# Calc I

Online Lecture Notes

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# Volumes

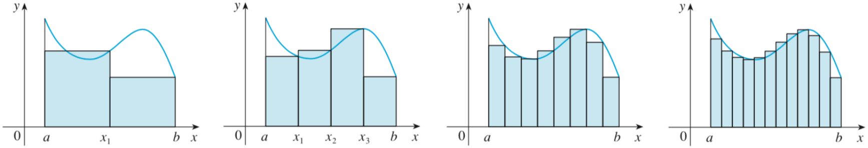
Up to this point we've talked ALOT about applying the integral to find area; but what about volume? Since the indefinite integral is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x,$$

surely we can find some way to repurpose it to find volume?

To start, let's recall what our equation is describing when we apply it to the question of area.

Recall, we began by dividing our interval up into $n$ many pieces of equal length, namely $\Delta x$. And then, we defined a rectangle on each interval of height $f(x_k^*)$, where $x_k^*$ was some $x$ in our $k^{th}$ interval. Next we calculated area of each rectangle and added them together. Finally, we let $n \to \infty$ and called that our integral.

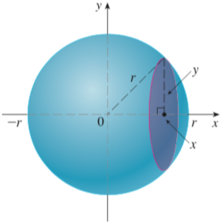


Couldn't we do something similar for volume? Sure! Except instead of using 2D rectangles, let's make them 3D, and because they're 3D we're going to define an area instead of a height.

So what does this look like?

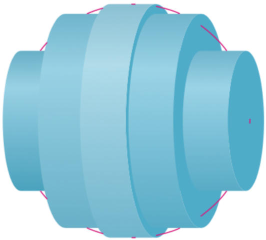
Since we know the equation to find the volume of a sphere, let's use that as our example... Because we can check out work in the end.

Imagine now, you have a sphere of some radius $r$. For any cross sectional plane(in the image, we choose planes parrallel to the yz-plane), we know the area of the circle is going to be $A_k = \pi \tilde{r}_k^2$, where $\tilde{r}_k$ is the radius of the circle.



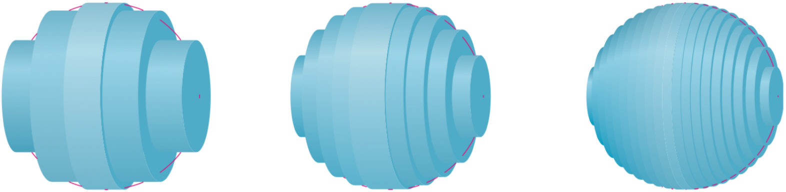

You may fine yourself asking, "what is, $\tilde{r}_k$?" We will return to that question at the end. For right now I want us to focus on the development of how we find volume.

Let's now imagine the divided our x-axis into n many pieces of equal length, namely $\Delta x$. Now, let's choose one of those circles in each subinterval and stratch it into a cylinder the length of it's subinterval.



Notice, the volume of each of these cylinders is, $A_k \Delta x$. So if we add the volume of each of the cylinders together, we get an approximation for the volume of our sphere.

And, just like was the case with finding area, if we let $n$ increase we get a better and better approximation.

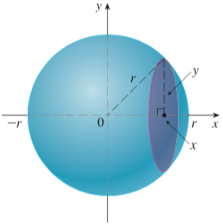


Hence, to find the volume of a solid, we calculate

$$V = \int_a^b A(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x.$$

**A note:** the function $A(x)$ is going to depend on the solid.

Returning to the question of $\tilde{r}$ in our example.




As shown in the image, we know that every point at the edge of the circle is on the edge of the sphere. So, if we imagine our cross section to be parrallel with the yz-plane, we can use the upper most point of our circle to see the we have $r^2 = x^2 + y^2$ and the radius of our circle($\tilde{r}$) is equal to $y$.

Hence it must be that

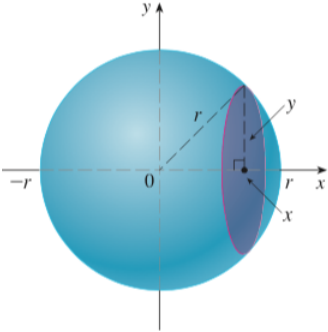
$$\tilde{r}^2 = r^2 - x^2$$

where $r$ is the radius of our sphere and $\tilde{r}$ is the radius of the cross-sectional circle.

