Calc I

Online Lecture Notes

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Volumes by Cylindrical Shells

Imagine, for a moment, that we wanted to find the volume of a cylinder or radius r_1 and height h, we know from algebra that it's volume is,

$$v_1 = h\pi r_1^2$$

Now, let's say we wanted to remove, from the center of it, another cylinder also of height h but with a smaller radius r_2.
Then the remaining volume would be,

$$V = v_1 - v_2 = h\pi r_1^2 - h\pi r_2^2 = h\pi \left(r_1^2 - r_2^2 \right)$$

If we massage the algebra some more we find,

$$V = h\pi\left(r_1^2 - r_2^2\right) = h\pi\left(r_1 - r_2\right)\left(r_1 + r_2\right)$$

Now let's multiply everything by $\frac{2}{2}$ and replace $r_1 - r_2$ with Δr, then we get

$$V = 2\pi h \Delta r \frac{r_1 + r_2}{2}$$

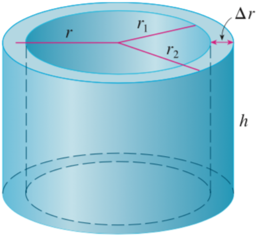
Notice, $\frac{r_1 + r_2}{2}$ give the average of the two radii, so it's really just another radius. Because of that why don't we replace it with r. So, we have,

$$V = 2\pi h r \Delta r$$

In other words, the volume of a **cylindrical shell** is

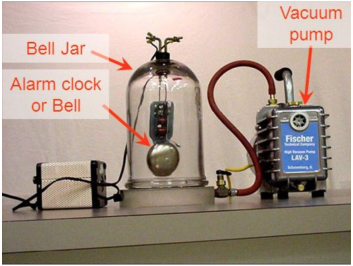
$$V = 2\pi hr\Delta r$$

where r is the average of its radii and Δr is it's width.



Now, let's put this to use.

In the 1800's it was common for scientists to use bell jars for experiements and to hold small specimens in, both of which are practices still used today. For example



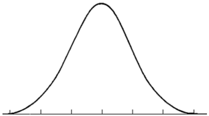
In the image is an experiment proving sound can not travel through a vaccume.

In this particular experiment one has to pump ari out of the jar, while ringing the bell at set intervals and measuring the intensity of of the sound each ring.

Naturally, in order for this experiment to be successful, the scientists running it need to know the volume of air inside the jar.

We're going to do that calculations here.

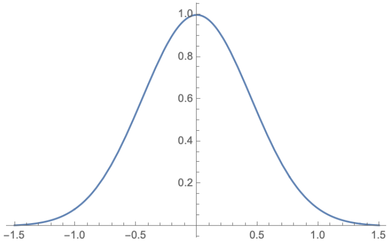
Luckily for us, the bell jar gets it's name from it's shape... a fact which it shares with the bell curve and a fact that we're going to exploit.



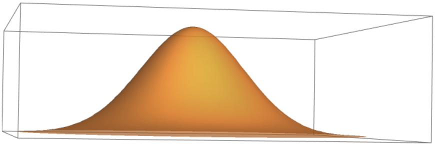
(a) image of a bell curve

(b) image of a bell jar

Notice, the interior of bell jar can be thought of a the bell curve rotated about it's center.



(a) Graph of the bell curve $f(x) = e^{\frac{-5x^2}{2}}$

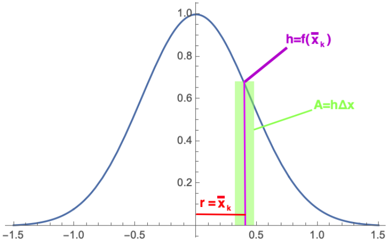


(b) Graph of the bell curve $f(x)$ revolved about $x = 0$

Notice, if we think of the interior of the bell jar as being the same shape and size as the solid formed by revolving the bell curve, then the height of the interios of the jar as any give point is the same as the solid, or

$$h = f(x) = e^{\frac{-5x^2}{2}}.$$

That also means, we can call x the radius.



