Calc I

Online Lecture Notes

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The First Part of the Fundemental Theorem of Calculus

The **first part of the Fundemental Theorem of Calculus** states,

If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

Example of Part 1 of The Fundemental Theorem of Calculus

Example 1

Let,

$$g(x) = \int_0^x \sqrt{1+t^2}dt.$$

We could try calculate it and find $g(x)$[1] then take the derivative. Or we can TRY to apply the first part of the Fundemental Theorem of Calculus.

[1] Using a method you'll learn in Calc 2, Trig sub, we find

$$g(x) = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\left[\ln|\tan(u) + \sec(u)|\right]_0^{\arctan(x)}$$

So, since $f(t) = \sqrt{1 + t^2}$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, we can apply the first part of the Fundemental Theorem of Calculus for $0 \leq x < \infty$.

In otherwords, $g'(x) = f(x) = \sqrt{1 + x^2}$.

Example 2

Let,

$$h(x) = \int_0^{x^4} \sec(t)dt.$$

Then,

$$h'(x) = \frac{d}{dx}\int_0^{x^4} \sec(t)dt;$$

then, by setting $u = x^4$ and applying the chain rule we get

$$\frac{d}{du}\left(\int_0^{u} \sec(t)dt\right)\frac{du}{dx} = \sec(u)\frac{du}{dx} = \sec(x^4)4x^3.$$

The Second Part of the Fundemental Theorem of Calculus

The **second part of the Fundemental Theorem of Calculus** states,

If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

Example of Part 2 of The Fundemental Theorem of Calculus

Example 1

Find $\int_{e}^{e^3} \frac{1}{r} dr$.

Since $\frac{1}{r}$ is continuous on $[e^1, e^3]$ and its antiderivative is

$$\int \frac{1}{r} dr = \ln(r) + C,$$

by the second part of The Fundemental Theorem of Calculus

$$\int_e^{e^3} \frac{1}{r} dr = (\ln\left(e^3\right) + C) - (\ln(e) + C) = \ln\left(e^3\right) - \ln(e) = 3 - 1 = 2$$

Example 2

Let's now return to the example discussed earlier about the distant travled by the center of a vibrating string with velocity $v(t) = \sin(t)$ in the interval $t \in [0, \frac{\pi}{2}]$. Recall, I claimed that the exact distance travled was 1. We will prove that now.

Since, $d(t) = \int_a^b v(t)dt$ where (a, b) in the interval of time we are concerned with we want to find,

$$\int_0^{\pi/2} \sin(t)dt.$$

We know, the funtion of velocity is continuous on our interval and its antiderivative is $\int \sin(t)dt = -\cos(t) + C$. So, together we get,

$$\int_0^{\pi/2} \sin(t)dt = \left(-\cos\left(\frac{\pi}{2}\right) + C\right) - (-\cos(0) + C)$$

$$= -\cos\left(\frac{\pi}{2}\right) + \cos(0) = 0 + 1 = 1$$

Together these two parts form **The Fundemental Theorem of Calculus**:

Suppose $f(x)$ is continuous on $[a, b]$

1. *If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.*
2. *$\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$, that is, $F'(x) = f(x)$.*

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$