FOURIER'S HEAT CONDUCTION EQUATION: HISTORY, INFLUENCE, AND CONNECTIONS

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Abstract. The equation describing the conduction of heat in solids has, over the past two centuries, proved to be a powerful tool for analyzing the dynamic motion of heat as well as for solving an enormous array of diffusion-type problems in physical sciences, biological sciences, earth sciences, and social sciences. This equation was formulated at the beginning of the nineteenth century by one of the most gifted scholars of modern science, Joseph Fourier of France. A study of the historical context in which Fourier made his remarkable contribution and the subsequent impact his work has had on the development of modern science is as fascinating as it is educational. This paper is an attempt to present a picture of how certain ideas initially led to Fourier's development of the heat equation and how, subsequently, Fourier's work directly influenced and inspired others to use the heat diffusion model to describe other dynamic physical systems. Conversely, others concerned with the study of random processes found that the equations governing such random processes reduced, in the limit, to Fourier's equation of heat diffusion. In the process of developing the flow of ideas, the paper also presents, to the extent possible, an account of the history and personalities involved.

1. INTRODUCTION

The equation describing the conduction of heat in solids occupies a unique position in modern mathematical physics. In addition to lying at the core of the analysis of problems involving the transfer of heat in physical systems, the conceptual-mathematical structure of the heat conduction equation (also known as the heat diffusion equation) has inspired the mathematical formulation of many other physical processes in terms of diffusion. As a consequence, the mathematics of diffusion has helped the transfer of knowledge relating to problem solving among diverse, seemingly unconnected disciplines. The transient process of heat conduction, described by a partial differential equation, was first formulated by Jean Baptiste Joseph Fourier (1768–1830) and presented as a manuscript to the Institut de France in 1807. At that time, thermodynamics, potential theory, and the theory of differential equations were all in the initial stages of their formulation. Combining remarkable gifts in pure mathematics and insights into observational physics, Fourier opened up new areas of investigation in mathematical physics with his 1807 masterpiece, *Théorie de la Propagation de la Chaleur dans les Solides*.

Fourier's work was subjected to review by some of the most distinguished scientists of the time and was not accepted as readily as one might have expected. It would be another 15 years before this major contribution would be accessible to the general scientific community through publication of his classic monograph, *Théorie Analytique de la Chaleur* (Analytic Theory of Heat) [*Fourier*, 1822]. Soon after this publication, the power and significance of Fourier's work was recognized outside of France. Fourier's method began to be applied to analyze problems in many fields besides heat transfer: electricity, chemical diffusion, fluids in porous media, genetics, and economics. It also inspired a great deal of research into the theory of differential equations. Nearly 2 centuries later, the heat conduction equation continues to constitute the conceptual foundation on which rests the analysis of many physical, biological, and social systems.

A study of the conditions that led to the articulation of the heat conduction equation and the reasons why that equation has had such a major influence on scientific thought over nearly 2 centuries is in itself instructive. At the same time, an examination of how the work was received and accepted by Fourier's peers and successors gives us a glimpse into the culture of science, especially during the nineteenth century in Europe. The present work has been motivated both by the educa-

8755-1209/99/1998RG900006$15.00

Reviews of Geophysics, 37, 1 / *February 1999*
pages 151–172
Paper number 1998RG900006

TABLE 1. Chronology of Significant Contributions on Diffusion

	Year	*Contribution*
Fahrenheit	1724	mercury thermometer and standardized temperature scale
Abbé Nollet	1752	observation of osmosis across animal membrane
Bernoulli	1752	use of trigonometric series for solving differential equation
Black	1760	recognition of latent heat and specific heat
Crawford	1779	correlation between respiration of animals and their body heat
Lavoisier and Laplace	1783	first calorimeter; measurement of heat capacity, latent heat
Laplace	1789	formulation of Laplace operator
Biot	1804	heat conduction among discontinuous bodies
Fourier	1807	partial differential equation for heat conduction in solids
Fourier	1822	*Théorie Analytique de la Chaleur*
Ohm	1827	law governing current flow in electrical conductors
Dutrochet	1827	discovery of endosmosis and exosmosis
Green	1828	formal definition of a potential
Graham	1833	law governing diffusion of gases
Thomson	1842	similarities between equations of heat diffusion and electrostatics
Poiseuille	1846	experimental studies on water flow through capillaries
Graham	1850	experimental studies on diffusion in liquids
Fick	1855	Fourier's model applied to diffusion in liquids
Darcy	1856	law governing flow of water in porous media
Dupuit	1863	potential theory applied to flow in groundwater basins
Maxwell	1867	diffusion equation for gases derived from dynamical theory
Pfeffer	1877	investigations on osmosis in biological and inorganic membranes
Edgeworth	1883	law of error and Fourier equation
Forchheimer	1886	flow nets for solving seepage problems using potential theory
van't Hoff	1887	theory of osmotic pressure by analogy with gas laws
Nernst	1888	interpretation of Fick's law in terms of forces and resistances
Lord Rayleigh	1894	random mixing of sound waves as a diffusion process
Roberts-Austen	1896	experimental measurement of solid diffusion
Bachelier	1900	option pricing and diffusion of probability
Einstein	1905	Brownian motion and diffusion equation
Pearson	1905	notion of random walk
Buckingham	1907	diffusion of multiple fluid phases in soils
Langevin	1908	framework for stochastic differential equation
Gardner	1922	measurement of potential in a multiple-fluid-phase porous medium
Terzaghi	1924	seepage in deformable clays as analogous to heat diffusion
Richards	1931	nonlinear diffusion of moisture in soils
Fermi	1936	neutron diffusion in graphite as analogous to heat diffusion
Bullard	1949	thermal gradient probe for the ocean floor
Taylor	1953	advective dispersion as a diffusion process

tional and historical importance of Fourier's work. Accordingly, the purpose of this paper is to explore how the framework of the heat conduction equation has come to help us understand an impressive array of seemingly unconnected natural processes and, in so doing, to gain historical insights into the manner in which scientific ideas develop. The focus of this paper is on the connections of concepts and ideas. An in-depth treatment of Fourier's 1807 manuscript can be found in the work of *Grattan-Guinness and Ravetz* [1972], and critical treatment of Fourier the man and the physicist, including important Fourier correspondence, can be found in the work of *Herivel* [1975].

The paper starts with scientific developments during the eighteenth century that set the stage for Fourier's work on heat conduction. Following this, details are presented of Fourier himself and his contribution, especially the 1807 manuscript. Fourier's influence has occurred along two lines. Experimentalists in electricity, chemical diffusion, and fluid flow in porous materials directly derived benefit from interpreting their experiments by analogy with the heat conduction phenomenon. Researchers in other fields such as statistical mechanics and probability theory indirectly established connections with the heat conduction equation by recognizing the similarities between the mathematical behavior of their systems and mathematical solutions of the heat conduction equation. These direct and indirect influences of Fourier's work are described next. The paper concludes with some reflections on the scientific atmosphere during the nineteenth century, a comparison of the different facets of diffusion, and a look beyond Fourier's solution strategy. A chronology of the important developments is presented in Table 1.

2. DEVELOPMENTS LEADING UP TO FOURIER

Before we describe the scientific developments of the eighteenth century that set the stage for Fourier's contribution, it is useful to briefly state the nature and content of the heat conduction process. The transient heat diffusion equation pertains to the conductive transport and storage of heat in a solid body. The body itself, of finite shape and size, communicates with the external world by exchanging heat across its boundary. Within the solid body, heat manifests itself in the form of temperature, which can be measured accurately. Under these conditions, Fourier's differential equation mathematically describes the rate at which temperature is changing at any location in the interior of the solid as a function of time. Physically, the equation describes the conservation of heat energy per unit volume over an infinitesimally small volume of the solid centered at the point of interest. Crucial to such conservation of heat is the recognition that heat continuously moves across the surfaces bounding the infinitesimal element as dictated by the variation of temperature from place to place within the solid and that the change in temperature at a point reflects the change in the quantity of heat stored in the vicinity of the point.

It is clear from the above that the notions of temperature, quantity of heat, and transport of heat, as well as the relation between quantity of heat and temperature, are fundamental to Fourier's heat conduction model. It is important to recognize here that these basic notions were still evolving when Fourier developed his equation. Therefore it is appropriate to begin by familiarizing ourselves with the evolution of these notions during the eighteenth century.

Since heat can be readily observed and measured only in terms of temperature, the development of a reliable thermometer capable of giving repeatable measurements was critical to the growth of the science of heat. Gabriel Daniel Fahrenheit (1686–1736), a German instrument maker and physicist, perfected the closed-tube mercury thermometer in 1714 and was producing it commercially by 1717 [*Middleton*, 1966]. By 1724 he had established what we now know as the Fahrenheit scale with the melting of ice at 32° and the boiling of water at 212°.

The next developments of interest were qualitative and conceptual, and of great importance. Joseph Black (1728–1799), a pioneer in quantitative chemistry, was known for his lectures in chemistry at Glasgow and was also a practicing physician. Around 1760 he noticed that when ice melts, it takes in heat without changing temperature. This observation led him to propose the term "latent heat" to denote the heat taken up by water as it changes its state from solid to liquid. He also noticed that equal masses of different substances needed different amounts of heat to raise their temperatures by the same amount. He coined the term "specific heat" to denote this type of heat. Although Black is said to have constructed an ice calorimeter, he never published his results. The precise measurement of latent heat and specific heat was left to Lavoisier and Laplace, some 20 years later. Another important development was the appearance of the book *Experiments and Observations of Animal Heat, and the Inflammation of Combustible Bodies* by Adair Crawford (1748–1795) in 1779. In this work, Crawford proposed that oxygen was involved in the generation of heat by animals during respiration and went on to discuss a method of measuring specific heat by a method of mixtures [*Guerlac*, 1982]. Crawford's idea of measuring specific heat by the method of mixtures would soon have a significant influence on Lavoisier and Laplace, although he himself was unable to measure these quantities accurately.

