Calc I

Day 1

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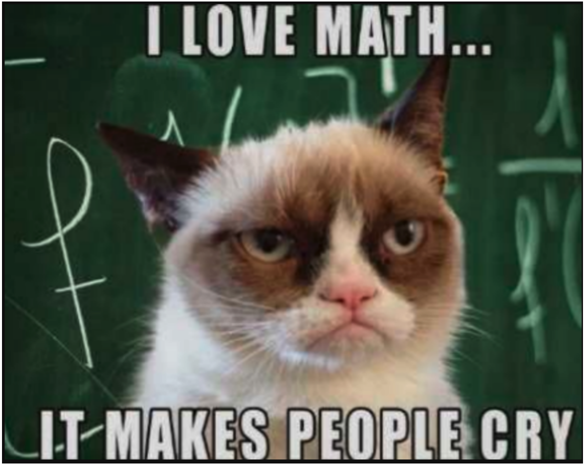
21 Jan. 2020


I LOVE MATH...
IT MAKES PEOPLE CRY

Review of Functions

Questions:

- What is a function?
- What sort of information can we gleam from functions?
- What rules MUST a function follow?
- **What happens when a function isn't continuous?**

Functions

Generally speaking, functions are equations that take in a value and give back exactly one value.

A common description of a function is, to think of it as a machine that takes an unique input and provides unique outputs.

FIGURE 2
Machine diagram for a function f

Representation of Functions

Functions
Representation of Functions
Notation and Anatomy of a Function
Domain and Range
Vertical Line Test
Example Problems
Piecewise Functions
More Example Problems
List of Common Functions
Inverse Functions
Again with the Example Problems!!!

There are 4 ways to represent a function,

- verbally
- numerically
- visually
- algebraically

Example(Verbally):

- The area of a circle is equal to π times the radius squared.

Example(Agebraically):

- $A = \pi r^2$

Example(Numerically and Visually):

r	A
1	π
2	4π
3	9π

Table: Numerically

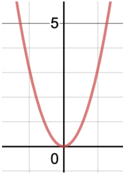
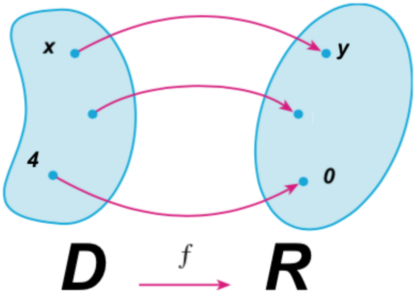


Figure: Visually

Another way to visually represent a function is with an ”arrow diagram”



Functions
Representation of Functions
Notation and Anatomy of a Function
Domain and Range
Vertical Line Test
Example Problems
Piecewise Functions
More Example Problems
List of Common Functions
Inverse Functions
Again with the Example Problems!!!

Notation and Anatomy of a Function

We traditionally use the notation $f(x) = y$ to denote functions.

Some notes:

- We can think of f as the "title" of our function.
- x is the **independent variable**
- y is the **dependent variable**

Functions
Representation of Functions
Notation and Anatomy of a Function
Domain and Range
Vertical Line Test
Example Problems
Piecewise Functions
More Example Problems
List of Common Functions
Inverse Functions
Again with the Example Problems!!!

Domain and Range

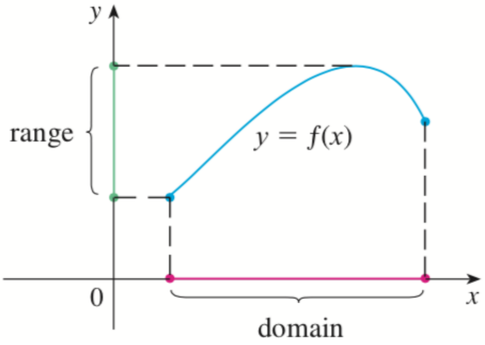
Cool story bro...

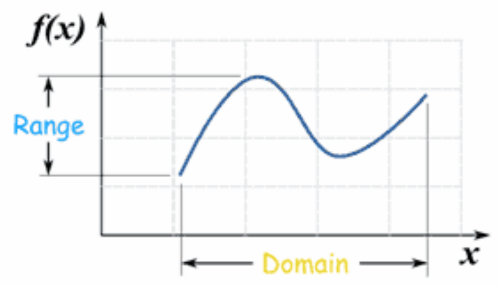
Thanks for the obvious stuff! But like... where do our numbers come from?

Well...

- The set of values that our **independent variable** could be, we call the **domain**.
- The set of values that our **dependent variable** could be, we call the **range**

You can, in some sense think of the **domain** and **range** as the ”shadows” of our function projected onto the x-, or y-, axis accordingly.

