Calc I

Online Lecture Notes

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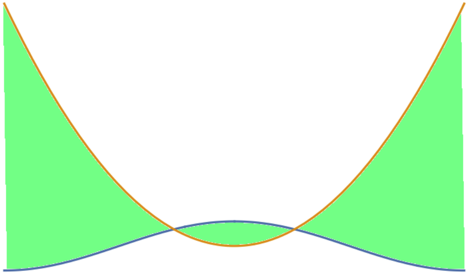
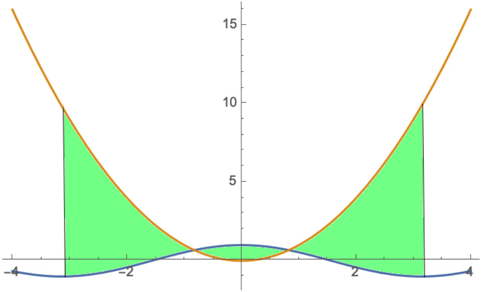
6 Apr. 2020

Area Between Curves

We already talked, in great length, about how the integral can be use to give the area under a curve/function over some interval; but, what about the area between two cuves like the green region in the image below?



Well, we first need to know something about the curves and the boundary. For example, in the case of our image, the functions associated with the curves were $f(x) = x^2$ and $g(x) = \cos(x)$ and the interval was $[-\pi, \pi]$.



We'll come back to this problem at the end of the lecture.

First, let's notice something, the integral only ever gives us the area between the function and a corresponding axis. And if the function is below the axis, we get that the area is negative.

For example, $\cos(x) \leq 0$ for $\pi/2 \leq x \leq 3\pi/2$ and

$$\int_{\pi/2}^{\pi} \cos(x)dx = \sin(x)]_{\pi/2}^{\pi} = \sin(\pi) - \sin(\pi/2) = 0 - 1 = -1$$

So the area between $\cos(x)$ and the x-axis on the interval comes out to be -1. But, area is "never" supposed to be reported as a negative, so we know that the area must actually be 1.

This is a particularly important fact if we wanted to find the area between cos(x) and the x-axis for the interval $[0, 2\pi]$.

Notice $\int_0^{2\pi} \cos(x)dx = 0$ and if we break this up into parts, the negative and the positive parts, we get

$$\int_0^{2\pi} \cos(x)dx = \int_0^{\pi/2} \cos(x)dx + \int_{\pi/2}^{3\pi/2} \cos(x)dx + \int_{3\pi/2}^{2\pi} \cos(x)dx = 0$$

because the negative parts cancel out the positive parts.

Well... this is frustrating. What's happening here? I thought the integral gave us the area!

It does!

It gives us what is called the **signed area** under a curve. Which is a mathematician's way of saying what we've already said,

> *"the integral only ever gives us the area between the function and a corresponding axis. If the function is below the axis, we get that the area is negative."*

To see why this is, recall

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x, \quad \Delta x = \frac{b-a}{n}.$$

So, if $f(x_k^*) < 0$ so is the area of the corresponding rectangle($A_k^* < 0$).

So, how do we fix this problem?

We apply the absolute value!

We say that the [propper/unsigned] **area** between a function and an axis is

$$\int_a^b |f(x)|dx = \lim_{n\to\infty} \sum_{k=1}^{n} |f(x_k^*)|\Delta x, \quad \Delta x = \frac{b-a}{n}.$$

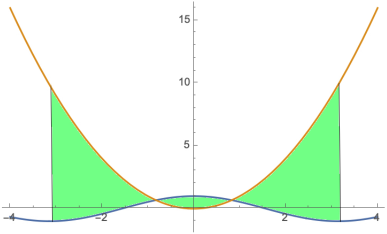
Revisiting the area between $\cos(x)$ and the x-axis for the interval $[0, 2\pi]$:

Notice, $\int |\cos(x)| dx \big]_0^{2\pi} = 0$. So we still have to break our interval up into parts. It is natural, in this case, to choose $[0, \pi/2]$, $[\pi/2, 3\pi/2]$, and $[0, 2\pi]$(what makes $\pi/2$ and $3\pi/2$ natural choices? Hint: take a peak at slides 14-16.).