In the wake of the contributions of Black and Crawford, what must be considered as one of the most important papers of modern chemistry and thermodynamics appeared in 1783. This was the paper entitled *Mémoire sur la Chaleur* coauthored by Antoine Laurent Lavoisier (1743–1794), the central figure of the revolution in chemistry of the latter half of eighteenth century, and Pierre Simon Laplace (1749–1827), one of the more influential mathematicians and theoretical physicists of modern science. *Lavoisier and Laplace* [1783] provided detailed descriptions of an ice calorimeter with which they measured, for the first time, the latent heat of melting of ice and the specific heats of different materials. All the measurements were made relative to water, the chosen reference. They also showed experimentally that animals release heat during respiration by placing a guinea pig within the calorimeter for several hours and measuring the quantity of ice melted. In a related set of experiments, they also demonstrated quantitatively that the process of respiration, in which oxygen is combined with carbon in the animal's body, is in fact combustion, resulting in the release of heat. During the late nineteenth century, when this work was done, the nature of heat was still a matter of debate. Some believed that heat was a fluid diffused within the body (referred to as "caloric") while others believed that heat was a manifestation of vibrations or motions of matter at the atomic level. Although Lavoisier and Laplace preferred the latter concept, they interpreted and presented their results in such a way that the experiments stood by themselves, independent of any hypothesis concerning the nature of heat. The significance of the Lavoisier-Laplace contribution to Fourier's equation is that it provided the notion of specific heat, which is fundamental to the understanding of time-dependent changes of temperature. Nonetheless, the significance of the work far transcends Fourier's equation. By experimentally quantifying latent heat and heats of reactions, the Lavoisier-Laplace work constitutes an essential component of the foundations of thermodynamics.

We now consider the process of transfer of heat in solids, that is, the process of heat conduction. The best known pre-Fourier work in this regard is that of Jean Baptiste Biot (1774–1862) who made important contributions in magnetism, optics, and celestial mechanics. *Biot* [1804] addressed the problem of heat conduction in a thin bar heated at one end. In the bar, heat not only was conducted along the length but was also lost to the exterior atmosphere transverse to the direction of conduction. Biot's starting point was Newton's law of cooling, according to which the rate at which a body loses heat to its surroundings is proportional to the difference in temperature between the bar and the exterior atmosphere. Biot, who was a student of Laplace's mechanistic school, believed in the philosophy of action at a distance between bodies. Accordingly, the temperature at a point in the heated rod was perceived to be influenced by all the points in its vicinity. Essentially, then, the mathematical problem of heat conduction came to be considered as one of a class of many-body problems. As was pointed out by *Grattan-Guinness and Ravetz* [1972], Biot's idealization of action at a distance involved only the difference in temperature between points and did not involve the distance between the points. Consequently, Biot's approach did not involve a temperature gradient, so necessary to the formulation of the differential equation. However, Biot did articulate the underlying concepts clearly by stating that when the heat content of the bar changes at each instant, the net accumulation of heat at a point causes a change in temperature. Biot also asserted that he experimentally found Newton's law concerning the loss of heat to be rigorous. Similar inferences of a qualitative nature had been drawn by *Lambert* [1779], who had experimentally studied heat conduction in a rod [*Herivel*, 1975].

Apart from these foundational developments relating to heat, developments relating to potential theory and differential equations during the nineteenth century deserve notice. The theory of potentials arises in many branches of science, such as electrostatics, magnetostatics, and fluid mechanics. Potential theory involves problems describable in terms of a partial differential equation in which the dependent variable is the appropriate potential (defined as a quantity whose gradient is force) and the sum of the second spatial derivatives of the potential in three principal directions is equal to zero. This equation was first formulated by *Laplace* [1789], although the term potential would be coined later by George Green (1793–1841), a self-educated mathematician, [*Green*, 1828]. *Laplace* [1789] formulated the equation in the context of the problem of the stability of Saturn's rings.

The eighteenth century also saw very active developments in the theory of ordinary and partial differential equations through the contributions of Daniel Bernoulli (1700–1782), Jean le Rond d'Alembert (1717–1783), Leonhard Euler (1707–1783), John-Louis Lagrange (1736–1813), and others. For the partial differential equation describing a vibrating string, Bernoulli had suggested, on physical grounds, a solution in terms of trigonometric series. Similar usage of trigonometric series was also made a little later by Euler and Lagrange. Yet d'Alembert, Euler, and Lagrange were not particularly satisfied with the trigonometric series. Their concerns were purely mathematical in nature, consisting of issues of convergence and algebraic periodicity of such series [*Grattan-Guinness and Ravetz*, 1972; *Herivel*, 1975].

In nineteenth-century Europe, two philosophical views of the physical world prevailed: the mechanistic school of Isaac Newton (1642–1727) and the dynamic school of Gottfried Wilhelm Leibniz (1646–1716). During the eighteenth and nineteenth centuries, a number of leading thinkers from France were fully committed to the mechanistic view and devoted their efforts to describing the physical world with greater detail in terms of Newton's laws. At the same time, Newton's contemporary Leibniz also had a major influence on the development of scientific thought. At the foundation of physics were the notions of force, momentum, work, and action. Although these notions are all related, Newton and Leibniz pursued two parallel but distinct avenues to understanding the physical world. Newton's approach was based on the premise that by knowing forces and momenta at every point or particle, one could completely describe a physical system. Leibniz, on the other hand, pursued the approach of understanding the total system in terms of work and action. One of the leading figures of Newton's mechanistic school was Laplace. Laplace, in turn, had many ardent followers, including Biot and Poisson. Among those who followed Leibniz's philosophy were Lagrange, Euler, and Hamilton. Although both approaches ultimately proved equivalent, the mathematics associated with each of them are very different. While the mechanistic school relied on the use of vector fields to describe the physical system, the dynamic school of Leibniz, remarkably, realized the same results through the use of energy and action, which are scalar quantities. Additionally, the thinking of the mathematical physicists of the late eighteenth century was also influenced by their intense interest in celestial mechanics, a field that had greatly captivated Galileo, Newton, and Kepler.

It was under these circumstances that observational data on heat, electricity, chemical reactions, and physiology of animals were being collected and great efforts were being made to understand them rationally in terms of force, momentum, energy, and work. As was already noted, at the turn of the nineteenth century, the nature of heat was still unresolved. Those of the mechanistic school, including Biot, believed that heat was a permeating fluid. On the other hand, those of the dynamic school believed that heat was essentially motion, consisting of rapid molecular vibrations. Those of the mechanistic school also believed that a cogent theory of heat should be rigorously built from a detailed description of motion at the level of individual particles. This approach, it appears, influenced the work of *Biot* [1804] and his use of action at a distance.

3. FOURIER'S CONTRIBUTION

The science of heat, the theory of potentials, and the theory of differential equations were all in their early stages of development by the time Fourier started his work on heat conduction. Opinions were still divided about the nature of heat. However, heat conduction due to temperature differences and heat storage and the associated specific heat of materials had been experimentally established. Potential theory had already been formulated. Finally, the representation of dynamic problems in continuous media with the help of partial differential equations (e.g., the problem of a vibrating string) and their solution with the help of trigonometric series were also known. In this setting, Fourier began working on the transient heat conduction problem.

Fourier's life and contributions are so unusual that a brief sketch of his career and the conditions under which he worked is worthwhile. For a comprehensive account, the reader is referred to *Grattan-Guinness and Ravetz* [1972] and *Herivel* [1975]. Fourier was born in 1768 in Auxerre in Burgundy, now the capital of Yonne department in central France. In 1789, about the time his mathematical talents began to blossom, the French Revolution intervened. In his native Auxerre he was socially and politically active, being a forceful orator. His outspoken criticism of corruption almost took him to the guillotine in 1794; he was saved mainly by the public outcry in the town and a deputation of local people on his behalf. Following this he taught mathematics for a few years at the Ecole Polytechnique in Paris. In 1798, Napoleon Bonaparte (1769–1821) was leading an expedition to Egypt, and Fourier was made Secrétaire Perpétuel of the newly formed Institut d'Egypte. In Egypt he held many important administrative and judicial positions and, in 1799, was made leader of a scientific expedition investigating monuments and inscriptions in Upper Egypt. In November 1801, Fourier returned to France upon the withdrawal of French forces from Egypt. However, his hopes for resuming his teaching duties at the Ecole Polytechnique were ended when Napoleon made him prefect of the department of Isère, near the Italian border, with its capital at Grenoble.

During his tenure as prefect, Fourier embarked on two very different major scholarly efforts. On the one hand, he played a leadership role on a multivolume work on Egypt, which would later form the foundation for the science of Egyptology. On the other, he began working on the problem of heat diffusion. It appears that Fourier started work on heat conduction sometime between 1802 and 1804, probably for no other reason than that he saw it as one of the unsolved problems of his time. Between 1802 and 1807 he conducted his researches into Egyptology and heat diffusion whenever he could find spare time from his prefectural duties.

Like Biot before him, Fourier initially formulated heat conduction as an n-body problem, stemming from the Laplacian philosophy of action at a distance. During these early investigations he was aware of Biot's work, having received a copy of the paper from Biot himself. For some reason that is not quite clear, Fourier abandoned the action at a distance approach around 1804 and made a bold departure from convention, which eventually led to his masterpiece, the transient heat conduction equation.

Essentially, Fourier moved away from discontinuous bodies and towards continuous bodies. Instead of starting with the basic equations of action at a distance, Fourier took an empirical, observational approach to idealize how matter behaved macroscopically. In this way he also avoided discussion of the nature of heat. Rather than assuming that the behavior of temperature at a point was influenced by all points in its vicinity, Fourier assumed that the temperature in an infinitesimal lamina or element was dependent only on the conditions at the lamina or element immediately upstream and downstream of it. He thus formulated the heat diffusion problem in a continuum.

In formulating heat conduction in terms of a partial differential equation and developing the methods for solving the equation, Fourier initiated many innovations. He visualized the problem in terms of three components: heat transport in space, heat storage within a small element of the solid, and boundary conditions. The differential equation itself pertained only to the interior of the flow domain. The interaction of the interior with the exterior across the boundary was handled in terms of "boundary conditions," conditions assumed to be known a priori. The parabolic equation devised by Fourier was a linear equation in which the parameters, conductivity, and capacitance were independent of time or temperature. This attribute of linearity enabled Fourier to draw upon the powerful concept of superposition to combine many particular solutions and thereby create general solutions [*Grattan-Guinness and Ravetz*, 1972]. The superposition artifice offered such promise for solving problems that mathematicians who followed Fourier resorted to linearizing differential equations so as to facilitate their subsequent solution.

Perhaps the most powerful and most daunting aspect of Fourier's work was the method of solution. Fourier was clearly aware of the earlier work of Bernoulli, Euler, and Lagrange relating to solutions in the form of trigonometric series. He was also aware that Euler, D'Alembert, and Lagrange viewed trigonometric series with great suspicion. Their opposition to the trigonometric series stemmed from reasons of pure mathematics: convergence and algebraic periodicity. Lagrange, in fact, had a particular preference for solutions expressed in the form of Taylor series [*Grattan-Guinness and Ravetz*, 1972]. Yet Fourier, who was addressing a well-defined physical problem with physically realistic solutions, did not allow himself to be held back by the concerns of his illustrious predecessors. He boldly applied the method of separation of variables and generated solutions in terms of infinite trigonometric series. Later, he would also generate solutions in the form of integrals that would come to be known as Fourier integrals. In the last part of his 1807 work, Fourier also presented some results pertaining to heat conduction in a cylindrical annulus, a sphere, and a cube.

