Calc I

Online Lecture Notes

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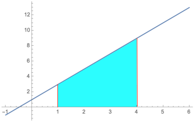
30 Mar. 2020

Area

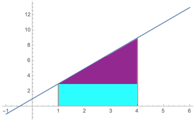
Question:

- For simple shapes and curves(i.e. a circle or a straight line), we generally know how to find a desired area; but, what about for more comple shapes and curves(i.e. the shape of a bell or x^2)? How might we find areas relating to them?

Let's imagine we wanted to find the area under the line, $f(x) = 2x + 1$ and above the x-axis, between $x = 1$ and $x = 4$.

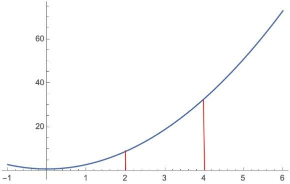


Well, we might cut up the shape into two parts.



Notice when we do, we find that it can be broke up into two regular shapes who's area we know how to calculate.

What if we asked the same question except the function was $g(x) = x^2$?



Notice, we can't do that same trick. So what can we do?

Let's try approximating the area with a couple of "small" rectangles with fixed widths(Δx), where the top right corner is equal to our function at that corresponding x value($f(x_k)$). Notice when we do that this, we seem to over "approximate" the area.

And if we do it again, but this time with the left corner($f(x_{k-1})$)... we seem to be under estimating.

Now, let's use rectangles with smaller widths(Δx).

(See video lecture, or the gif titled "prabola_sum" on BlackBoard, for demonstration.)

Did you notice that as we added rectangles by making Δx smaller we go better approximations?

Well, this seems useful so let's go ahead and try to define it mathematically...

Let's start by looking at a single rectangle, with the top right corner equal to our function.

We know, each rectangle has a height of $f(x_m)$ and a width Δx. So, each rectangle has an area of $A_m = f(x_m)\Delta x$.

Using summation notation, that means the approximate area of our function is

$$\sum_{m=1}^{n} f(x_m)\Delta x = \sum_{m=1}^{n} A_m \approx A.$$

Now, what exactly is x_m and Δx.

Well, first we divided the interval in to n-many slices of equal length and set each of those slices to be the width of each rectangle. So, if we started at $x = a$ and ended at $x = b$, we get that $\Delta x = \frac{b-a}{n}$

It also must mean, since we're setting the top right corner equal to our function, that $x_m = a + m\Delta x$. Notice, that means $b = a_n = a + n\Delta x$. Which makes sense because $x = b$ would be the line forming the right side of our last rectangle.

Putting it all together we have that:

to approximate the area between a function and the x-axis with n-many rectangles who's top right corner is equal to our function we have,

$$A \approx \sum_{m=1}^{n} f(x_m)\Delta x = \sum_{m=1}^{n} f(a + m\Delta x)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$.

Note, if we wanted to use the left corner instead, we would just have to start at $m = 0$ **and end at** $m = n - 1$

Now, let's apply this to our original example, $g(x) = x^2$ from $x = 1$ to $x = 4$.

Since we're asking for the area from $x = 1$ to $x = 4$, we know $a = 1$ and $b = 4$, so it must be that $\Delta x = \frac{4-1}{n} = \frac{3}{n}$

Therefore, using the right corners,

$$A \approx \sum_{m=1}^{n} f(x_m)\Delta x = \sum_{m=1}^{n} f(a + m\Delta x)\Delta x = \sum_{m=1}^{n} f(1 + m\frac{3}{n})\frac{3}{n}$$

or

$$A \approx \sum_{m=1}^{n} \left(1 + m\frac{3}{n}\right)^2 \frac{3}{n} = \sum_{m=1}^{n} \frac{3\,(n+3m)^2}{n^3}$$

And, using the leftt corners,

$$A \approx \sum_{m=0}^{n-1} \left(1 + m\frac{3}{n}\right)^2 \frac{3}{n} = \sum_{m=0}^{n-1} \frac{3\,(n+3m)^2}{n^3}$$

Let’s now let’s divide our interval into 4 parts, so $n = 4$.