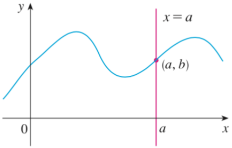

y
range
y = f(x)
0
x
domain


f(x)
Range
Domain
x

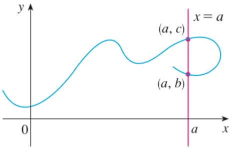
Vertical Line Test

Remember how functions spit out exactly one value for every unique value we put in? We can visuallise this property as the vertical line test.

Example:



(a) This curve represents a function.



(b) This curve doesn't represent a function.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Functions
Representation of Functions
Notation and Anatomy of a Function
Domain and Range
Vertical Line Test
Example Problems
Piecewise Functions
More Example Problems
List of Common Functions
Inverse Functions
Again with the Example Problems!!!

Example Problems

Functions
Representation of Functions
Notation and Anatomy of a Function
Domain and Range
Vertical Line Test
Example Problems
Piecewise Functions
More Example Problems
List of Common Functions
Inverse Functions
Again with the Example Problems!!!

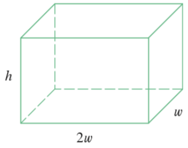
Prob 1:

Example 1 A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

Functions
Representation of Functions
Notation and Anatomy of a Function
Domain and Range
Vertical Line Test
Example Problems
Piecewise Functions
More Example Problems
List of Common Functions
Inverse Functions
Again with the Example Problems!!!

Setup:

We want to derive a function for the cost of materials in terms of width, $C(w)$.



Setup:

We know,

- The volume of the box is $10m^3$
- The box is twice as long as it is wide, $L = 2w$

So, we can say

$$v = wLh = w(2w)h = 10$$

Setup:

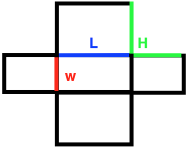
We also know

- For the base, the material cost is $\$10/m^2$
- For each of the side, the material cost is $\$6/m^2$

Setup:

Before we go any further lets expand on our image of the box by sketching an ”unfolded” view of it to more easily identify each side.

Setup:



From this we can see the areas of our sides are,

- The base(center): Lw
- Two of the sides(left and right) are: wh
- The other two sides(top and bottom) are: Lh

Setup:

So, we can say

- The cost of the base is, \$10($2w^2$)
- The cost of the sides is, 2(\$6Lh) + 2(\$6wh)
  - Remember: $L = 2w$

So we have that the totall cost of our box is,

$$c = 10(2w^2) + 2(6(2w)h) + 2(6wh)$$

Setup:

Since we want to to find $C(w) = c$ and we know

- the volume is $(2w^2)h = 10$, and
- the cost is $c = 10(2w^2) + 6\left(2((2w)h) + 2(wh)\right)$

we can find $C(w)$.

Take a few minutes to try this on your own.

So, we should have found the **c**ost as a function of the **w**idth to be

$$C(w) = 20w^2 + \frac{180}{w}$$

...are we done?

Not quite yet, for this particular problem, we need to define the domain of our solution.

Luckily the real world does this part for us since the width of something has to be greater than 0.

Solution 1:

SOLUTION We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3. Thus

$$w(2w)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C, we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C(w) = 20w^2 + \frac{180}{w} \qquad w > 0$$

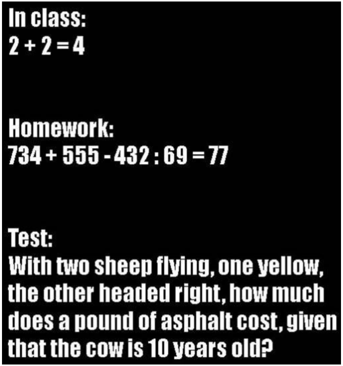
expresses C as a function of w. ■

Prob 2:

EXAMPLE 6 Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

"Wait... but like how? You haven't even shown us that yet!"



When solving for domain try asking yourself,

”what can’t ’x’ be?”

In the case of $\frac{1}{x}$, x can't be 0. So to find when x is 0 and include everything but that.

Similarly, for $\sqrt{x}$, x can't be negative. So we solve for, $x \geq 0$.

Setup:

So applying the rules from the previous slides,

1. for $f(x) = \sqrt{x+2}$, the domain is the soltion to $x + 2 \geq 0$
2. for $g(x) = \frac{1}{x^2 - x}$, the domain is the soltion to $x^2 - x \neq 0$

Solution 2:

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geqslant 0$. This is equivalent to $x \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x-1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is

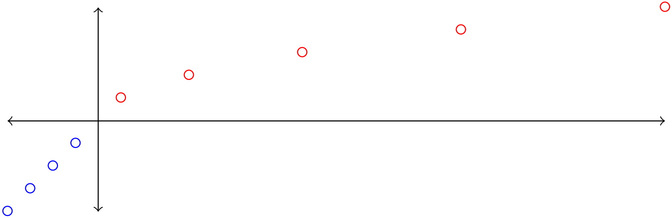
$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

Piecwise Functions

So, what if we have want to find the equation that defines some data, but it looks like there are two parts?



How might we define this function?

We can use functions that are called piecewise functions.