Fourier submitted his manuscript to the French Academy in December 1807. As was the practice, the secretary of the Academy appointed a committee of reviewers consisting of four of the most renowned mathematicians of the time, Laplace, Lagrange, Monge, and Lacroix. The manuscript was not well received, particularly by Laplace and Lagrange, for the mathematical reasons alluded to above. Although Laplace would later become sympathetic to Fourier's method, Lagrange would never change his mind. Because of the lack of approval by his peers, the possible publication of Fourier's work by the French Academy was delayed indefinitely. In the end, Fourier took it upon himself to expand the work and publish it on his own in 1822 under the title *Théorie Analytique de la Chaleur*; it is now an avowed classic.

4. THE HEAT CONDUCTION EQUATION

It is appropriate to introduce here the transient heat conduction equation of Fourier. In modern notation, this parabolic partial differential equation may be written as,

$$\nabla \cdot K \nabla T = c \frac{\partial T}{\partial t}, \tag{1}$$

where K is thermal conductivity, T is temperature, c is specific heat capacity of the solid per unit volume, and t is time. The dependent variable T is a scalar potential, while thermal conductivity and specific heat capacity are empirical parameters. Physically, the equation expresses the conservation of heat per unit volume over an infinitesimally small volume lying in the interior of the flow domain. The exchange of heat with the external world is to be taken into account with the help of either temperature or thermal fluxes prescribed on the boundary. Also, it is assumed that the distribution of temperature over the domain is known at the initial time $t = 0$. For the particular case when the temperature over the flow domain does not change with time and K is independent of temperature, (1) reduces to Laplace's equation.

In (1), thermal conductivity K is physically a constant of proportionality, which relates the quantity of heat crossing a unit surface area in unit time to the spatial gradient of temperature perpendicular to the surface. This relationship is now known as Fourier's law. In his 1807 manuscript, Fourier formulated thermal conductivity mathematically rather than experimentally. As was pointed out by *Grattan-Guinness and Ravetz* [1972] and *Herivel* [1975], Fourier arrived at this concept gradually, as he was making the transition from discontinuous bodies to a continuous body. The notion of heat flux was yet a new concept, and Fourier would fully clarify it only in 1810, in a letter to an unknown correspondent [*Herivel*, 1975].

The concept of specific heat capacity, proposed experimentally by *Lavoisier and Laplace* [1783], is an essential part of the transient heat diffusion process. It helps convert the rate at which heat is accumulating in an elemental volume to an equivalent change in temperature. Thermal conductivity and thermal capacity are two different attributes of a solid, one governing transport in space and the other governing change in storage in the vicinity of a point. Together, these two parameters govern the ability of the solid to respond in time to forces that cause the thermal state of the solid to change. Sometimes, it is found mathematically convenient to combine the two parameters into a single parameter known as thermal diffusivity, $\eta = K/(\rho c)$, where ρ is density of the solid. The higher the diffusivity, the faster the tendency of the material to respond to externally imposed perturbations.

5. INFLUENCE AND CONNECTIONS

Soon after the publication of the analytic theory of heat in 1822, the general scientific community became aware of the significance of Fourier's work, not merely for the science of heat, but in general as a rational framework for conceptualization for other branches of science. Within a few years the heat conduction analogy was brought to the study of electricity, and later it was applied to the analysis of molecular diffusion in liquids and solids. The dynamical theory of gases directly led to the analogy between diffusion of gases and diffusion of heat. The investigation of the flow of blood through capillary veins and the flow of water through porous materials led to the adaption of Fourier's heat conduction model to the flow of fluids in geologic media. The study of random motions of particles led to the interpretation of Fourier's equation in terms of stochastic differential equations.

Simultaneously, Fourier's work began also to be recognized by the establishments of the intellectual world [*Grattan-Guinness and Ravetz*, 1972]. He was made a foreign member of the Royal Society in 1823, and in 1827 he was elected to the Académie Française and the Académie de Médicine. He succeeded Laplace as the president of the Council of Prefects of the Ecole Polytechnique. He also became the Secrétaire Perpétuel of the Académie des Sciences.

For the sake of completeness it may be mentioned here that Fourier's political career came to an end with the fall of Napoleon at Waterloo in 1815. His pension was refused and, close to 50 years old, he was virtually without an income. However, thanks to a former student of his at the Ecole Polytechnique in 1794 who was a prefect of the department of Seine, Fourier was given the directorship of the Bureau of Statistics in Paris. Later, in 1817, he was elected to a vacancy in physics in the Académie des Sciences. With these appointments, Fourier had a secure income for the rest of his life and he could find plenty of time for conducting research. During the 1820s Fourier also had an influential and distinguished following: Sturm, Navier, Sophie Germain, Dirichlet, and Liouville.

To gain an understanding of Fourier's influence over the past nearly 2 centuries, it is convenient to organize the discussions into the following general subheadings: electricity, molecular diffusion, flow in porous materials, and stochastic diffusion.

5.1. Electricity

The nature of electricity and its relation to magnetism were not completely understood at the time Fourier published his analytic theory, nor were the relations between electrostatics and electrodynamics (galvanic electricity). Quantities such as current strength and intensity were not precisely defined. At this time, Georg Simon Ohm of Germany (1787–1854) set himself the task of removing the ambiguities about galvanic electricity with mathematical rigor, supported by experimental data. He published four papers on galvanic current between 1825 and 1827, of which the most well-known is his 1827 pamphlet, *Die galvanische Kette, mathematisch Bearbeitet*. Ohm's work, which is considered to be one of the most important fundamental contributions to electricity, was largely inspired by Fourier's heat conduction model. *Ohm* [1827] started with three "laws." According to his first law, the communication of electricity from one particle takes place only directly to the particle next to it, so that no immediate transition from that particle to any other situated at a greater distance occurs. Recall that Fourier made this important idealization when making the transition from action at a distance to the continuous medium. The second law was that of Coulomb, relating to the effect of a charge at a distance in a dielectric medium. The third law was that when dissimilar bodies touch one another, they constantly maintain the same difference of potential at the surface of contact. This assumption is quite important because it points to a significant difference between the processes of heat conduction and conduction of electricity. In the case of heat conduction, temperature is continuous at material interfaces, whereas in the case of galvanic electricity the potential, namely, voltage, is discontinuous, as is implied by this assumption of Ohm.

Ohm's careful experiments showed that the current in a galvanic circuit did not vary with time (steady flow), the intensity of the electric current (measured with a torsion magnetometer) was directly proportional to the drop in "electrostatic force" (measured with an electroscope) along the conductor in the direction of flow and inversely proportional to the resistance of the conductor. In turn, the resistance of the conductor was a function of the material of which the conductor is made and of its form (resistance was found to be inversely proportional to the cross-sectional area of the conductor). Equally important, Ohm showed that the resistance of the conductor was independent of the magnitude of the current itself or the magnitude of the electrostatic force. Ohm gave a precise meaning to flux (current) and resistance. However, he erred in his use of electrostatic force, which he considered to be the quantity of electricity contained in an unit volume. Although electrostatic force so defined is an intensive quantity, it was left to *Kirchhoff* [1849] to establish that Ohm's law should properly be expressed in terms of potential (voltage) difference rather than difference in electrostatic force [*Archibald*, 1988]. Following Ohm's work, the measurement of the electrical resistance of various materials with great precision became a fundamental task in physics [*Maxwell*, 1881].

Ohm took the analogy with heat conduction farther and considered the flow of electricity to be exactly analogous to the flow of heat and wrote a transient equation of the form similar to (1),

$$\gamma \frac{du}{dt} = \chi \frac{d^2u}{dx^2} - \frac{bc}{\omega} u, \tag{2}$$

where γ is a quantity analogous to heat capacity, u is the electrostatic force, χ is electrical conductivity, b is a transfer coefficient associated with the atmosphere to which electricity is being lost by the conductor according to Coulomb's law, c is the circumference of the conductor, and ω is the area of cross section of the conductor along the x direction. (Unless otherwise stated, the notations used in this paper are those of the referenced authors.) Ohm was not confident about this equation and admitted that no experimental evidence for γ was as yet forthcoming.

James Clerk Maxwell (1831–1879) derived the same equation in a different context and showed that Ohm was in error in proposing (2) the way he did. *Maxwell* [1881] considered a long conducting wire (such as a transoceanic telegraph cable) surrounded by an insulator. In this case, the insulator, which is a dielectric material, functions as a condenser and possesses the electrical capacitance property analogous to heat capacitance. Moreover, if the insulator is not perfect, some amount of electricity would be lost to the surroundings, as is indicated by the second term on the right-hand side of (2). *Maxwell* [1881, p. 422] expressed Ohm's error thus: "Ohm, misled by the analogy between electricity and heat, entertained an opinion that a body when raised to a high potential becomes electrified throughout its substance, as if electricity were compressed into it, and was thus by means of an erroneous opinion led to employ the equations of Fourier to express the true laws of conduction of electricity through a long wire, long before the real reason of the appropriateness of these equations had been suspected." Indeed, it is fundamental to the nature of electricity that capacitance is an electrostatic phenomenon and only insulators possess that property. Electricity, as *Maxwell* [1881, p. 336] pointed out, behaves like an incompressible fluid, and hence conductors do not possess the property of capacitance.

It is interesting to note that Ohm formulated his flux law in terms of a difference in potential and a resistance, rather than in terms of the infinitesimal notion of a gradient as was done by Fourier. The resistance, in Ohm's law is an integral that combines the material property as well as the geometry of the conductor of finite size through which current is flowing. Fourier's method of separating material property from geometry was of the right mathematical form to pose the problem as a differential equation. In contrast, Ohm's approach of dealing with resistance and potential difference is more naturally suited for appreciating the diffusion problem directly in terms of integrals involving work, energy, and action.

Ohm's work is now accepted as one of the most important contributions in the science of electricity. Yet recognition did not come to him readily. Although physicists such as Fechner, Lenz, Weber, Gauss, and Jacobi drew upon Ohm's work in their own research soon after Ohm published *Die galvanische Kette*, Ohm's work came under criticism from an unexpected quarter. His experimental approach to finding order in nature was heavily criticized by Georg Poul [*Gillispie*, 1981], a physicist who was a follower of Hegel's philosophy of pure reason. However, due recognition came to Ohm after a few years when he was elected to the Academies at Berlin and Munich and the Royal Society conferred on him the Copley Medal in 1841.