Then the sum using the right corners is,

$$A \approx \sum_{m=1}^{4} \frac{3\left(4+3m\right)^2}{4^3} = \frac{861}{32}$$

And the sum using the leftt corners is,

$$A \approx \sum_{m=0}^{3} \frac{3\left(4+3m\right)^2}{4^3} = \frac{501}{32}$$

Distance

Question:

- For constant velocity($v(t) = v_i$), it's pretty straight forward how to calculate the distance traveled over some interval of time($d = v_i t$); but, what if velocity wasn't constant($v(t) = v_i + at$)? How would we calculate the distance traveled then?

Note, here the "i" subscript is meant to denote "initial velocity." It is NOT meant to be an index.

Well, we solve it in much the same way as previously mentioned area problem. We breakup our interval of time into subintervals, and use some velocity value in each of the corresponding subintervals to calculate the approximate distance travled for the subinterval. And then we add each of those distances togeter.

Wait... what? Let's see this in practice.

Imagine you had a taut string and you decided to pluck it so it would vibrate. Now, let's say you collected some data and found that at any point in time the middle of the string had a velocity, in the y-direction, of $v(t) = \sin(t)$

Figure: The string tautly held in place

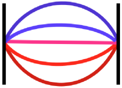


Figure: The string at different positions after being plucked

Let's try to find how far the string travles from $t = 0$ to $t = \frac{\pi}{2}$ using 4 subintervals and measureing the velocity at the middle of each interval.

So, we want to find the value of,

$$D \approx \sum_{k=1}^{n} d_k = \sum_{k=1}^{n} v(t_k)\Delta t.$$

Since $v(t) = \sin(t)$, $n = 4$, $\Delta t = \frac{\pi}{8}$, and we are using the velocity measured the middle of each interval to find the approximate distance traveled during the corresponding subinterval so $t_k = (2k-1)\frac{\pi}{16}$.

We know,

$$D \approx \sum_{k=1}^{4} \sin\Big((2k-1)\frac{\pi}{16}\Big)\left(\frac{\pi}{8}\right)$$

Question:

Why is $t_k = (2k-1)\frac{\pi}{16}$? Try drawing out the line $t = 0$ to $t = \frac{\pi}{2}$ with each subinterval, and finding where the center of each subinterval is.

So,

$$D \approx \sum_{k=1}^{4} \sin\Big((2k-1)\frac{\pi}{16}\Big)(\frac{\pi}{8}) =$$

$$(\frac{\pi}{8})\sin\Big(\frac{\pi}{16}\Big) + (\frac{\pi}{8})\sin\left(\frac{3\pi}{16}\right) + (\frac{\pi}{8})\sin\left(\frac{5\pi}{16}\right) + (\frac{\pi}{8})\sin\left(\frac{7\pi}{16}\right) =$$

$$(\frac{\pi}{8})(\sin\frac{\pi}{16} + \sin\frac{3\pi}{16} + \sin\frac{5\pi}{16} + \sin\frac{7\pi}{16}) \approx (\frac{\pi}{8})(2.56291) \approx 1.006$$

In other words, we found that the center string will travel a distance of approx 1 unit in that time interval, $t \in (0, \pi/2)$.

(We will later find that the center of the string in fact travels exactly 1 unit in that interval!)

So, what exactly have we done here? Well,

- we broke our interval of time up into 4 subintervals of length $\Delta t = \frac{\pi}{8}$,
- calclated the approximate distance travled in each of those intervals using the velocity in the middle of those subintervals $f(t_k) = \sin\big((2k-1)\frac{\pi}{16}\big)$,
- and added them together to get the totall approximate distance.

And if we wanted a more accurate approximation of the totall distance traveled we would use smaller subintervals.

More specifically, we showed that the distance an object travels for so interval of time, is the area under the velocity function $v(t)$ of the object for that same inteval of time!

Definite Integrals

Question:

- Is there a maximum number of subintervals we can use(n)?
- Is there a minimum width we can use for Δx?
- Can we, let $n \to \infty$ or $\Delta x \to 0$?

Before we aswer those questions, let's look back at what we just did. We defined three series and used them to calculate an approximation of the desired distance or area.

