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 8

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 11 through 13

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 29 and 30