From this, we get that the volume of a single cylindrical shell is,

$$v_k = 2\pi \overline{x}_k f(\overline{x}_k)\Delta x$$

Adding up all of the cylindrical shells and letting the number of shells go to infinity we get,

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi \overline{x}_k f(\overline{x}_k)\Delta x = \int_a^b 2\pi x f(x) dx, \quad \text{where } 0 \le a \le b.$$

This is called the **cylindrical shell method**.

Applying it to our problem, if the max radius of the interior of the jar is $b = 1.5$, we get

$$V = \int_0^{1.5} 2\pi x e^{\frac{-5x^2}{2}}\,dx.$$

To evaluate this integral, we're going to have to use a method known as **u-Substitution**.

u-Substitution:

The substitution rule says,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

You can find two other examples on Study Guide 3.

To apply ”u-sub,” here

$$V = \int_0^{1.5} 2\pi x e^{\frac{-5x^2}{2}} dx,$$

let $f(u) = e^u$ and $g(x) = -\frac{5}{2}x^2 = u$. Then $\frac{du}{dx} = -5x$, so $xdx = \frac{-du}{5}$. Turning our attention now to the bounds, $g(0) = 0$ and $g(1.5) = -\frac{5}{2}(1.5)^2 = -\left(\frac{5}{2}\right)\left(\frac{9}{4}\right) = -\frac{45}{8}$ Putting all of this together we get,

$$V = \int_0^{1.5} 2\pi x e^{\frac{-5x^2}{2}} dx = \int_{g(0)}^{g(1.5)} 2\pi e^u \frac{-1}{5} du = \frac{-2}{5}\pi \int_0^{-\frac{45}{8}} e^u du$$

Recall, one of the propperties of the integral is

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

So, we can rewrite our integral as

$$V = \frac{-2}{5}\pi \int_0^{-\frac{45}{8}} e^u du = \frac{2}{5}\pi \int_{-\frac{45}{8}}^0 e^u du$$

Evaluating the integral now, we get

$$V = \frac{2}{5}\pi \int_{-\frac{45}{8}}^{0} e^u du = \frac{2}{5}\pi \; e^u\Big]_{-\frac{45}{8}}^{0}$$

$$= \frac{2}{5}\pi(e^0 - e^{-\frac{45}{8}}) = \frac{2}{5}\pi(1 - e^{-\frac{45}{8}})$$

So there is, $\frac{2}{5}\pi(1 - e^{-\frac{45}{8}})$ ”units” of air in the jar.

Appendex

Derivative Identities:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\tan(x) = \sec(x)$

Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

   where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a ”family” of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, ”the integral(’$\int$’) of f(x) with respect to x(’dx’).”

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
- $\int (f(x)+g(x))\,dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n}x^{n+1}$
- $\int \frac{1}{x}dx = ln(x) + C$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, "the integral('$\int$') from *a* to *b* of f(x) with respect to x('*dx*')."

Properties of the Definite Integral:

- $\int_a^b c\, dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a,b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Area between Two Curves:

The area between two curves is,

$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k^*)-g(x_k^*)|\Delta x=\int_a^b|f(x)-g(x)|dx$$

Volume of a Solid:

To find the volume of a solid, we calculate

$$V = \int_a^b A(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x.$$

A note: the function $A(x)$,the cross-sectional area of our solid at some x value, is going to depend on the solid.

Volume of a Solid of Revolution

The Disk Method:

When restricted to the volume of a solid of revolution(a solid formed by revolving a region, under a curve, about an axis), we calculate

$$V = \int_a^b \pi f(x)^2 dx.$$

A note: here $A(x) = \pi f(x)^2$, the cross-sectional area of a disk at x formed by revolving our function about the axis.

Volume of a Solid of Revolution

The Washer Method:

To the find the volume between two solids of revolution, we evaluate

$$V = \int_a^b \pi \left| f(x)^2 - g(x)^2 \right| dx.$$

A note: here $A_1(x) = \pi f(x)^2$ and $A_2(x) = \pi g(x)^2$, are the cross-sectional area of disks at x formed by revolving our functions about the axis. Notice too, that since π is positive we just factored it out side of the absolute value.

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 5

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 8

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 11 through 13

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 29 and 30