William Thomson (1824–1907), also known as Lord Kelvin, was greatly influenced by Fourier's work. Thomson's first two articles, written at ages 16 and 17, were in defense of Fourier's mathematical approach. Later, he demonstrated the similarities between the mathematical structures of Fourier's heat conduction equation and the equations of electrostatics stemming from the works of Laplace and Poisson [*Thomson*, 1842]. For example, potential was analogous to temperature, a tube of induction was analogous to a tube of heat flow, the electromotive force was in the direction of the gradient of potential and the flux of heat was in the direction of temperature gradient.

While physical analogies serve a useful purpose, *Maxwell* [1888, pp. 52–53] emphasized that caution was in order to prevent the analogies from being carried too far. He pointed out that the analogy with electric phenomena applied only to the steady flow of heat. Even here, differences exist between electricity and heat. For steady flow, heat must be kept up by a continuous supply, accompanied by its continuous loss. However, in electrostatics a set of electrified bodies placed in a perfectly insulating medium might remain electrified forever without any supply from external sources. Moreover, there is nothing in the electrostatic system that can be described as flow. Note also that the temperature of a body cannot be altered without altering the physical state of the body, such as density, conductivity, or electrical properties. On the contrary, bodies may be strongly electrified without undergoing any physical change.

It is pertinent here to mention a major geological controversy of the nineteenth century in which Lord Kelvin and the heat conduction model played a part. Fourier himself had maintained on more than one occasion [*Herivel*, 1975] that the phenomenon of terrestrial heat motivated him to develop a theory for heat conduction in solids. On the basis of geological observations, contemporary geologists were of the opinion that the Earth was very old. For example, Charles Darwin had estimated that the age of the Earth was about 300 million years, based on assumed erosional rates of sediments. Kelvin analyzed the problem from a different basis, assuming that the Earth was initially a solid sphere at a high uniform temperature which gradually lost heat by conduction to reach the present state. Accordingly, he estimated that the Earth cannot be older than about 100 million years [*Hallam*, 1983]. Based on this he severely criticized Darwin and other geologists for grossly overestimating the age of the Earth. However, Kelvin's cooling Earth model was eventually invalidated by with the discovery of radiogenic heat in the Earth's crust. The radioactive heat source of the Earth enabled Earth scientists to extend the age of the Earth to many billion years during the early twentieth century.

Despite his erroneous estimate of the age of the Earth, Kelvin's conceptualization of global heat transport was very perceptive. It drew the attention of Earth scientists to the fact that the Earth is a heat engine and that observations of temperature and heat flow near the Earth's surface are essential for understanding the internal structure and the evolution of the Earth. Prior to the 1950s, however, few heat flow measurements were available from continental boreholes, and practically nothing was known about the natural loss of heat from the oceanic floors occupying over 70% of the Earth's surface. The prevalent untested view was that because the oceanic crust was known to have very low content of radioactive minerals (compared to the high content of the continental granitic rocks), heat loss from the floors of the oceans must be significantly smaller than that of the continental surface.

A major breakthrough in the field of terrestrial heat flow studies was the design and development of a probe to measure temperature gradients on the deep ocean floor by Edward Crisp Bullard (1907–1980) in the summer of 1949 [*Bullard et al.*, 1956]. This device produced the first set of heat flow data from the Pacific Ocean in 1952 and the Atlantic Ocean in 1954. Surprisingly, the data showed that the heat loss from beneath the oceans was comparable in magnitude to that from the continents, and it became necessary for Earth scientists to rethink their fundamental notions about the thermal structure of the Earth. These initial observations provided a major impetus to marine geophysicists to the measurement of heat flow beneath the deep oceans.

This was also the period during which the notion of plate tectonics was taking root in the Earth sciences on the basis of diverse field observations: magnetic reversals, young ages of oceanic crust, patterns of submarine seismicity and volcanism. Within a decade of the first heat flow measurements by Bullard and his coworkers, by the mid 1960s, several hundred heat flow measurements became available from the Atlantic, the Pacific and the Indian Oceans. The spatial patterns of heat flow on the ocean floors revealed by these measurements contributed greatly to the emerging notions of plate tectonics. Fundamental to the new view was the hypothesis that the oceanic crust is created at the mid-oceanic ridges by molten rock welling up from the Earth's interior and that upon cooling, the rigid oceanic plate spreads away from the ridges with time. Thus the rocks of the seafloor become older with distance away from the ridge.

To analyze this problem, *McKenzie* [1967] considered the oceanic crust to be a plate of finite thickness (about 50 km), in which conductive heat transport occurred in two dimensions. The bottom (550°C) and the top of the plate (0°C) were treated as constant temperature boundaries. The lateral spread of the plate away from the ridge at a finite velocity of a few centimeters per year was handled with the help of an advection (linear translation) term. Heat flow observations from the Atlantic, the Pacific and the Indian Oceans agreed reasonably well with the estimates based on the solutions of this advection-diffusion problem. The fact that the spatial heat flow patterns from the ocean floors were consistent with a spreading seafloor with vertical heat conduction through the rigid crust was an important corroborative factor in the establishment of plate tectonics as a viable theory.

We saw earlier that Ohm had attempted unsuccessfully to formulate a time-dependent electrical flow equation by direct analogy with Fourier's equation. Later work, stemming from Maxwell's equations, established that transient heat conduction and transient electricity flow are very different in nature. Transient flow of electricity typically arises in the case of alternating current as opposed to the steady state direct current with which Ohm was concerned. In the case of alternating current, the change in electric field is intrinsically coupled with an induced magnetic field in a direction perpendicular to the direction in which current is flowing. The nature of the coupled phenomena is such that when the frequency of the alternating current is low, Maxwell's electromagnetic equations may be described in the form of an equation which looks mathematically similar to the heat conduction equation, in that one side of the equation involves the Laplace operator (second derivative in space) and the other involves the first derivative in time. However, the resemblance is only superficial because the dependent variable in this equation is a vector potential, whereas the dependent variable in the heat conduction equation is a scalar potential.

5.2. Molecular Diffusion

Molecular diffusion is the process by which molecules of matter migrate within solids, liquids and gases. The phenomenon of diffusion was observationally known to chemists and biologists during the eighteenth century. In the early nineteenth century, experimental chemists began paying serious attention to molecular diffusion, and the publication of Fourier's work provided the chemists with a logical framework with which to interpret and extend their experimental work. The following discussion on molecular diffusion starts with diffusion in liquids, followed by solids and gases.

5.2.1. Diffusion in liquids. Among the earliest observations that attracted the attention of chemists to diffusion in liquids is the phenomenon of osmosis. In 1752, Jean Antoine (Abbé) Nollet (1700–1770) observed and reported selective movement of liquids across an animal bladder (semipermeable membrane). Between 1825 and 1827, Joachim Henri René Dutrochet (1776–1847) made pioneering contributions in the systematic study of osmosis. A physiologist and medical doctor by training, Dutrochet spent most of his career in the study of the physiology of animals and plants. About this time, Poisson had attempted to explain osmosis in terms of capillary theory. *Dutrochet* [1827] strongly disagreed with Poisson and, on the basis of experimental evidence, argued that two currents (solute and solvent) simultaneously occur in opposite directions during osmosis, one of them being stronger than the other, and that the understanding of osmosis required something more than a simple physical mechanism such as capillarity. He speculated on the possible role of electricity in the osmotic phenomenon. He also coined the terms "endosmosis" for the migration of the solvent toward the solution and the term "exosmosis" for the reverse process.

The next major work on liquid diffusion was that of Thomas Graham (1805–1869). *Graham* [1850] presented data on the diffusibility of a variety of solutes and solvents in his Bakerian Lecture of the Royal Society. Despite the wealth of data he collected, Graham did not attempt to elicit from them a unifying fundamental statement of the process of diffusion in liquids. That Graham restricted himself essentially to the collection of experimental data on diffusion in liquids proved to be a catalyst for one of the most influential papers of molecular diffusion, that of *Fick* [1855a, b]. Adolf Fick (1829–1901) was a Demonstrator in Anatomy at the University of Zurich and, in addition to his professional training in medicine, had a sound background in mathematics and physics. Fick expressed regret that Graham failed to identify any fundamental law of diffusion from his substantial experimental data. In seeking to remedy the situation, Fick saw a direct analogy between the diffusion of heat in solids and the diffusion of solutes in liquids.

By direct analogy with Fourier, *Fick* [1855b] wrote down the parabolic equation for transient diffusion of solutes in liquids in one dimension thus:

$$D\left(\frac{\delta^2 c}{\delta x^2} + \frac{1}{Q}\frac{dQ}{dx}\frac{\delta c}{\delta x}\right) = \frac{\delta c}{\delta t}, \tag{3}$$

where D is the diffusion coefficient, c is aqueous concentration, and Q is the area of cross section. Note that Fick made a novel departure from Fourier in writing the one-dimensional equation. The second term on the left-hand side of (3) accounts for the variation of the area of cross section along the flow path (the x axis). Intrinsically, Fick's equation is valid for a flow tube of arbitrary shape involving a curvilinear x axis. Indeed, *Fick* [1855a, b] presented data from a diffusion experiment in an inverted-funnel-shaped vessel, solved (3) for the geometric attributes of the particular cone-shaped vessel, and found that his mathematical solution compared favorably with the steady state concentrations at different locations within the vessel. For a flow tube with constant area of cross section, (3) simplifies to Fourier's equation. One can readily verify that (3) leads also to appropriate differential equations for radial and spherical coordinates. Upon reflection, it becomes apparent that integration of (3) along curvilinear flow tubes leads to the evaluation of resistances within finite segments of flow tubes and that the evaluation of resistances thus provides a link between the approaches of Fick and Ohm.

According to Fick, concentration is analogous to temperature, heat flux is analogous to solute flux and thermal diffusivity is analogous to chemical diffusivity. If concentration in the aqueous phase is defined as mass per unit volume, then specific chemical capacity (analogous to specific heat) equals unity and chemical diffusivity is equal to chemical conductivity.

In the second part of his paper, *Fick* [1855a, b] went on to analyze flow across a semipermeable membrane by idealizing it as a collection of cylindrical pores of radius ρ. As was suggested earlier by *Dutrochet* [1827], two simultaneous currents will occur through the capillaries; the solute current will occur toward the solvent and the solvent current will occur toward the solution. Fick reasoned that because of the affinity of water to the material composing the membrane, the water current will be organized more toward the walls of the pores and the solute will be organized toward the axis of the pores. Incidentally, a remarkably similar reasoning was employed by *Taylor* [1953], who studied solute diffusion in capillary tubes with moving water. When the radius of the pore becomes sufficiently small, the flow of the solute will be arrested and osmosis will involve one current, that of the solvent.

