Calc I

Online Lecture Notes

Matt Tucker

14 Apr. 2020

Volume

To find the volume of a solid, we calculate

$$V = \int_a^b A(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x.$$

A note: the function $A(x)$, the cross-sectional area of our solid at some x value, is going to depend on the solid.

When generalized to the volume of a solid of revolution(a solid formed by revolving a region, under a curve, about an axis), we calculate

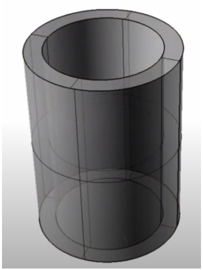
$$V = \int_a^b \pi f(x)^2 dx.$$

A note: here $A(x) = \pi f(x)^2$, the cross-sectional area of a disk at x formed by revolving our function about the axis.

This is called the disk method

Volumes between Two Solids

Let's imagine we have a solid and we removed a chunk of it, so for example a cylinder formed by extruding a ring/washer or by revoling a rectangle about an axis.



How might we find that volume?

As I alluded to in the previous slide, we treat it as, the large volume minus the smaller volume.

So, in much the same way as we area between two curves to be found by taking the integral of the absolute value of their difference

$$A = \int_a^b |f(x) - g(x)| dx,$$

the volume between two surfaces is found by taking calculating the integral over an interval of the absolute value of the difference of their cross-sectional areas

$$V = \int_a^b |A_1(x) - A_2(x)| dx$$

When generalized to the volumes of solids of revolution, we calculate

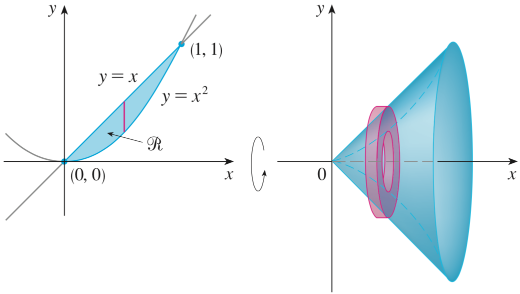
$$V = \int_a^b \pi \left| f(x)^2 - g(x)^2 \right| dx.$$

A note: here $A_1(x) = \pi f(x)^2$ and $A_2(x) = \pi g(x)^2$, are the cross-sectional area of disks at x formed by revolving our functions about the axis. Notice too, that since π is positive we just factored it out side of the absolute value.

This is called the washer method

Example 1:

Let's consider the volume formed by revolving the region between $f(x) = x$ and $g(x) = x^2$ on the interval $[0, 1]$ about the x-axis.



Notice on the interval $[0, 1]$ it is always the case that $x^2 \leq x$; so, we can drop the absolute value, and get

$$V = \int_0^1 A_1(x) - A_2(x)dx = \int_0^1 \pi(f(x)^2 - g(x)^2)dx = \int_0^1 \pi(x^2 - x^4)dx$$

$$= \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 = \frac{2\pi}{15}$$

Example 2:

Let's consider the volume formed by revolving the region between $f(x) = x^2$ and $g(x) = x^3$ on the interval $y = 0$ to $y = 1$ about the line $x = -1$.