Therefore we have, $A_k = \pi(r^2 - {x_k^*}^2)$ and

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \pi(r^2 - {x_k^*}^2)\Delta x = \int_a^b \pi(r^2 - x^2)dx.$$

Notice, in our example $[a, b] = [-r, r]$.




So, the integral becomes

$$V = \int_{-r}^{r} \pi(r^2 - x^2)dx.$$

Thanks to symmetry we can divide our sphere into 2 hemispheres of equal volume by splitting it in half at the yz-plane, giving us

$$V = 2\pi \int_0^r (r^2 - x^2)dx = 2\pi \left[ xr^2 - \frac{1}{3}x^3 \right]_0^r = 2\pi \frac{2}{3} r^3 = \frac{4}{3}\pi r^3.$$

# Example 2

Find the volume of $f(x) = \cos(x)$ from $x = 0$ to $x = \pi/2$ revolved about the x-axis.

Before we start, let's take a look at the function $f(x)$ and the solid formed by revolving it around the x-axis.

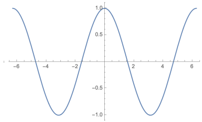



Figure: Graph of $f(x) = \cos(x)$

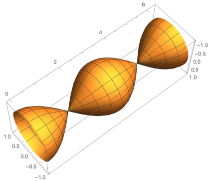



Figure: Graph of $f(x) = \cos(x)$ revolved about the x-axis

Applying our restriction, $0 \leq x \leq \pi/2$:

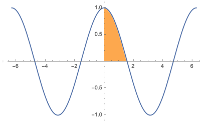



Figure: Graph of $f(x) = \cos(x)$

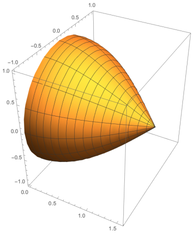



Figure: Graph of $f(x) = \cos(x)$ revolved about the x-axis

So, what we want to find is the volume of the cone like shape.

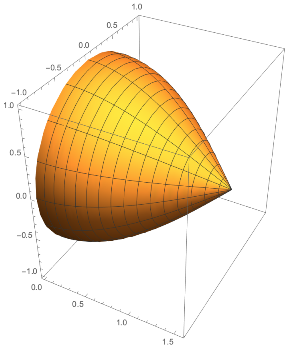



Figure: Graph of $f(x) = \cos(x)$ restricted to $0 \leq x \leq \pi/2$ and revolved about the x-axis

Notice, we can again break our interval up into subintervals and form cylinders the width of each subinterval, $\Delta x$.

Now, if we revolve $f(x)$ about the x-axis to from a cone or cylinder -like solid, notice the at an point along the x-axis our solid has a radius, $f(x)$. So, from this, we can deduce that the cross-sectional area of our cylinder in the $k^{th}$ subinterval is $A_k = \pi f(x_k^*)^2$.

From that this we have,

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \pi \cos^2(x_k^*)\Delta x = \int_0^{\pi/2} \pi \cos^2(x) dx.$$

Applying the identity $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$ we get

$$V = \frac{\pi}{2}\int_0^{\pi/2} 1 + \cos(2x) dx = \frac{\pi}{2}\int_0^{\pi/2} 1 dx + \frac{\pi}{2}\int_0^{\pi/2} \cos(2x) dx.$$

The first integral $\frac{\pi}{2}\int_0^{\pi/2} 1 dx$ is pretty straight forward,

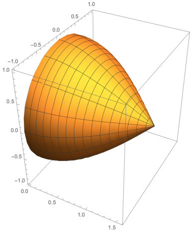
$$\frac{\pi}{2}\int_0^{\pi/2} 1 dx = \frac{\pi}{2}\; x\Big]_0^{\pi/2} = \left(\frac{\pi}{2}\right)^2$$

To calculate the second integral, we have to apply a subsitution $u = 2x$ which means $\frac{du}{dx} = 2 \implies du = 2dx$ or $dx = \frac{du}{2}$. Also changing or bounds, $x = 0 \implies u = 0$ and $x = \pi/2 \implies u = \pi$. So, we get that

$$\frac{\pi}{2}\int_0^{\pi/2} \cos(2x)dx = \frac{\pi}{2}\int_0^{\pi} \cos(u)\frac{du}{2} = \frac{\pi}{4}\int_0^{\pi} \cos(u)du$$

$$= \frac{\pi}{4}\sin(u)]_0^{\pi} = 0$$

So, the volume of $f(x) = \cos(x)$ from $x = 0$ to $x = \pi/2$ revolved about the x-axis is