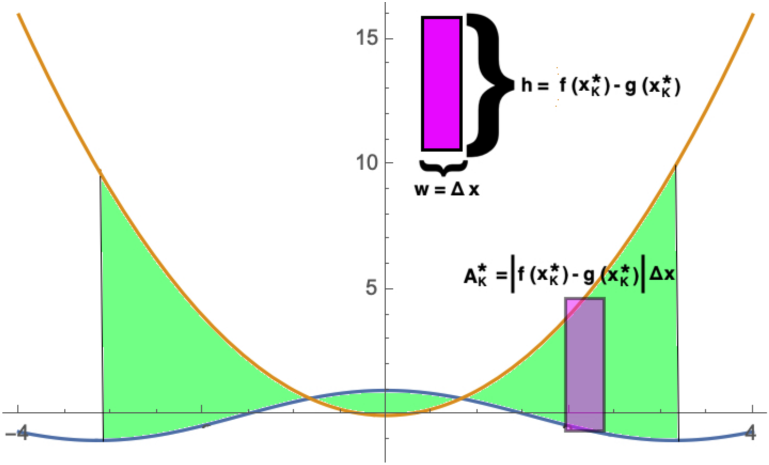
$$\int_0^{2\pi} |\cos(x)| dx = \int_0^{\pi/2} |\cos(x)| dx + \int_{\pi/2}^{3\pi/2} |\cos(x)| dx + \int_{3\pi/2}^{2\pi} |\cos(x)| dx = 4$$

So, the area between $\cos(x)$ and the x-axis for the interval $[0, 2\pi]$ is 4.

Getting back to our original question, "how to find the area between curves/functions."



Let's imagine our region broken up into tiny rectangles, each of width Δx and height $f(x_k^*) - g(x_k^*)$. Then the area of the k^{th} rectangle is $A_k^* = |f(x_k^*) - g(x_k^*)|\Delta x$, and is depicted on the next slide.


15
10
5
−4
4
h = f (x*K) - g (x*K)
w = Δ x
A*K = |f (x*K) - g (x*K)| Δx

So, we get that the area between two curves is,

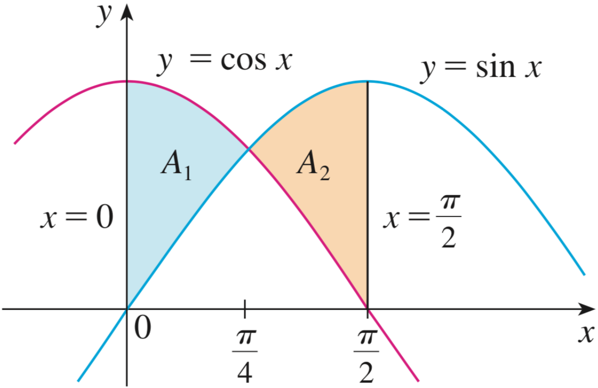
$$\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k^*)-g(x_k^*)|\Delta x=\int_a^b|f(x)-g(x)|dx$$

Now, before we look at an example, notice

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x), & f(x) \geq g(x) \\ g(x) - f(x), & g(x) \geq f(x) \end{cases}$$

Example 1:

Consider the example, $f(x) = \cos(x)$ and $g(x) = sin(x)$ and we want the area between the curves on the interval $[0, \pi/2]$.



So, the area is

$$\int_0^{\pi/2} |\cos(x) - \sin(x)|dx = \int_0^{\pi/4} \cos(x) - \sin(x)dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)dx$$

$$= (\sin(x) + \cos(x))]_0^{\pi/4} + (-\cos(x) - \sin(x))]_{\pi/4}^{\pi/2}$$

$$= \Big((\sqrt{2}/2 + \sqrt{2}/2) - (0+1)\Big) + \Big((0-1) - (-\sqrt{2}/2 - \sqrt{2}/2)\Big) = 2\sqrt{2} - 2$$

Example 2:

Now, recall in our original example the curves were $f(x) = x^2$ and $g(x) = \cos(x)$ and the interval was $[-\pi, \pi]$.

So, by taking the difference and applying the absolutevalue we find

$$\left|x^2 - \cos(x)\right| = \begin{cases} \cos(x) - x^2, & \frac{-\pi}{4} \leq x \leq \frac{\pi}{4} \\ x^2 - \cos(x), & x \leq \frac{-\pi}{4} \; OR \; \frac{\pi}{4} \leq x \end{cases}$$

As a note, we could equivalently have written

$$\left|x^2-\cos(x)\right| = \begin{cases} \cos(x)-x^2, & |x| \le \frac{\pi}{4} \\ x^2-\cos(x), & \frac{\pi}{4} \le |x| \end{cases}$$

Combining all of this together we have that the area between the curves $f(x) = x^2$ and $g(x) = \cos(x)$ on the interval $[-\pi, \pi]$ is,

$$\int_{-\pi}^{\pi} \left| x^2 - \cos(x) \right| dx =$$

$$\int_{-\pi}^{-\pi/4} x^2 - \cos(x) dx + \int_{-\pi/4}^{\pi/4} \cos(x) - x^2 dx + \int_{\pi/4}^{\pi} x^2 - \cos(x) dx$$