The study of liquid diffusion was soon to take a very important place in the field of biophysics through the investigations of Wilhelm Pfeffer (1845–1920). After receiving a doctoral degree in chemistry from the University of Göttingen when he was 20 years old, Pfeffer grew interested in the study of biological processes and brought his experimental and analytical skills to bear on the study of mass transfer in plant cells. Broadly, treating the outer layer of the cell as a semipermeable membrane, Pfeffer devised sophisticated techniques to measure osmotic pressure within cells and went on to develop and test several hypotheses concerning the diffusion of nutrients within and across cells.

Pfeffer found osmosis experiments on plant cells to be limiting and sought to conduct measurements on controlled inorganic membranes. Along these lines he pioneered the use of thin layers of ferrocyanide deposited on ceramic substrates as semipermeable membranes. Using such membranes he measured osmotic pressure of various solutions as a function of concentration as well as temperature. Pfeffer's data, published in his 1877 classic, *Osmotische Untersuchungen*, would later help van't Hoff to lend credibility to his theory of osmotic pressure. Dutrochet, Pfeffer, Fick, and other biophysicists of the time strongly supported the view that physiological processes must be elucidated and understood in terms of inorganic (nonbiological) processes.

By the time Pfeffer published his book on cell mechanics, a wealth of data had been collected on osmosis, both from physiological and inorganic materials. Many hypotheses were in vogue, and a rational description of osmosis in terms of known principles of physics and chemistry was lacking. Jacobus Hendricus van't Hoff (1852–1911), an influential physical chemist of the second half of the nineteenth century, filled this gap by providing a theoretical foundation for osmotic pressure based on well-established laws of chemistry. *Van't Hoff* [1887] started with and justified the proposition that the physical behavior of solutions and the associated osmotic pressure can be rationally understood by treating solutions as analogous to gases and by applying Boyle's law, Gay-Lussac's law, and Avogadro's law to solutions. He formally defined osmotic pressure as the excess pressure that would develop in a solution contained in a vessel that communicates with a reservoir of a solvent across a perfect semipermeable membrane. By using the aforesaid laws and the second law of thermodynamics, van't Hoff was able to draw many inferences about relationships between the magnitude of osmotic pressure on the one hand and the nature of the solute concentration and temperature on the other. He demonstrated that the experimental data of previous workers, especially *Pfeffer* [1877], substantially justified his theoretical framework.

In osmosis, two opposing currents of flow are involved, each being driven by its own force: the solvent by spatial variations in its fluid potential, and the solute by the spatial variations of osmotic pressure. Therefore it is convenient to conceptualize the total pressure in the solution as a sum of the water phase pressure and the osmotic pressure. Thus in the solution, the pressure in the water phase $p_w = p_{total} - p_{osmotic}$. The stronger the concentration of the solution, the lesser the water phase pressure and the stronger will be the solvent current toward the solution should the solution communicate with the solvent. Analogously, the solute will be driven in the opposite direction because osmotic pressure decreases in the direction of the solvent.

Closely following van't Hoff, Walther Hermann Nernst (1864–1941) examined the process of solute diffusion in the context of osmotic pressure as defined by the former. *Nernst* [1888] pointed out that the diffusion of solutes in the direction of decreasing concentration had been suggested earlier by *Berthollet* [1803] and that Fick established it rigorously with mathematics, supported by experimental data. Nernst found Fick's approach to be formal and lacking in the elucidation of the forces that impelled the solute diffusion process. To overcome this deficiency, he looked at diffusion in terms of impelling forces and resistive forces, the former stemming from spatial variations of osmotic pressure and the latter stemming from the collision of molecules with the solvent molecules and even among the solute molecules themselves.

Nernst [1888] considered the force due to osmotic pressure acting on a molecule of the solute and defined a coefficient of resistance K representing the force required to move 1 gram-molecule of the solute through the solvent at a velocity of 1 cm s^{-1}. Combining these, he expressed the flux of solute in terms of the gradient of the osmotic pressure and the reciprocal of the coefficient K. He then recognized that for dilute solutions, osmotic pressure is linearly related to concentration by a simple relation, $p = p_0 c$, where p_0 is the osmotic pressure in a solution containing a gram molecular weight of the solute and c is concentration. As a result, for dilute solutions, the ratio p_0/K becomes part of the diffusion coefficient and flux becomes proportional to the gradient of concentration, as was proposed by *Fick* [1855a, b]. By extending the analysis to concentrated solutions, Nernst pointed out that in such solutions the solute will encounter greater resistance to flow because of mutual collision among the solute molecules in addition to the solvent molecules. Therefore in concentrated solutions the diffusion coefficient will be a function of concentration. As a consequence, the relevant differential equation of diffusion becomes nonlinear.

For electrolytes in which individual ions will migrate separately, Nernst suggested that in order that the ions composing a given solute may migrate at the same velocity, the differences in ion velocities induced by osmotic pressure will be compensated by electrostatic forces.

5.2.2. Solid diffusion. An early documented observation of solid diffusion is attributed to Robert Boyle (1627–1691) [*Barr*, 1997], who succeeded in 1684 in making zinc diffuse into one of the faces of a copper farthing, leading to the formation of brass. By carefully filing the face, Boyle showed that zinc had indeed diffused into the body of copper. Yet controlled diffusion measurements would not become possible until 200 years later.

The first measurements of the diffusion of one solid metal into another was made by William Roberts-Austen (1843–1902), who was Chemist and Assayer of the British Mint. He took up the challenge of extending Graham's work on liquid diffusion to metals. His progress was considerably hampered by difficulties in accurately measuring the temperature at which diffusion was taking place in the solid state. By adopting Le Chatelier's platinum-based thermocouples, he succeeded in studying the diffusion of gold in solid lead at different temperatures. The results were analyzed in terms of Fourier's model of one-dimensional diffusion [*Roberts-Austen*, 1896].

Solid diffusion is a process of great importance in many natural and industrial processes. In the lithosphere of the Earth it influences the genesis of minerals, ores, and rocks. As an illustration, we consider the role of solid diffusion in metamorphism, one of three major rock-forming processes. Metamorphism constitutes the physical, chemical, and structural adjustments undergone by solid rocks in response to temperature, pressure, and other environmental changes consequent to burial at depths within the Earth's crust. Metasomatism is a metamorphic process by which a new mineral may grow in the body of an old mineral or mineral aggregate, often occurring at constant volume, with little disturbance of original texture or structure of the mineral (palimpsest textures).

Many geologists of the first half of this century, especially from Europe, were persuaded by field evidence to believe that solid diffusion must have been responsible for the genesis of large bodies of granite and granite-looking rocks, which were previously thought to have solidified from a molten magma. Although their inference was justified qualitatively, the magnitude of the role of solid diffusion in the genesis of large rock masses could not be gauged reliably. This was due to a lack of instruments to observe the effects of diffusion on the microscopic scale at which solid diffusion occurs within mineral grains. Fortunately, the 1950s saw rapid developments in the field of X-ray spectrometry, leading to the development of the "electron-microprobe." This remarkable instrument enabled quantitative chemical analysis in situ within a mineral grain, in the immediate vicinity of a location. By the mid-1960s, mineralogists were able to quantitatively study variations of chemical composition within a single mineral grain at a spatial resolutions of ≤ 1 μm. New possibilities opened up in the application of the diffusion equation to address fundamental questions related to the age of rocks and their thermal history.

Over the past 3 decades, diffusion-based mathematical models have been used to decipher the history of certain potassium-rich minerals such as feldspars using the decay of radioactive ^{40}K to ^{40}Ar. Similarly, techniques have also been developed to analyze the decay of uranium in naturally occurring zircon to lead. In both cases the assumption is that the radioactive clock starts to count when the mineral of interest (feldspar or zircon) begins to crystallize from a melt at a certain critical temperature. Once the crystal has formed, the decay process will release the daughter product, as dictated by the appropriate half-life. For different chemical reasons, the daughter products will tend to be excluded from the lattice structure and diffuse out of the crystal, at a rate that depends on the geometry of the crystal (sheet, sphere, cylinder), the diffusion coefficient, and the thermal environment. In minerals, diffusion coefficient is an exponential function of temperature (the Arrhenius relationship), and diffusion will become practically negligible below some threshold temperature. Thus solid diffusion in these minerals is constrained to a temperature window. Consequently, the measurement of diffusion profiles within a single mineral grain may provide valuable information about the thermal history of the mineral and its environment, that is, whether it cooled rapidly or very slowly. In quantitatively interpreting observed diffusion profiles, it is customary to use solutions to Fourier's heat diffusion equation for assumed simple geometries of the mineral grains (sheet, sphere, cylinder), initial conditions and boundary conditions.

Although looking at minerals at extremely fine spatial resolution is of great value, rocks and their origin have to be understood on a much larger field scale, on the scale of rock masses. On a large scale, one may wish to look at metasomatism as a macroscopic solid diffusion process. Intuitively, this may be reasonable because the porosity of rocks is extremely small, comprising microfissures and grain boundaries. However, such a macroscale idealization may lead to quantitatively inaccurate inferences because the fluids in the micropores and grain boundaries may have diffusivities that are many orders of magnitude larger than the corresponding diffusivities of the solid minerals. Thus, in these larger bodies, diffusion over long distances may be dictated by the fluids in the fine pores, and solid diffusion may occur locally, within the mineral grains on the small scale. This entails the application of Fourier's diffusion equation to systems with interpenetrating continua (e.g., fractures on the one hand, and the solid on the other) with radically different diffusion properties.

We may end this discussion on solid diffusion with the work of Enrico Fermi (1901–1954) on neutron diffusion. Fermi was the first to successfully achieve, in 1942, a sustained release of energy from a source other than the Sun by bombarding and splitting uranium atoms with the help of neutrons slowed down in a matrix of solid graphite. Critical to the design of the experiment was the calculation of the slowing down of neutrons and the absorption of thermal neutrons by the carbon host. The slowing down of the neutrons was described as a diffusion process [*Anderson and Fermi*, 1940], and the corresponding diffusion constants were calculated on the basis of experimental data. The approach was one Fermi had already perfected earlier [*Fermi*, 1936]. The diffusion theory developed by Fermi would later be known as the "age theory."

5.2.3. Diffusion of gases. The earliest experimental work on the diffusion of gases was by *Graham* [1833]. When two or more gases are mixed together in a closed vessel, the natural tendency is for the gases to redistribute themselves by diffusion in such a way that the mixture has a uniform composition everywhere. Graham showed experimentally that the rate at which each of the gases diffuses is inversely proportional to the square root of its density. This observation is known as Graham's law. When we compare gas diffusion with liquid diffusion or the conduction of heat or electricity, we find that in these latter cases we are concerned with conductive transport of the permeant in different host materials, whereas in this case of gas diffusion we are concerned with the conduction of gas in free space. In the case of nongaseous conduction, the transport coefficient (conductivity or diffusivity) is experimentally estimated for different materials on the basis of Fourier's law. Thus conductivity is a property of the material rather than the permeant. In contrast, in the case of pure gaseous diffusion, diffusivity is a property that stems solely from the attributes of the permeating fluid, the gas.