Those three series were,

- $A_R = \sum_{k=1}^{n} f(x_k)\Delta x$, the top right corner is equal to our function so $x_k = a + k\Delta x$
- $A_L = \sum_{k=1}^{n} f(x_{k-1})\Delta x = \sum_{k=0}^{n} f(x_k)\Delta x$, the top left corner is equal to our function so $x_k = a + k\Delta x$
- $A_M = \sum_{k=1}^{n} f(x_k)\Delta x$, the center of the top edge of our rectangle is equal to our function so $x_k = a + (2k-1)\frac{\Delta x}{2}$

where $\Delta x = \frac{b-a}{n}$

Notice, all three of these can be generallized to the form,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and x_k^* is some x value in the interval $[a+(k-1)\Delta x, a+k\Delta x]$. Notice the last condition can equivallently be written as the inequality, $a+(k-1)\Delta x \leq x_k^* \leq a+k\Delta x$.

We call this general for a **Riemann Sum**, named after the German mathematician Bernhard Riemann(for my West World fans, he's not a robot).

We previously observed that as n gets larger we get a more approximate solution for the area or distance. Notice then, something special happens if we let, $n \to \infty$... We get an "exact" solution for the area under the function $f(x)$ for our interval $[a, b]$!

This brings us to the definition of the **Definite Integral**

We say,

*the **Definite Integral** of a function $f(x)$ over some interval $[a, b]$ with respect to x is,*

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)dx,$$

where $\Delta x = \frac{b-a}{n}$ and $a + (k-1)\Delta x \le x_k^* \le a + k\Delta x$. Provided the limit exists and gives the same value for all possible choices of x_k^*.

And we say, if it does exist,

*$f(x)$ is **integrable** on $[a, b]$.*

Now, an important theorem about the definite integral!

Theorem

*If, $f(x)$ is continuous on $[a, b]$, or if $f(x)$ has **ONLY A FINITE** number of jump discontinuities, then $f(x)$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.*

Let's take a moment to compare that to the derivative.

Recall: a function is continuous if, $\lim_{x\to a} f(x) = f(a)$.

The derivative is defined as,

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

So, inorder for a function to be differentiable in a domain, it had to be continuous in that domain. But, a function isn't nessicarily differentiable in a domain just because it's continuous there.

So, differentiability implies continuity; but, continuity doesn't imply differentiability.

Where as for a function to be integrable in a domain, it can have at most a finite number of jump discontinuities in that domain!

So, being integrable does not imply continuity; but, continuity implies being integrable.

I REPEAT

Differentiability implies continuity; but, continuity doesn't imply differentiability.

And being integrable does not imply continuity; but, continuity implies being integrable.

THESE ARE IMPORTANT FACTS!

Midpoint Rule

An important rule about the definite integral is the Midpoint Rule. But before we go in that let's recall,

*the **Definite Integral** of a function $f(x)$ over some interval $[a, b]$ with respect to x is,*

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)dx,$$

where $\Delta x = \frac{b-a}{n}$ and $a + (k-1)\Delta x \leq x_k^* \leq a + k\Delta x$. Provided the limit exists and gives the same value for all possible choices of x_k^*.

If however, we don't impose the limit and hold that x_k^* is the midpoint of the k-th subinterval, that is

$$x_k^* = \frac{x_{k-1} + x_k}{2}$$

which we write as $\overline{x}_k$ then we get the following:

$$\sum_{k=1}^{n} f(\overline{x}_k)\Delta x \approx \int_a^b f(x)dx,$$

where $\Delta x = \frac{b-a}{n}$. This is known as the **Midpoint Rule**.

Notice, we applied the "midpoint rule" solving the distance problem.

Go to Problem

Properties of the Definite Integral:

We will quickly justify three properties of the definite integral and list the remaining properties.

Given that,

*the **Definite Integral** of a function $f(x)$ over some interval $[a, b]$ with respect to x is,*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)dx,$$

where $\Delta x = \frac{b-a}{n}$ and $a + (k-1)\Delta x \leq x_k^* \leq a + k\Delta x$. Provided the limit exists and gives the same value for all possible choices of x_k^*.

We can deduce three properties immediately.

1. If $a = b$, then $\Delta x = 0$ and

$$\int_a^a f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} 0 = 0.$$

2. If $a > b$, then $\Delta x = \frac{b-a}{n} < 0$; so,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

3. $\int_a^b (f(x) \pm g(x))\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} (f(x_k^*) \pm g(x_k^*))\Delta x =$
$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \pm \lim_{n\to\infty} \sum_{k=1}^{n} g(x_k^*)\Delta x =$
$\int_a^b f(x)dx \pm \int_a^b g(x)dx$

Other Properties of the Definite Integral:

- $\int_a^b c\,dx = c(b-a)$, c is a constant
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a, b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

The Fundemental Theorem of Calculus

Before we state either part of the Fundemental Theorem of Calculus, we are going to prove them.