Now we can apply the rule

$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x)) dx$$

So,

$$\int_{-\pi}^{\pi} \left|x^2 - \cos(x)\right| dx =$$

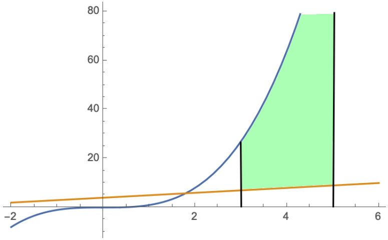
$$\int_{-\pi}^{-\pi/4} x^2 dx - \int_{-\pi}^{-\pi/4} \cos(x) dx + \int_{-\pi/4}^{\pi/4} \cos(x) dx -$$

$$\int_{-\pi/4}^{\pi/4} x^2 dx + \int_{\pi/4}^{\pi} x^2 dx - \int_{\pi/4}^{\pi} \cos(x) dx$$

$$= 2\sqrt{2} + \frac{31\pi^3}{48} \approx 22.853$$

Example 3:

Let $f(x) = x^3$ and $g(x) = x + 4$ and let's say we want the area between them on the interval $[3, 5]$.

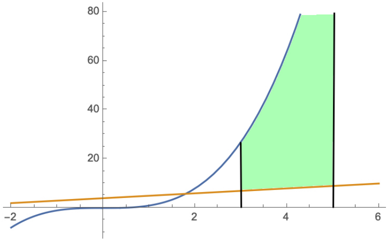


Example 3:

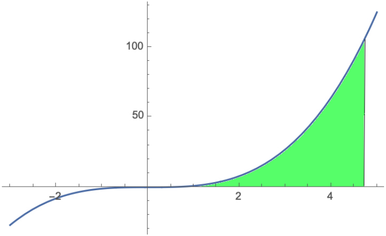
Then,

$$\int_3^5 \left|x^3 - (x+4)\right| dx = \int_3^5 \left|x^3 - x - 4\right| dx = \int_3^5 x^3 - x - 4 dx$$

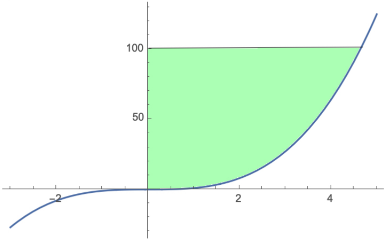
$$= \frac{1}{4}x^4 - \frac{1}{2}x^2 - 4x \Big]_3^5 = 120$$



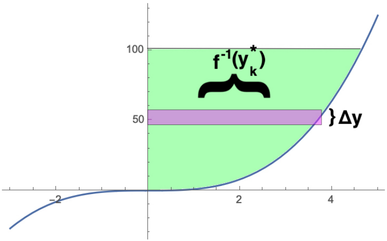
What if, instead of finding the area ”under” a curve,



I wanted to find the area ”beside” the curve?



Notice, now we want our rectangles to be of length $f^{-1}(y_k^*)$ and width Δy.



So, the signed area would be

$$\lim_{n\to\infty} \sum_{n}^{k=1} f^{-1}(y_k^*)\Delta y = \int_{f^{-1}(a)}^{f^{-1}(b)} f^{-1}(y)dy$$

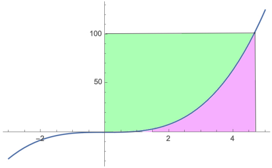
So, in the graphed example here, provided that $f(x) = x^3$ and the interval is $[0, \sqrt[3]{100}]$ to find the signed area of the green region we would calculate,

$$\int_{f^{-1}(0)}^{f^{-1}\sqrt[3]{100}} f^{-1}(y)dy = \int_0^{100} \sqrt[3]{y}dy = 75\sqrt[3]{100}$$

We can verify our result by obersving

$$\int_0^{100} \sqrt[3]{y}\,dy + \int_0^{\sqrt[3]{100}} x^3\,dx = (100)(\sqrt[3]{100}),$$

where $(100)(\sqrt[3]{100})$ is the area of the corresponding rectangle,



and checking if

$$(100)(\sqrt[3]{100}) - \int_0^{100} \sqrt[3]{y}\,dy = \int_0^{\sqrt[3]{100}} x^3\,dx$$

Appendex

Derivative Identities:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\tan(x) = \sec(x)$

Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

   where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a "family" of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, "the integral('$\int$') of f(x) with respect to x('dx')."

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
- $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n}x^{n+1}$
- $\int \frac{1}{x}dx = ln(x) + C$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, "the integral('$\int$') from a to b of f(x) with respect to x('dx')."

Properties of the Definite Integral:

- $\int_a^b c\, dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a,b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 5

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 8

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 11 through 13

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 29 and 30