With the advances that were taking place in molecular physics and chemistry during the middle of the nineteenth century, a great deal of effort was made by researchers to directly estimate the properties of gases such as viscosity, specific heat, thermal conductivity, diffusion coefficient, and diffusivity by starting with force, momentum, and energy at the molecular level and statistically integrating these quantities in space and time to estimate the macroscopic properties of interest. Among the earliest researchers in this regard was Maxwell whose work on the dynamical theory of gases is of fundamental importance. *Maxwell* [1867] assumed molecules to be small bodies or groups of small bodies which possess forces of mutual repulsion varying inversely as the fifth power of distance. Macroscopically, he described the diffusion of a mixture containing two gases in terms of an equation with the same form as Fourier's transient heat conduction equation. In this case, the diffusion coefficient is describable by Dalton's law of partial pressures and densities of the two gases and is inversely proportional to the total pressure. Maxwell generated a solution of this equation for the case of a particular column experiment conducted by Graham involving carbonic acid and air and found some agreement with the diffusion coefficient independently estimated by Graham.

5.3. Flow of Water in Porous Materials

Fourier's heat conduction equation has had an enormous influence in the study of fluid flow processes in the Earth, especially water and petroleum in porous media.

In applying the equation to these processes, the following analogies can be made: temperature corresponds to scalar fluid potential, heat corresponds to mass of fluid, thermal conductivity corresponds to hydraulic conductivity and, heat capacity corresponds to hydraulic capacity. However, unlike electricity, heat, and solutes, the potential of water has a very special attribute, namely, gravity. This attribute renders the extension of heat analogy to the Earth sciences particularly interesting.

5.3.1. Steady flow of water. By the time of Fourier's work, fluid mechanics was a well-developed science, and the concept of a fluid potential, defined as energy per unit mass of water was already established through the seminal contribution of Bernoulli in hydrodynamics, early in the eighteenth century. The flow of water in open channels was being rigorously studied by civil engineers. In addition to civil engineers, many physiologists were also interested in the study of water flow through capillary tubes to better understand, by analogy, the flow of blood through narrow vessels.

Among the earliest experimentalists to study the slow motion of water through capillary tubes was Jean Léonard Marie Poiseuille (1799–1869), a physician and physiologist. Not satisfied with the contemporary understanding of blood circulation in veins, he embarked on a study of the flow of water in narrow capillary tubes under carefully controlled conditions. Using a sophisticated laboratory setup, Poiseuille studied the flow of water in horizontal capillary tubes varying in diameter from ~50 μm to ~150 μm and measured fluxes as low as 0.1 cm^3 over several hours. In the absence of gravity, he found that water flux was directly proportional to the pressure difference between the inlet and the outlet and inversely proportional to the length of the capillary. These observations were very similar to those made by Ohm in the case of galvanic current. Although the work was completed in 1842, it was not published until a few years later [*Poiseuille*, 1846]. Similar observations had been made earlier in Germany by *Hagen* [1839].

One of the most influential works on the flow of water in porous media during the nineteenth century was that of *Darcy* [1856]. Henry Darcy (1803–1858) was a highly recognized civil engineer who is credited with designing and completing, in 1840, the first ever protected town water supply system in the world. Dissatisfied with the unhealthy sources of drinking water in his native town of Dijon, he helped bring and distribute water from a perennial spring located several kilometers away from the town. Later, presumably to build a water purification system, Darcy conducted a series of experiments in vertical sand columns to develop a quantitative relationship for estimating the rate of flow of water through sand filters. Darcy's experiment was novel in that it included gravity and involved a natural material (sand) rather than an engineered material such as a capillary tube. He too, like Ohm and Poiseuille before him, found that the flux through the column was directly proportional to the drop in potentiometric head, $h = z + \psi$, where z is elevation with reference to datum and ψ is pressure head, directly proportional to the area of cross section and inversely proportional to the length of the column. Darcy's law plays a fundamental role in many branches of Earth sciences such as hydrogeology, geophysics, petroleum engineering, soil science, and geotechnical engineering.

During the middle of the nineteenth century, potential theory was recognized as providing a useful conceptual-mathematical basis for understanding artesian wells and other manifestations of deep groundwater circulation in France and elsewhere in Europe.

Soon the heat conduction model of Fourier began to be used for analyzing circulation of water in groundwater basins. The earliest studies in this regard restricted themselves to the steady motion of groundwater. Unlike the problems of electricity and molecular diffusion, the problems of groundwater involved large spatial scales (many tens or even hundreds of kilometers laterally and hundreds of meters vertically). Two of the most distinguished engineers of this era were Jules-Juvenal Dupuit (1804–1866) in France and Philipp Forchheimer (1852–1933) in Austria. *Dupuit* [1863] developed the basic theoretical framework for analysis of flow in groundwater systems and of the flow of water to wells. *Forchheimer* [1886] formally stated the steady seepage of water in terms of the Laplace equation and initiated the use of complex variable theory to the solution of two-dimensional problems in flow domains of complicated geometry that occur in the vicinity of dams and other engineering structures. He also pioneered the use of flow nets as practical, graphical means of solving seepage problems in complex flow domains.

5.3.2. Flow of multiple fluid phases. A significant development in the study of flow in porous media was the work of Edgar Buckingham (1867–1940). From 1902 to 1905 he was an Assistant Physicist with the Bureau of Soils, U.S. Department of Agriculture. In this brief period he not only introduced himself to a totally new field, the science of soils, but also made one of the most important contributions to soil physics in particular and the study of multiphase fluid flow in general. Soon after, he moved to the newly formed National Bureau of Standards (now National Institute of Standards and Technology) and became well known for, among other achievements, his π theorem of dimensional analysis.

In soils close to the land surface, where plant roots thrive, both water and air coexist. An understanding of the dynamics of the occurrence and movement of moisture in the soil is critical to judicious agricultural management. When water and air coexist in soils, the contacts between air and water in the minute pores are curved menisci in which energy is stored. As a result, the pressure in the water phase is less than that in the air phase and the difference is the capillary pressure. The physics and the mathematics of capillarity had been enunciated a hundred years earlier by Laplace and by Thomas Young (1773–1829). *Buckingham* [1907] brought together the work on capillary pressure with that of Fourier and Ohm on diffusion, and defined the capillary potential in the water phase as a sum of work to be done per unit mass against gravity and fluid pressure. He stated that moisture moves in soils in response to spatial variations in potential and that moisture flux density is directly proportional to the gradient of capillary potential, the proportionality constant being hydraulic conductivity. Although this statement resembles the laws of Fourier, Ohm, and Darcy, there exists a very important difference. In soils that contain water and air, the capillary potential is directly related to water saturation, and as water saturation decreases, the flow paths available for moisture movement decrease. Therefore the conductivity parameter is strongly dependent on capillary potential, instead of being constant or nearly so as is the case with the laws of Fourier, Ohm, or Darcy. The strong dependence of hydraulic conductivity on capillary potential renders the study of moisture diffusion in soils a very difficult mathematical problem. In an earlier work, *Buckingham* [1904] also applied the diffusion equation to the migration of gas in soils and analyzed the dynamic vertical migration of air from the land surface to the water table in response to fluctuating atmospheric pressure. Buckingham's work helped resolve a contemporary paradox in agriculture. In arid regions, where evaporation rates are very high, the soils are found to be wetter and hold their moisture for much longer periods than do the soils of humid areas in dry seasons. Part of the reason for this counterintuitive observation is to be found in the dependence of hydraulic conductivity on capillary potential, or, equivalently, water saturation. In arid areas, as evaporation rapidly desaturates the uppermost soil, the hydraulic conductivity drops practically to zero and further evaporative loss from deeper zones is virtually eliminated.

Note that specific heat, originally defined and measured by *Lavoisier and Laplace* [1783], is an extremely important physical attribute of materials and occupies an important position in the transient heat conduction equation. It influences the rapidity with which a material will respond thermally to externally imposed perturbations: the smaller the capacitance, the faster the response. Analogously, in the phenomenon of fluid flow in porous media, hydraulic capacitance plays a very important role. Indeed, the slope of the variation of water saturation as a function of capillary potential contributes to the hydraulic capacitance of a soil. As a consequence, Buckingham's work lies at the foundation of the dynamics of multiple fluid phases in porous media.

Although Buckingham defined a capillary potential theoretically, he could measure it only indirectly in vertical columns in which water moves down solely by gravity. He recognized that new instruments would have to be developed to measure capillary potential under dynamic conditions of flow. Such an instrument was invented over a decade later by Willard Gardner (1883–1964). This ingenious instrument is called the tensiometer. The key component of this device is a porous ceramic cup that is completely saturated with water. Such a porous cup acts like a semipermeable membrane, allowing the flow of water from the soil into the cup, but not allowing the flow of air. The cup is connected to a long, water-filled tube, which is connected to a manometer. The tensiometer is set into a natural soil, and through exchange of water between the soil and the cup, fluid pressure inside of the cup is allowed to attain equilibrium with that in the soil. The equilibrium pressure represents the capillary potential. The first measurements from this instrument were reported by *Gardner et al.* [1922]. This strong dependence capacitance on capillary potential introduces a strong nonlinearity into the differential equation of moisture movement in soils.

5.3.3. Hydraulic capacitance. The attribute of hydraulic capacitance of a naturally occurring porous material such as a soil or a rock arises also for reasons other than the rate of change of saturation with potential. Earth materials are deformable in response to changes in the stresses which act on the porous skeleton. The ensuing rate of change of pore volume (which is occupied by water) in response to changes in fluid potential also contributes to hydraulic capacitance. The measurement of pore volume as a function of fluid potential was elucidated through the work of Karl Terzaghi (1883–1963), who founded the discipline of soil mechanics. In presenting his experimental results on the deformation of water saturated clays, *Terzaghi* [1925] postulated that in water-saturated earth materials, change in pore volume is to be related to the difference between skeletal stresses and water pressure. Thus when skeletal stresses remain unchanged, volume change is directly attributable to change in fluid pressure or, equivalently, the fluid potential. Extensive experimental work following Terzaghi has shown that earth materials invariably exhibit nonelastic deformation behavior.

In addition, a third component also contributes to hydraulic capacitance of porous materials, the compressibility of the fluid itself. Thus hydraulic capacitance in natural geological materials arises from the ability of the porous medium to deform as a result of changing fluid pressure, the ability of the fluid to dilate, and the desaturation of the pores due to changing capillary pressure. Whereas the specific heat of most known materials does not vary by more than a factor of a hundred it is not uncommon for the hydraulic capacitance of soils and rocks to vary by a factor of a million.

