Calc I

Online Lecture Notes

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Work

Formally defined, a physicist would say:

work *is the scalar product of force and the displacement of an object caused by that force.*

Generally speaking, what this means is,

work is,

$$W = Fd$$

where W is work, F is force, and d is displacement(the distance an object was moved by the force).

So, as an example, moving a block by pushing on it with 10_N(N is Newtons, the SI measurement of force and has the dimensions $kg\frac{m}{s^2}$) over a distance of 12_m. Then the work done is

$$10_N\ 12_m = 120_J,$$

where J is for Joules the SI measurement of work and energy and has the dimensions $kg\frac{m^2}{s^2}$.

As another example, let's say we tap a block on a sheet of ice and it goes sliding. Notice, after we tapped the block, we were no longer appling a force on it. So, no matter how far it goes we did 0_J of work.

Now this is great and everything. But, what if the force we aplied wasn't constance? What if, the force the force changed with distance, and thus could be thought of as a function of distance, $F = f(x)$?

We can't exactly say $W = Fd$ any more.

Now, in the case where $F = f(x)$, let's say we move the block from $x = a$ to $x = b$ then

$$W \approx \sum_{k=1}^{n} W_k = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Where $W_k = f(x_k^*)\Delta x$ is the approximate work done on the k^{th} interval of $x = a$ to $x = b$ diveded into n many subintervals of length $\Delta x = \frac{b-a}{n}$.


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And as we've several times now, as we let $n \to \infty$ we get

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)dx = \int_a^b F\, dx.$$

Example 1:

According to hooks law, the force applied by a spring is $F = kx$, where x is how far a spring has been stretched or compressed and k is the spring **constant**.

Now, imagine you hung a block which weighed 11_N on to a spring and the spring stretched 4_m. How much work was done by the block falling?

Because $F = kx$ and spring was stretched to a final resting postion of 4_m, we know

$$W = \int_0^4 kxdx.$$

So now we just need to solve for k to calculate the integral and find the work done.

Since the block weighed 11_N and streched the spring was stretched to a final resting postion of 4_m, we know

$$F = 11_N = k\ 4_m.$$

Solving then for k we find,

$$k = \frac{11_N}{4_m}$$

Plugging $k = \frac{11_N}{4_m}$ into our integral, we get

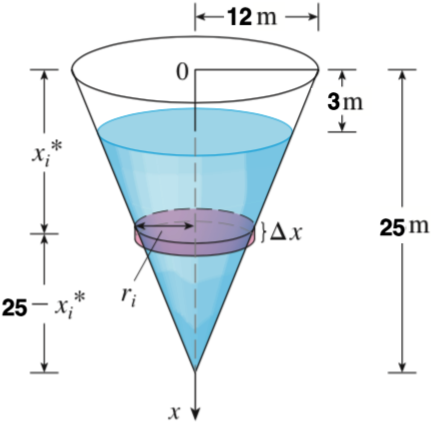
$$W = \int_0^4 \frac{11_N}{4_m} x dx = \frac{11_N}{4_m}\left[\frac{1}{2}x^2\right]_0^4 = \frac{11_N}{8_m}x^2\Big]_0^4 = 16_{m^2}\frac{11_N}{8_m} = 22_{N*m} = 22_J.$$

So, the work done by the block stretching the spring as it fell was $W = 22_J$.

Example 2:

Imagine you had a tank, in the shape of an inverted cone, full of water that you want pumped out. Imagine too, that you wanted to know how much work was needed to pump it all out(maybe your electric bill was way too high so you're trying to prove the electric company they overcharged you. Bet! Gotta save that money!).

Now, let's imagine the tank is 25_m high and 12_m wide at the top; and, that water extends from a depth of 3_m to a depth of 25_m.


