Calc I

Online Lecture Notes

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Indefinite Integral

Definition of the Indefinite Integral

Given the function $F(x)$ with the derivative $\frac{d}{dx}F(x) = f(x)$, the definition of the **indefinite integral** is

$$\int f(x)dx = F(x) + C,$$

where C is a constant.

Look familiar? It's, for all practical purposes, the same idea as the "antiderivative" with **one slight difference**. The antidericative gives us $F(x) + k$ where k is a specific constant. But, the indefinite integras gives us $F(x) + C$ where C is a general constant.

What's the difference?

Let's say we wanted to find the **indefinite integral** of $g(x) = x^2$, then

$$G(x) = \int x^2 dx = \frac{1}{3}x^3 + C, \text{ where } C \text{ is any constant.}$$

But, if we were to ask for an **antiderivative** of $g(x)$ we would define C. So one **antiderivative** of $g(x)$ would be,

$$\int x^2 dx = \frac{1}{3}x^3 + \pi$$

and another would be

$$\int x^2 dx = \frac{1}{3}x^3 - 36.$$

The reason for this is because, the **indefinite integral** of $f(x)$ is asking for the **family of functions** whoes derivative is $f(x)$.

Where as, the antiderivative is asking that we undo the derivative.

It may help to think of integrals and the antiderivative in terms of what their results are:

Operation	Notation	Result
Indefinite Integrals	$\int f(x)dx$	A family/set of functions
Antiderivative	$\int f(x)dx$	A function
Definite Integrals	$\int_a^b f(x)dx$	A number(with notable exceptions)

A note on notation:

You may see $F(x)]_a^b$, sometimes written as $F(x)|_a^b$, they both mean

$$F(x)]_a^b = F(b) - F(a) = F(x)|_a^b$$

So, we can rewrite the definite integral in terms of the indeinite integral

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)]_a^b = \int f(x)dx\Big]_a^b$$

Together these last 6 slides give us new interpretations of the Fundemental Theorem of Calculus:

If $f(x)$ is continuous on $[a, b]$

1. *then the function $F(x)$, defined*

$$F(x) = \int_a^x f(t)dt \quad a \leq x \leq b,$$

   is an antiderivative of $f(x)$

2. $\int_a^b f(x)dx = F(b) - F(a) = F(x)]_a^b = \int f(x)dx\Big]_a^b$, *where $F(x)$ is an antiderivative of $f(x)$.*

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
- $\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)dx$
- $\int kdx = kx + C$
- $\int x^n dx = \frac{1}{n}x^{n+1}$
- $\int \frac{1}{x}dx = ln(x) + C$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ (sometimes written, using substitution, as $\int e^u du = e^u + C$)

More Common Indefinite Integral:

- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \sec(x)^2 dx = \tan(x) + C$
- $\int \csc(x)^2 dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

Net Change Theorem

Recall, in the past we've seen that the derivative of a function evaluated at a point is the slope of the line tangent to our function at that point. And, in that sense, the derivative of a function gives the slope of that function at a particular point. So, in that way just like the slope is the rate of change of a straight line, the derivative is a function which yields the rate of change of our function at the points for which the derivative is defined.

Net Change Theorem states:

*Provided $F(x)$ is the antiderivative of $f(x)$, that is $\frac{d}{dx}F(x) = f(x)$, and $f(x)$ is continuous on the interval $[a, b]$; then, the **net change** of $F(x)$ over the interval $[a, b]$ is*

$$\int_a^b f(x)dx = F(b) - F(a).$$

We could equvalently say,

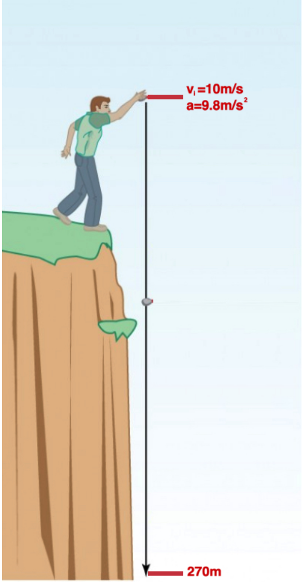
"The integral of a rate of change is the net change."

Example:

Consider a ball that's been thrown straight down from a edge of a cliff(there's no drag or coriolis effect) that's 270m high. If the ball is thrown so that it has an intial velocity of $\vec{v}_i = 10m/s$ then it's velocity, as a function of time is, $\vec{V}(t) = 10 + 9.8t$, where $9.8m/s^2$ is the acceleration due to gravity.

Questions we could solve using the "Net Change Theorem:"

1. How far will it travel after the first $5s$?
2. How far will it have traveled between $t = 3s$ and $t = 5.3s$?
3. How long will it take to hit the ground?


v_i =10m/s
a=9.8m/s²
270m

Solutions:

1. How far will it travel after the first $5s$?

$$\int_0^5 \vec{V}(t)dt = \int_0^5 10 + 9.8tdt = (10t + \frac{1}{2}9.8t^2)\Big]_0^5 = 172.5m$$

2. How far will it have traveled between $t = 3s$ and $t = 5.3s$?
3. How long will it take to hit the ground?

Solutions:

1. How far will it travel after the first $5s$?

2. How far will it have traveled between $t = 3s$ and $t = 5.3s$?

$$\int_{3}^{5.3} \vec{V}(t)dt = \int_{3}^{5.3} 10 + 9.8tdt = (10t + \frac{1}{2}9.8t^2)\Big]_{3}^{5.3} = 116.541m$$

3. How long will it take to hit the ground?

Solutions:

1. How far will it travel after the first $5s$?
2. How far will it have traveled between $t = 3s$ and $t = 5.3s$?
3. How long will it take to hit the ground?

$$\int_0^x \vec{V}(t)dt = \int_0^x 10 + 9.8tdt = (10t + \frac{1}{2}9.8t^2)\Big]_0^x = 270m$$

$$\implies 10x + \frac{1}{2}9.8x^2 = 270$$

In other words we need to solve the equation
$10x + \frac{1}{2}9.8x^2 = 270.$

Appendex

Derivative Identities:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\tan(x) = \sec(x)$

Derivative Rules and Identities:

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$
- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

   where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a "family" of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, "the integral('$\int$') of f(x) with respect to x('dx')."

Common Indefinite Integral:

- $\int cf(x)dx = c\int f(x)dx$
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- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

The Definite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int_a^b f(x)dx = F(b) - F(a).$$

A Note about the Definite Integral:

Given $\int_a^b f(x)dx = F(b) - F(a)$, we say that this is, "the integral('$\int$') from a to b of f(x) with respect to x('dx')."

Properties of the Definite Integral:

- $\int_a^b c\,dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a, b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 4

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 7 through Chapter 10

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 8 through 10

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 22 through Chapter 24