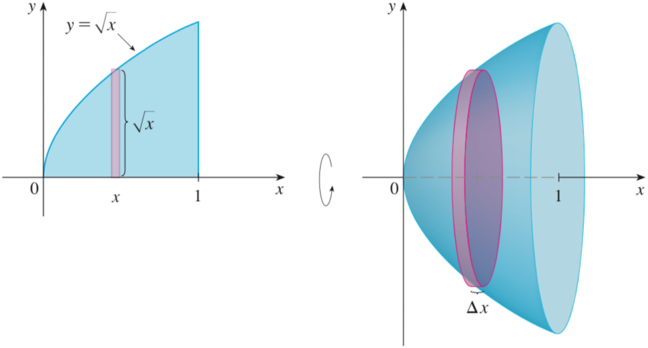



$$V = \int_0^{\pi/2} \pi \cos^2(x)dx = \frac{\pi}{2}\int_0^{\pi/2} 1 + \cos(2x)dx$$

$$= \left(\frac{\pi}{2}\right)^2 + 0 = \frac{\pi^2}{4}.$$

# Example 3

Verify the volume of the solid formed by revolving $f(x) = \sqrt{x}$, restricted to $0 \leq x \leq 1$, is $\frac{\pi}{2}$.



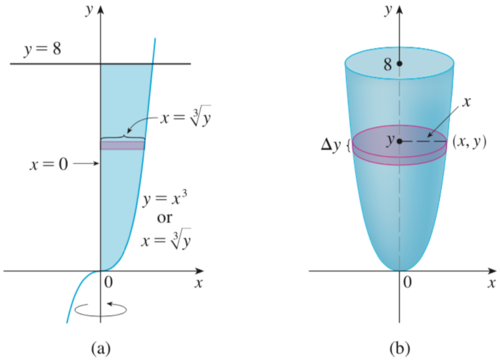
Given that solid is formed by revoloving $f(x)$, restricted to $0 \leq x \leq 1$, about the x-axis we know for any cross-section parrallel to the yz-plane has an area $A(x) = \pi x$.

So the volume must be,

$$\int_0^1 \pi x dx = \pi \left.\frac{x^2}{2}\right]_0^1 = \pi/2$$

# Example 4

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.



The first thing to notice, we're going to wnt to integrate with respect to $y$ this time. So, our integral should be of the form,

$$\int_{\alpha}^{\beta} A(y)dy$$

And since $x^3 = y$ we need to solve for $x$ so we can write everything in terms of $y$, $x = \sqrt[3]{y}$.

Now, since our solid is formed by revolving our function $f(y) = \sqrt[3]{y}$ about the y-axis we have $A(y) = \pi y^{2/3}$.

Next, we were given the bounds $y = 8$ and $x = 0$, we need to "fix" our lower bound so it's writen in terms of $y$. Well, since $x = 0$ we have $y = 0^3 = 0$. So our bounds are $0 \leq y \leq 8$

So, putting all of this together and solving, we get

$$V = \int_0^8 A(y)dy = \int_0^8 \pi y^{2/3} dy = \pi \left. \frac{3}{5} y^{5/3} \right]_0^8 = \frac{96\pi}{5}$$

# Appendix

# Derivative Identities:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\tan(x) = \sec(x)$

# Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

## The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on $(a, b)$, and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

## The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a "family" of functions with the derivative $f(x)$ and they all differ only by a constant.

# A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, "the integral( '$\int$' ) of f(x) with respect to x( '$dx$' )."

# Common Indefinite Integral:

- $\int cf(x)dx = c \int f(x)dx$
- $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n} x^{n+1}$
- $\int \frac{1}{x} dx = ln(x) + C$
- $\int f'(x)e^{f(x)} dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

# More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

# The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

## A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, "the integral( '$\int$' ) from $a$ to $b$ of f(x) with respect to x( '$dx$' )."

# Properties of the Definite Integral:

- $\int_a^b c\,dx = c(b-a)$, $c$ is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, $c$ is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a,b)$

## Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all $x$ such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all $x$ such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all $x$ such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

# Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 5

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 8

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 11 through 13

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 29 and 30