If we now look at Fourier's diffusion equation as the basis for the flow of water in soils and rocks, we see at once that hydraulic conductivity may vary significantly as a function of fluid potential, as does the hydraulic capacitance. Thus, unlike the heat problem, which is generally characterized by a linear differential equation, the diffusion equation pertaining to flow of water in soils and rocks is characterized by a highly nonlinear differential equation. Drawing upon the contribution of Buckingham and Gardner, Lorenzo Richards (1904–1993) presented the transient capillary conduction of water in porous media [*Richards*, 1931],

$$\nabla \cdot K(\psi)\nabla(\phi + \psi) = \rho_s A \frac{\partial \psi}{\partial t}, \qquad (4)$$

where $\phi = gz$ is the potential in the gravity field (in which g is gravitational acceleration and z is elevation above datum), $\psi = \int dp/\rho$ is capillary potential (where p is pressure in the water phase and ρ is density of water), ρ_s is dry bulk density of the soil, and A is the rate of change of moisture content with respect to capillary potential, referred to as the capillary capacity of the medium.

5.3.4. Hydrodynamic dispersion. Geoffrey Taylor (1886–1975) who studied the advective transfer (transport by the bulk movement of water) of dissolved solutes by water in thin capillary tubes made an interesting conceptual-mathematical addition to Fourier's diffusion equation. In a tube through which water is flowing, the velocity of water is practically zero at the walls and is at a maximum along the central axis of the tube. Thus although we may be satisfied with an average velocity to quantify the total flux of water in the tube, we cannot ignore the velocity variation within the tube if we wish to understand the migration of a solute dissolved in water. The process is complicated by molecular diffusion which will cause the solute to spread perpendicular to the direction of advective transport. By an elegant mathematical analysis of the problem, *Taylor* [1953] showed that after a sufficient period of time the distribution of the solute will exhibit a diffusion-like profile along the direction of flow and that the effective diffusion coefficient is a function of the average velocity as well as the geometrical attributes of the capillary tube. Recall that *Fick* [1855a, b] considered, in a similar fashion, the variation of concentration as a function of capillary radius in osmotic membranes.

Taylor's work inspired the concept of hydrodynamic dispersion, widely used to analyze the migration of contaminants in groundwater systems. Hydrodynamic dispersion is a macroscopic diffusion-like process by which contaminants dissolved in water in a porous medium spread (principally along the general flow direction) owing to random variations in flow velocity on a microscopic scale.

In considering the migration of solutes with moving water in porous materials, it is important to focus on another attribute of these systems that has direct relevance to the heat diffusion equation. Many solutes that occur in groundwater also have affinities for the solid surface. Hence they tend to partition themselves between the aqueous phase and the solid surface by a process of adsorption. Adsorption, in turn, is proportional to concentration in the aqueous phase. When one writes the molecular diffusion equation only for the aqueous phase in such systems, the portion of the solute taken up by the solid surface is accommodated in the form of a chemical capacitance term, usually referred to as the retardation factor because this process effectively slows down spreading caused by the advective process. In the mathematics of diffusion, the retardation factor plays the same role as specific heat in the heat diffusion equation.

5.4. Stochastic Diffusion

5.4.1. Random walk. It is evident from the foregoing discussion that the analyses of the flow of electric current; diffusion in liquids, solids, and gases; and the flow of fluids in porous materials were all directly influenced by Fourier's heat conduction model during the nineteenth century. In these analyses, Fourier's model was used in an empirical way, to interpret experimental data from macroscopic systems. In marked contrast to this empirical transfer of concept, the second half of the nineteenth century saw the extension of the heat diffusion equation to problems of a more theoretical nature, involving the gross manifestation of random processes. During the early twentieth century this extension would lead to the development of a new field of considerable interest, namely, stochastic processes. The beginnings of stochastic diffusion were latent in the work of four quite different workers: the theory of sound of Lord Rayleigh, the law of error of Edgeworth, the theory of speculation of Bachelier, and the theory of Brownian motion of Einstein. The appellation "random walk" to denote these processes was coined by Karl Pearson (1857–1936), a biometrician. The transition from random walk to stochastic differential equations would be catalyzed by the work of Langevin. It is now of interest to consider in some detail, the conceptual-mathematical developments that led to the theoretical view of the heat conduction equation as an ensemble manifestation of fine-scale random processes.

Pearson [1905, p. 242] sought the help of the journal *Nature* with the request, "Can any one of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in this matter. A man starts from a point O and walks L yards in a straight line; he then turns through any angle whatever and walks another L yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O." Within a couple of weeks there were several responses to his request. One of them provided a solution in the form of elliptic integrals for $n = 3$. In another, Lord Rayleigh (John W. Strutt, 1842–1919) brought to Pearson's attention his own work [*Rayleigh*, 1894] on a problem in sound, which gave a simple solution for very large n. Pearson thanked the correspondents and stated [*Pearson*, 1905, p. 342]. "I ought to have known it, but my reading of late years has drifted into other channels, and one does not expect to find the first stage in a biometric problem provided in a memoir on sound." He ended his response by stating, "The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!" Pearson was probably the earliest to formally state the problem of random walk or random flight. By analogy, however, the representation of the random walk problem in the form of a diffusion equation for very large n goes back to Lord Rayleigh.

5.4.2. Theory of sound. *Rayleigh* [1880] addressed the problem of estimating the amplitude and intensity of the resultant of the mixing of n vibrations of the same period and amplitude but of arbitrarily chosen phase. He solved this problem using Bernoulli's theorem on probability and obtained an expression in terms of the exponential of $-x^2/2n$ for the probability that the resultant amplitude would be between x and $x + \delta x$ after a large number of trials. Recognizing the similarity of this solution to that of Fourier's heat equation, *Rayleigh* [1894] solved the same problem by a different method, obtained the same results, and showed that the resultant of random mixing satisfies Fourier's heat conduction equation on an average after a large number of trials.

Rayleigh started with a simple case in which only two phases were possible, positive and negative. In this instance, if all the n cases had the same phase, the resulting intensity would be n^2, but if half of them had one phase and half had the other phase, the resultant intensity will be 0. Rayleigh investigated the question: What is the expectation that the amplitude will be between x and $x + \delta x$, given n is large? Here, "expectation" denotes the mean value that can be expected from a large number N of such experiments, with the number of waves mixed in each case being n. Let $f(n, x)$ denote the number of combinations in which the resultant amplitude is x. Suppose the number of waves mixed is increased to $n + 1$. What is the number of combinations in which the resultant is x? If the phase is restricted to $+1$ or -1, the number of combinations that can have a value x after mixing $n + 1$ will depend on $f(n, x - 1)$ and $f(n, x + 1)$. In fact, if the choice is purely random, we must have,

$$f(n + 1, x) = \tfrac{1}{2}f(n, x - 1) + \tfrac{1}{2}f(n, x + 1). \qquad (5)$$

By subtracting $f(n, x)$ from both sides of (5),

$$f(n + 1, x) - f(n, x) = \tfrac{1}{2}f(n, x - 1) + \tfrac{1}{2}f(n, x + 1) - f(n, x). \qquad (6)$$

Note that (6) is the classical finite difference form of the diffusion equation. Thus for large n, (6) reduces to,

$$\frac{df}{dn} = \frac{1}{2}\frac{d^2f}{dx^2}. \qquad (7)$$

Subject to the condition that $f(0, 0) = 0$, the solution to this is the probability density function,

$$f(n, x) = \frac{1}{\sqrt{(2\pi n)}} \exp\left(-\frac{x^2}{2n}\right). \qquad (8)$$

The arithmetic mean of the intensity of a large number of trials is given by,

$$\frac{1}{\sqrt{2\pi n}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2n}\right) dx = n. \qquad (9)$$

Rayleigh now relaxed the assumption of allowing only two phases, positive and negative, and let the phases take all values from 0 to 2π. By using suitable transformations between polar coordinates and cartesian coordinates, he then showed that the appropriate equation takes on the two-dimensional form of Fourier's equation,

$$\frac{df}{dn} = \frac{1}{4}\left[\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2}\right]. \qquad (10)$$

5.4.3. Law of error. Francis Ysidro Edgeworth (1845–1926) was a statistician who played a major role in the development of mathematical economics by incorporating probability and statistics into the analysis of social economic data [*Stigler*, 1978]. *Edgeworth* [1883] derived the differential equation governing the behavior of compound error, which he termed the law of error. He started by assuming that compound error is a linear function of indefinitely numerous elements, each element being sampled from facility function $f(z)$ assumed to be symmetric. Let $u_{x,s}$ be the function describing the compound error where x is the extent of the error and s is the number of elements. Given this definition of u, Edgeworth expressed $u_{x,s+1}$ in terms of $u_{x,s}$ as,

$$u_{x,s+1} = \int_{-\infty}^{\infty} f(z)u_{x+z,s}\, dz. \qquad (11)$$

The left-hand side of (11) may be expressed as, $u + du/ds$. The right hand side of (11) may be expanded into a Taylor series. Therefore (11) leads to

$$u_{x,s+1} = u_{x,s} \int_{-\infty}^{\infty} f(z)\, dz + \frac{du_{x,s}}{dx} \int_{-\infty}^{\infty} zf(z)\, dz + \frac{1}{2}\frac{d^2u_{x,s}}{dx^2} \int_{-\infty}^{\infty} z^2f(z)\, dz. \qquad (12)$$

Noting that the middle term on the right-hand side of (12) is equal to zero, (11) leads to

$$\frac{du}{ds} = \frac{c^2}{4}\frac{d_2u}{dx^2}, \qquad (13)$$

where $c^2 = 2 \int z^2 f(z)\, dz$. Equation (13) is referred to as the law of error and provides an approximate asymptotic solution to the recursive relation (11). Note that the time dimension of Fourier's equation is replaced by the number of samples in the equations of Rayleigh (equation (7)) and Edgeworth (equation (13)).