The basics:

- A piecewise function defines a function within regions.
- We write piecewise functions as, $f(x) = \begin{cases} \mathit{Rule}\ 1 & \mathit{Region}\ 1 \\ \mathit{Rule}\ 2 & \mathit{Region}\ 2 \\ \mathit{Rule}\ 3 & \mathit{Region}\ 3 \\ \mathit{etc.} & \mathit{etc.} \end{cases}$

Functions
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Example:

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Figure: Try graphing it!

Example:

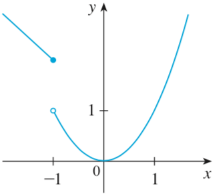


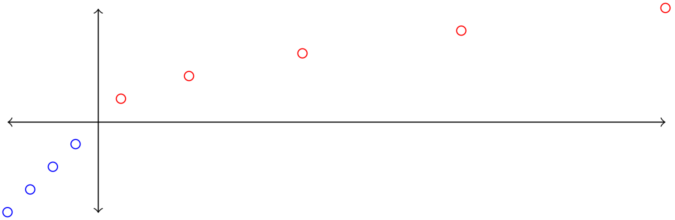
Figure: Is this what you drew?

Is this a function?

Functions
Representation of Functions
Notation and Anatomy of a Function
Domain and Range
Vertical Line Test
Example Problems
Piecewise Functions
More Example Problems
List of Common Functions
Inverse Functions
Again with the Example Problems!!!

More Example Problems

Prob 3:



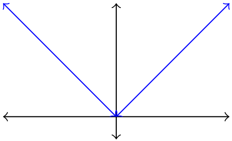
How would we define the function, $g(x)$, that fits this data, given the blue part($x < 0$) can be defined as $x = y$ and the red part as $\sqrt{x} = y$?

Solution 3:

$$g(x) = \begin{cases} x, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$$

Prob 4:

Try defining the absolute value of x, $|x|$, as a piecwise function.



Solution 4:

$$g(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

See if you can figure this one out at home.

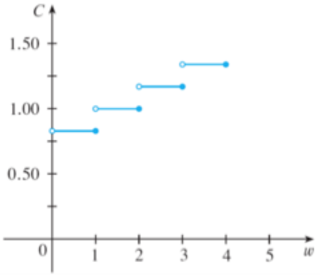


Figure: For those who are curious, this is an example of what's called a "Step Function." These play an important role in math, physics, and engineering!

List of Basic Functions

List of Basic Functions(The Pumpkin Spice Lattes of Functions)**:**

- $\sum_{n=0}^{k} a_n x^n, \quad a_k \neq 0$ **(Polynomial)** ***WHAT EVEN IS THIS!?***
- e^x **(Exponential)**
- $\frac{1}{x}$ **(Rational)**
- $\log x$ **-OR-** $\ln x$ **-OR-** $\log_k x$ **(Logorithim)**
- $\sin x$, $\cos x$, $\tan x$ **(Trig)**
- $|x|$ **(Absolute Value)**

Inverse Functions

Question:

If $f(x) = y$ then what is it's inverse?

The inverse function ”undoes” the function!

Functions
Representation of Functions
Notation and Anatomy of a Function
Domain and Range
Vertical Line Test
Example Problems
Piecewise Functions
More Example Problems
List of Common Functions
Inverse Functions
Again with the Example Problems!!!

So it answers the question:

If y is this, what x value "made it"

One way to think about functions is like ”maps.”

- The function gets you to your destination
- The inverse function takes you back to where you started.

And we write it as, $f(x)^{-1}$

Example Problems

Prob 5:

Let $f(x) = x + 5$. Find $f^{-1}(x)$.

Setup:

The general setup for finding an inverse function is-

1. Write out the equation of your function $f(x) = y$
2. Swap the variables
3. Solve for the independent variable
4. Rewriting it as the inverse function

Setup:

Applying that to our problem

1. $x + 5 = y$ Writing out our equation
2. $y + 5 = x$ Swapping out our variables
3. $y = x - 5$ Solving
4. $f^{-1}(x) = x - 5$ Rewritting the it as the inverse function

Solution 6:

$$f^{-1}(x) = x - 5$$

Prob 6:

Let $f(x) = x^2 - 4$. Find $f^{-1}(x)$.

Setup:

1. $x^2 - 4 = y$ Writing out our equation
2. $y^2 - 4 = x$ Swapping out our variables
3. $y^2 = x - 4$ Solving
4. $y = \pm\sqrt{x - 4}$ Solving
5. But wait! $y = \pm\sqrt{x - 4}$ fails the vertical line test so what do we do?????

Solution 6:

$$f^{-1}(x) = \sqrt{x-4}$$

Calculus 8th edition by James Stewart:

- Chapter 1

The Calculus Story A Mathematical Adventure by David Acheson:

- Chapter 1 through Chapter 3

The Cartoon Guide to Calculus by Larry Gonick:

- Chapter -1 and 0