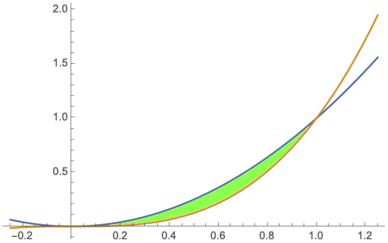
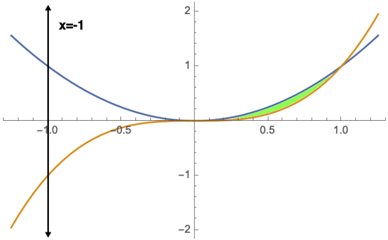
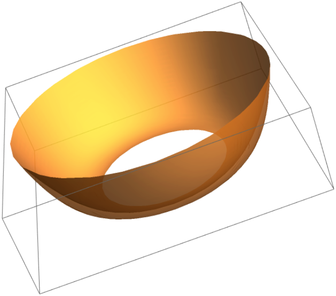
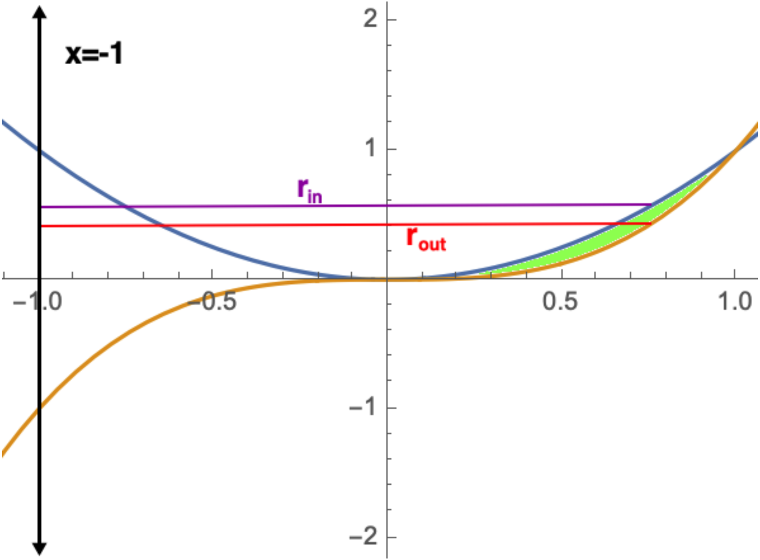


Figure: The region shaded green is the region being revolved


x=-1
2
1
−1
−2
−1.0
−0.5
0.5
1.0



Before we do anything else, notice the line, $x = -1$ is parallel to the y-axis. That means are radius is horizontal so we have to solve for the inverse functions $f^{-1}(y) = x$ and $g^{-1}(y) = x$; and, we are going to have to integrate with respect to y.


x=-1
r_{in}
r_{out}
2
1
−1
−2
−1.0
−0.5
0.5
1.0

Solving for $f^{-1}(y)$ and $g^{-1}(y)$ we find,

$$f^{-1}(y) = y^{1/2} \quad \text{and} \quad g^{-1}(y) = y^{1/3}.$$

So, as $f^{-1}(y) \leq g^{-1}(y)$ on $[0, 1]$, the inside radius is $r_{in} = f^{-1}(y) + 1 = y^{1/2}$ and the outside radius is $r_{out} = g^{-1}(y) + 1 = y^{1/3} + 1$.

From the previous slide we get that the corresponding cross-sectional areas are

$$A_{in}(y) = \pi(y^{1/2} + 1)^2 \quad \text{and} \quad A_{out}(y) = \pi(y^{1/3} + 1)^2.$$

Applying all of this now, we find

$$V = \int_0^1 \pi \left((g^{-1}(y) + 1)^2 - (f^{-1}(y) + 1)^2\right) dy$$

$$= \int_0^1 \pi \left((y^{1/3} + 1)^2 - (y^{1/2} + 1)^2\right) dy = \frac{4\pi}{15}$$

Appendex

Derivative Identities:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\tan(x) = \sec(x)$

Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

   where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a ”family” of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, ”the integral('$\int$') of f(x) with respect to x('dx').”

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
- $\int (f(x)+g(x))\,dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n}x^{n+1}$
- $\int \frac{1}{x}dx = ln(x) + C$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, ”the integral('$\int$') from *a* to *b* of f(x) with respect to x('*dx*').”

Properties of the Definite Integral:

- $\int_a^b c\,dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a,b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Area between Two Curves:

The area between two curves is,

$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k^*)-g(x_k^*)|\Delta x=\int_a^b|f(x)-g(x)|dx$$

Volume of a Solid:

To find the volume of a solid, we calculate

$$V = \int_a^b A(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x.$$

A note: the function $A(x)$,the cross-sectional area of our solid at some x value, is going to depend on the solid.

When generalized to the volume of a solid of revolution(a solid formed by revolving a region, under a curve, about an axis), we calculate

$$V = \int_a^b \pi f(x)^2 dx.$$

A note: here $A(x) = \pi f(x)^2$, the cross-sectional area of a disk at x formed by revolving our function about the axis.

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 5

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 8

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 11 through 13

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 29 and 30