12 m
0
3 m
x_i^*
$\}\Delta x$
25 m
r_i
$25 - x_i^*$
x

- We will need to know that the density of water is $\rho_{water} = 1000_{kg/m^3}$, so the mass for some volume of water is $m = V\,1000_{kg/m^3}$
- We will need to know the acceleration due to gravity is $9.8_{m/s^2}$, so force due to gravity is $F_{gravity} = m\,9.8_{m/s^2}$
- Our interval is, $x = 3$ to $x = 25$. To set up the problem by imagine our interval evenly broken up into subintervals
- Notice, the ratio of the radius of the cone and the corresponding height is constant so,

$$\frac{r_k}{25 - x_k^*} = \frac{12}{25} \implies r_k = \frac{12}{25}\left(25 - x_k^*\right)$$

Since, $r_k = \frac{12}{25}(25 - x_k^*)$ we have that

$$V_k = \pi \left(\frac{12}{25}(25 - x_k^*) \right)^2 \Delta x$$

and hence,

$$m_k = 1000_{kg/m^3} \pi \left(\frac{12}{25}(25 - x_k^*) \right)^2 \Delta x = \frac{1152\pi}{5} (25 - x_k^*)^2 \Delta x$$

Therefore

$$F_k = 9.8_{m/s^2}\ \frac{1152\pi}{5}\left(25 - x_k^*\right)^2 \Delta x$$

and

$$W_k = F_k\ x_k^* \approx x_k^*\ 9.8_{m/s^2}\ \frac{1152\pi}{5}\left(25 - x_k^*\right)^2 \Delta x$$

$$= x_k^*\ \frac{11289.6\pi}{5}\left(25 - x_k^*\right)^2 \Delta x.$$

Thus,

$$W \approx \sum_{k=1}^{n} x_k^*\ \frac{11289.6\pi}{5}\left(25 - x_k^*\right)^2 \Delta x.$$

Applying the limit, $n \to \infty$ we get that the work is

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} x_k^* \, \frac{11289.6\pi}{5} \left(25 - x_k^*\right)^2 \Delta x = \int_3^{25} x \, \frac{11289.6\pi}{5} \left(25 - x\right)^2 dx$$

$$= \frac{11289.6\pi}{5} \int_3^{25} x \ \left(25 - x\right)^2 dx = \frac{11289.6\pi}{5} \int_3^{25} x^3 - 50x^2 + 625x dx$$

$$= \frac{11289.6\pi}{5} \left[\frac{x^4}{4} - \frac{50x^3}{3} + \frac{625x^2}{2}\right]_3^{25} = \frac{11289.6\pi}{5} \left(\frac{90508}{3}\right)$$

$$= 68,119,941.11999\pi_J \approx 2.14 \times 10^8_J$$

Average Value of a Function

If we had a set of numbers, say

$$\{y_1, y_2, y_3, ..., y_{n-1}, y_n\},$$

and we're to find their average we would calculate

$$y_{ave} = \frac{1}{n}\sum_{k=1}^{n} y_k$$

Let's imagine now, $y_k = f(x_k^*)$ and x_k^* was from the k^{th} evenly spaced(Δx) subinterval of $x = a$ to $x = b$.

Notice then, since $[a, b]$ is evenly spaced, we can say

$$\Delta x = \frac{b-a}{n} \implies n = \frac{b-a}{\Delta x}.$$

Using our oberservations from the last slide, we can rewrite the average as,

$$y_{ave} = \sum_{k=1}^{n} \frac{y_k}{n} = \sum_{k=1}^{n} \frac{f(x_k^*)}{\left(\frac{b-a}{\Delta x}\right)} = \sum_{k=1}^{n} \frac{f(x_k^*)}{b-a} \Delta x$$

Letting $n \to \infty$ we find

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{f(x_k^*)}{b-a} \Delta x = \frac{1}{b-a} \int_a^b f(x)dx = f_{ave},$$

and thus the average of a function on some interval $[a, b]$.