To begin, we are going to choose two arbitrary functions, $f(x)$ and $g(x)$, such that

- $f(x)$ is continuous on $[a, b]$
- there exist an $x \in (a, b)$, so $a < x < b$.
- $g(x) = \int_a^x f(t)dt$.

Then, there exist an $h \neq 0$ such that $x + h \in (a, b)$, so $a < x + h < b$; and,

$$g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt =$$

$$\left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt\right) - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt.$$

Since we assumed $h \neq 0$ we have,

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$

Recall, "Extreme Value Theorem" (pg 206, Chapter 3 Section 1) says,

If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains an absolute maximum value $f(v) = M$ and an absolute minimum value $f(u) = m$ at some numbers u and v in $[a, b]$.

So, since we required "$f(x)$ is continuous on $[a, b]$," Go to Conditions we know there exist a u and v in $[x, x+h]$, such that $f(u) \leq f(t) \leq f(v)$ for all $t \in [x, x+h]$; and, assuming $h > 0$(we would just flip the inequalities if $h < 0$), it follows from the third comparison property of the definite integral that

$$hf(u) \leq \int_{x}^{x+h} f(t)dt \leq hf(v) .$$

Go to Properties

But, by dividing by h we can see that this means,

$$f(u) \leq \frac{1}{h} \int_{x}^{x+h} f(t)dt \leq f(v);$$

and, we already saw that $\frac{g(x+h)-g(x)}{h} = \frac{1}{h} \int_{x}^{x+h} f(t)dt$, Go to Result so

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v).$$

Now, since u and v are in $[x, x+h]$ we get that as $h \to 0$, $u \to x$ and $v \to x$; and, the limit yields,

$$f(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

But we chose $g(x) = \int_a^x f(t)dt$, Go to Conditions so

$$g'(x) = \frac{d}{dx} \int_a^x f(t)dt;$$

and we get that

$$f(x) = \frac{d}{dx} \int_a^x f(t)dt. \qquad \blacksquare$$

Thus we have just proven the first part of the Fundemental Theorem of Calculus!

The first part of the Fundemental Theorem of Calculus states,

If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

Now, to prove the second part, we are going to choose two arbitrary functions, $f(x)$ and $g(x)$, such that

- $f(x)$ is continuous on $[a, b]$
- there exist an $x \in (a, b)$, so $a < x < b$.
- $g(x) = \int_a^x f(t)dt$.

From part 1 we know, $g'(x) = f(x)$; hence, $g(x)$ is an antiderivative of $f(x)$.

If $F(x)$ is any other antiderivative of $f(x)$ on $[a, b]$, then we know, from the Corollary[1] on pg219 Chapter 3 Section 2, that

$$F(x) = g(x) + c$$

for $x \in (a, b)$.

[1] If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f(x) - g(x)$ is constant on (a, b) that is, $f(x) = g(x) + c$ where c is a constant.

And, since both $F(x)$ and $g(x)$ are continuous on $[a, b]$ we get the the limits $x \to a^+$ and $x \to b^-$ exists. Hence,

$$F(x) = g(x) + c$$

for $x \in [a, b]$.

Since $F(x) = g(x) + c$ for $x \in [a, b]$, we have that,

$$F(b) - F(a) = (g(b) + c) - (g(a) + c) = g(b) - g(a).$$

But $g(a) = \int_a^a f(t)dt$ and by one of the properties of definite integrals Go to Properties we have that $g(a) = 0$ so,

$$F(b) - F(a) = g(b) - g(a) = g(b) = \int_a^b f(t)dt. \qquad \blacksquare$$

Thus we have just proven the second part of the Fundemental Theorem of Calculus!

The second part of the Fundemental Theorem of Calculus states,

If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

Together these two parts form **The Fundemental Theorem of Calculus**:

Suppose $f(x)$ is continuous on $[a, b]$

1. *If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.*
2. *$\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$, that is, $F'(x) = f(x)$.*

The Fundemental Theorem of Calculus

1. If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

2. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$, that is, $F(x)$ is a function such that $F'(x) = f(x)$

You will be tested on this! You do need to know this! This is the **FUNDEMENTAL THEOREM OF CALCULUS**

Suppose $f(x)$ is continuous on $[a, b]$

1. *If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.*
2. *$\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$, that is, $F'(x) = f(x)$.*

Appendex

Derivative Identities:

- $\frac{d}{dx} x^n = nx^{n-1}$
- $\frac{d}{dx} \cos(x) = -\sin(x)$
- $\frac{d}{dx} \sin(x) = \cos(x)$
- $\frac{d}{dx} \tan(x) = \sec(x)$
- $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Derivative Identities:

- $\frac{d}{dx}f(x)g(x) = g(x)f'(x) + f(x)g'(x)$
- $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$
- $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

The Anatomy of Summation Notation:

To write $x_1 + x_2 + \cdots + x_{n-1} + x_n$ in abbreviated notation we use summation notation, and write

$$\sum_{i=1}^{n} x_i \; .$$

- We call i the index, or dummy varriable, of our series.
- We call, $i = 1$ the lower index and $i = n$ the upper index. These tell us which index to choose when adding our summation elements x_i.

Examples:

- $1+2+3+4+5=\sum_{i=1}^{5} i$
- $\sum_{k=1}^{5} k^2 = 1^2+2^2+3^2+4^2+5^2 = 1+4+9+16+25$
- $\sum_{j=0}^{n}(4j+5) = (4(0)+5)+(4(1)+5)+\cdots+(4(n-1)+5)+(4(n)+5)$
- $\sum_{i=5}^{8} \frac{2i}{i^3} = \frac{2(5)}{(5)^3}+\frac{2(6)}{(6)^3}+\frac{2(7)}{(7)^3}+\frac{2(8)}{(8)^3}$

Notes on Series:

- If $\sum_{i=m}^{n} x_i$ is equal to a value, we say the series converges, otherwise we say it diverges.

Common Series:(pay attention to the starting index!)

- $\sum_{i=0}^{m} ar^i = a\frac{1-r^m}{1-r}$, $0 < |r| < 1$ (Geometric Series)
- $\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$
- $\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$
- $\sum_{i=1}^{m} i^3 = (\frac{m(m+1)}{2})^2$
- $\sum_{i=1}^{m} c = mc$
- $\sum_{i=1}^{m} ca_i = c\sum_{i=1}^{m} a_i$
- $\sum_{i=1}^{m}(a_i \pm b_i) = \sum_{i=1}^{m} a_i \pm \sum_{i=1}^{m} b_i$

Common Infinite Series:

- $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$, $0 < |r| < 1$ (Geometric Series)
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ (isn't $\sin(x)$ an "odd function?")
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ (isn't $\cos(x)$ an "even function?")
- $e^x = \sum_0^{\infty} \frac{x^n}{n!}$ (isn't that **almost** like, $\cos(x) + \sin(x)$?)

The Indefinite Integral:

Given a function $F(x)$ with a derivative $\frac{d}{dx}F(x) = f(x)$, we say

$$\int f(x)dx = F(x) + C.$$

This is because there are a "family" of functions with the derivative $f(x)$ and they all differ only by a constant.

A Note about the Indefinite Integral:

Given $\int f(x)dx = F(x) + C$, we say that this is, "the integral('$\int$') of f(x) with respect to x('dx')."

Common Indefinite Integral:

- $\int x^n dx = \frac{1}{n} x^{n+1}$
- $\int cos(x)dx = sin(x) + C$
- $\int sin(x)dx = -cos(x) + C$
- $\int \frac{1}{x} dx = ln(x) + C$
- $\int f'(x)e^{f(x)} dx = e^{f(x)} + C$
- $\int e^x dx = e^x + C$ (Special case of the last one, it's when $f(x) = x$ and $f'(x) = 1$)

Properties of the Definite Integral:

- $\int_a^b c\,dx = c(b-a)$, c is a constant
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a, b)$

Comparison Properties of the Definite Integral:

- If $f(x) \geq 0$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for all x such that $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M$ for all x such that $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Supplemental Reading

Calculus 8th edition by James Stewart:

- Chapter 4

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 7 through Chapter 10

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter 8 through 10

Schaum's Outline - Calculus 6th Edition by Frank Ayres, Jr, PhD and Elliott Mendelson, PhD:

- Chapter 22 through Chapter 24