This brings us to one of the important theorems of Calculus, **The Mean Value Theorem for Integrals**!

The Mean Value Theorem for Integrals states,

If f is continuous on $[a, b]$, then there exists at least one number c in $[a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c)$$

or

$$\int_a^b f(x)dx = (b-a)f(c)$$

Example 1:

Let's use this to find the average of $\cos(x)$ on $[0, \pi/2]$.

$$\frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2}\cos(x)dx = \frac{2}{\pi}\sin(x)\Big]_0^{\pi/2} = \frac{2}{\pi}(1-0) = \frac{2}{\pi}$$

And if we wanted to find that a c, such that $0 \leq c \leq \frac{2}{\pi}$ and $\cos(c) = \frac{2}{\pi}$, we would solve

$$c = \arccos\left(\frac{2}{\pi}\right) = 0.88068\ldots \text{ (in degrees it comes out to } 50.46^\circ\text{)}$$

Example 2:

For a less esoteric example, let's find the average of $f(x) = x^3 - x$ on the interval $[1, 6]$.

$$f_{ave} = \frac{1}{6-1}\int_1^6 x^3 - x\ dx = \frac{1}{5}\left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_1^6$$

$$\frac{1}{5}\left(\frac{1}{4}6^4 - \frac{1}{2}6^2\right) - \frac{1}{5}\left(\frac{1}{4}1^4 - \frac{1}{2}1^2\right) = \frac{245}{4}$$

Now to solve for c, such that $1 \leq c \leq 6$ and $f(c) = \frac{245}{4}$, we set up the equation $f(c) = c^3 - c = \frac{245}{4}$ and solve for c accordingly.

Example 3:

Let's find the average of $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

$$f_{ave} = \frac{1}{2-(-1)} \int_{-1}^{2} 1 + x^2 \, dx = \frac{1}{3} \left[x + \frac{1}{3}x^3 \right]_{-1}^{2} = 2$$

Now to solve for c, such that $-1 \leq c \leq 2$ and $f(c) = 2$, we set up the equation $f(c) = 1 + c^2 = 2$ and solve for c accordingly to get that $c = -1$ or $c = 1$. So, in this particular case, we had two soltutions for c given that c had to be in $[-1, 2]$.

Appendex

Derivative Identities:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\tan(x) = \sec(x)$

Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and

   $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

   where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a ”family” of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, "the integral('$\int$') of f(x) with respect to x('dx')."

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
- $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n}x^{n+1}$
- $\int \frac{1}{x}dx = ln(x) + C$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, "the integral('$\int$') from *a* to *b* of f(x) with respect to x('*dx*')."

Properties of the Definite Integral:

- $\int_a^b c\,dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a, b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Area between Two Curves:

The area between two curves is,

$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k^*)-g(x_k^*)|\Delta x=\int_a^b|f(x)-g(x)|dx$$

Volume of a Solid:

To find the volume of a solid, we calculate

$$V = \int_a^b A(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x.$$

A note: the function $A(x)$,the cross-sectional area of our solid at some x value, is going to depend on the solid.

Volume of a Solid of Revolution

The Disk Method:

When restricted to the volume of a solid of revolution(a solid formed by revolving a region, under a curve, about an axis), we calculate

$$V = \int_a^b \pi f(x)^2 dx.$$

A note: here $A(x) = \pi f(x)^2$, the cross-sectional area of a disk at x formed by revolving our function about the axis.

Volume of a Solid of Revolution

The Washer Method:

To the find the volume between two solids of revolution, we evaluate

$$V = \int_a^b \pi \left| f(x)^2 - g(x)^2 \right| dx.$$

A note: here $A_1(x) = \pi f(x)^2$ and $A_2(x) = \pi g(x)^2$, are the cross-sectional area of disks at x formed by revolving our functions about the axis. Notice too, that since π is positive we just factored it out side of the absolute value.

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 5

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 8

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 11 through 13

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 29 and 30