5.4.4. Theory of speculation. At the turn of the twentieth century, Louis Bachelier (1870–1946) showed that probable values of stock option prices could be described in terms of a diffusion equation by making some assumptions about the randomness of stock prices. A student of Henri Poincaré at the Sorbonne, *Bachelier* [1900] presented a dissertation entitled "*Theory of Speculation*" in which he applied principles of probability to economic problems of stock option pricing. Pursuing an approach very similar to that of *Rayleigh* [1894] involving discrete difference equations, he introduced the notion of "radiation of probability," which is conceptually analogous to Fourier's law of heat conduction. Thus *Bachelier* (p. 39 of 1964 translation) states, "Each price x during an element of time radiates towards its neighboring price an amount of probability proportional to the difference of their probabilities." Accordingly, the probability p of price x at moment t is given as,

$$p = -d\mathcal{P}/dx, \tag{14}$$

where $\mathcal{P}$ is the probability that the price exceeds x. Using this to evaluate the "probability exchanged through x during the period Δt," Bachelier wrote down the diffusion equation,

$$c^2 \frac{\partial \mathcal{P}}{\partial t} - \frac{\partial^2 \mathcal{P}}{\partial x^2} = 0. \tag{15}$$

Bachelier's work remained unnoticed for over half a century, until its relevance to warrant pricing was demonstrated by *Samuelson* [1965]. Since then it has been very influential in the development of econometric models based on stochastic calculus, which are widely used to rationally price stock options and corporate assets subjected to highly volatile stock market conditions.

5.4.5. Brownian motion. The fourth paper of historical interest that inspired the development of stochastic differential equations was that on Brownian motion by Albert Einstein (1879–1955). During the 1820s, Robert Brown (1773–1858), a renowned British botanist, discovered that pollen and other fine particles suspended in water exhibited continuous and permanent random motions. It was soon recognized by physicists that the random motions were sustained by the impacts of the molecules of the liquid on the suspended particles.

Unlike Rayleigh, Edgeworth, and Bachelier, who devoted attention to the mathematics of randomness, *Einstein* [1905] saw in Brownian motion an opportunity to test the validity of the molecular-kinetic theory of heat. He started with the proposition that colloidal particles suspended in liquids exert osmotic pressure, just as dissolved solute molecules do, and that an equal number of suspended colloidal particles and nonelectrolyte solute molecules exert the same osmotic pressure in dilute solutions. Such osmotic pressure arises from the random motion of the particles as they are impelled by their random collisions with the vibrating liquid molecules. The kinetic energy transferred in the process is directly related to the temperature of the liquid (analogous to Nernst's idealization of osmotic pressure in solutes). In their random movement the particles are decelerated by the viscous resistive forces of the liquid. By analogy with *van't Hoff*'s [1887] equation for osmotic pressure for nonelectrolytic solutes, the osmotic pressure associated with suspended particles is given by

$$p = \frac{RT}{N} \nu, \tag{16}$$

where p is the osmotic pressure, R is the universal gas constant, T is temperature, N is Avogadro's number, and ν is the number of suspended particles per unit volume of the liquid. Combining (16) with Stokes' law for viscous resistance, Einstein derived a macroscopic flux law (analogous to Fourier's law for thermal conduction) for the flux of particles crossing a unit area in unit time,

$$D \frac{\partial \nu}{\partial x} = \frac{\nu K}{6\pi k P}, \tag{17}$$

where D, the diffusion coefficient, is given by $D = RT/(6\pi k P N)$, in which k is coefficient of viscosity and P is radius of the particle.

Following a procedure similar to that of *Edgeworth* [1883], Einstein then proceeded to derive the partial differential equation for the distribution of particles at time $t + \tau$, given that the distribution at time t is $\nu = f(x, t)$. During a time interval τ there exists a finite probability $\phi(\Delta)d\Delta$ that the x coordinate of a single particle will change by an amount Δ. This leads to the recursive relation similar to (11),

$$f(x, t + \tau) = \int_{-\infty}^{\infty} f(x + \Delta, t)\phi(\Delta)\, d\Delta. \tag{18}$$

Upon using Taylor series expansion, this leads to the one-dimensional diffusion equation,

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}, \tag{19}$$

where $D = (1/\tau) \int_{-\infty}^{\infty} (\Delta^2/2)\phi(\Delta)\, d\Delta$.

Einstein extended $f(x, t)$ to represent the probability that n particles, each with its own coordinate system, will have displacements between $x + \Delta x$ after time t and showed that the function can be expressed as

$$f(x, t) = \frac{n}{\sqrt{4\pi D}} \frac{\exp -(x^2/4Dt)}{\sqrt{t}}. \tag{20}$$

The implication of (20) is that $f(x, t)$ represents the mean value of $\phi(\Delta, t)$ after a very large number of trials.

Einstein [1905] (p. 16 of 1926 translation) recognized the similarity between this function and that representing the distribution of random error by stating, "The probable distribution of the resulting displacements in a given time t is therefore the same as that of fortuitous error, which was to be expected."

Finally, Einstein integrated (20) and obtained the following expression for the arithmetic mean of the squares of displacement of a large number of particles,

$$\bar{x}^2 = \frac{1}{n}\int_{-\infty}^{\infty} x^2 f(x, t)\, dx$$

$$= \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{4Dt}\right) dx = 2Dt. \tag{21}$$

Note that Einstein's equation (20) and (21) are the same as Rayleigh's equations (8) and (9) if we set $D = 0.5$.

Soon thereafter, Paul Langevin (1872–1946) developed an alternate approach to obtain (21), the mean of the square of displacements. Rather than solving the differential equation (as Einstein did), *Langevin* [1908] started with the forces acting on a single particle. A Brownian particle is impelled by the momentum transferred to it by the liquid molecules that collide with it. In turn, the particle is retarded by the viscous resistance offered to it by the liquid. Thus the net force on the particle equals the sum of a "systematic" drag force and a stochastic force (Langevin referred to it as complementary force),

$$F(t) = -6\pi\mu a u + X(t), \tag{22}$$

where μ is viscosity, a is the radius of the spherical particle, u is velocity, and X is the stochastic force. Recognizing that for a large number of particles the stochastic component will cancel out, Langevin obtained the same result as (21) by equating force expressed as mass times acceleration with the first term on the right-hand side of (22). He also showed that after a "relaxation" time τ of the order of $m/(6\pi\mu a)$, where m is the mass of the particle, Einstein's diffusion equation is valid. Langevin's elegant separation of the deterministic and the stochastic components of the force was soon recognized by physicists to be a strong foundation for handling randomness. Thus Langevin's short paper catalyzed the subsequent development of stochastic calculus.

6. REFLECTIONS

It is now useful to reflect on what we have gained from this historical, integrated survey of a broad topic of interest. This reflection should enable us to understand the manner in which ideas pertaining to the foundations of modern science evolved during the nineteenth century and the manner in which some of the leading thinkers of modern science communicated with each other in making their contributions. Revisiting some of the original contributions from a broad perspective should also help us become better aware of the intent and the content of the many equations we routinely use in our day-to-day scientific work.

6.1. Scientific Atmosphere

Many of the leading thinkers of the eighteenth and nineteenth century were natural philosophers, who did not limit their work by disciplinary boundaries. Lagrange, Euler, Laplace, Biot, Poisson, Thomson, Maxwell and others made simultaneous contributions in mechanics, electricity, magnetism, optics, fluid mechanics, and celestial mechanics. Others such as Fourier, went beyond the sciences and made contributions in the humanities. Major contributions in the physical sciences were made by biologists and men of medicine, notable among them being Black, Crawford, Dutrochet, Poiseuille, Fick, and Pfeffer. The difficulties that Fourier encountered in the acceptance of his ideas by Lagrange and Laplace or the difficulties Ohm encountered in the acceptance of his work by his contemporaries in Germany show that even these distinguished scientists arrived at their final results only in stages, as the ideas became progressively clarified. Despite the remarkable talents in pure mathematics that were being brought to the analysis of physical problems by Laplace, Lagrange, Fourier, and others, empirical observations and their intuitive interpretation (e.g., heat capacity, osmosis, heat conduction, electrical conduction) provided much impetus for mathematical innovation. Although intricate mathematical developments tend to overshadow the importance of more descriptive contributions in science, *Maxwell* [1873, p. 398] spoke eloquently about the importance of intuition and its relation to mathematics in his praise of Faraday, who made his greatest discoveries through mental imagery and used practically no mathematics at all: "The way in which Faraday used his lines of force in co-ordinating the phenomena of magneto-electric induction shews him to have been in reality a mathematician of a very high order—one from whom the mathematicians of the future may derive valuable and fertile methods."

It is interesting that nineteenth century science was noted for its eagerness to discover new "laws" of nature. The laws of Ohm, Fick, and Darcy were experimental observations that showed that measured fluxes could be approximately described, on the observed scale, by a simple, linear mathematical relation. Despite the limited set of experiments, all these authors daringly used the word "law" in their papers. These empirical laws were then used as justification for developing mathematical solutions to problems of complexity in terms of domain geometry, heterogeneity, and forcing functions. Although the mathematical solutions themselves may be very precise, the value of the solutions for understanding the physical problems on hand must necessarily depend on the faithfulness with which the mathematical idealization represents the mundane problem of interest. This is true especially in the Earth sciences, where Fourier's model is used to solve problems of fluid flow, chemical diffusion, and electrical resistivity on field scales that are often extremely large in comparison with the laboratory scale on which the laws are based. In these situations, the ultimate value of Fourier's equation depends on our ability to make intuitive judgements on the reasonableness of the mathematical idealization to represent the empirically describable field system.

6.2. Similarities and Differences Among Diffusion Processes

Despite the similarity of mathematical form of heat diffusion, electrical conduction, molecular diffusion and flow in porous media, it is necessary to recognize that in practical application of mathematics to solve particular problems proper consideration must be accorded to the physical traits peculiar to the particular system. To this end it useful to recognize some of the similarities and differences among the phenomena unified by Fourier's heat conduction equation.

The parameters conductivity and capacity are intrinsic to Fourier's diffusion model. How well these critical parameters may be known in the practical use of the Fourier model depends on the type of system of interest. Physicists and engineers who deal with engineered materials may able to fabricate materials whose conductivity or capacity may be controlled with great confidence by manipulating the purity of the materials and their structural arrangement. However, a very different situation exists in the Earth sciences and in the biological sciences, in which one has to work with materials in place in their natural heterogeneous state, whose geometry and structure may be known only in sketchy detail. Therefore the fundamental parameters of the diffusion model can only be estimated empirically, accompanied by uncertainty. In these systems, consequently, the connection between the mathematics and the physics has to be tempered by intuition and judgement.

Maxwell [1881, p. 334] stressed the importance of Ohm's law by noting that the electrical resistance of a conductor can be measured with great precision because the physical nature of the conductor is unaffected by the potential difference or the absolute value of the potential. Thus if the conductor is made of a homogeneous material and the area of cross section is constant, the potential drop will be as perfectly linear, as one may expect in a natural system. However, in the case of thermal conduction, molecular diffusion, and flow in porous media, the physical properties of the host are subject to modification by the magnitude of temperature, chemical concentration, or fluid pressure. Thus, in these cases, even when a single homogeneous material of uniform cross section is involved, the profile of potential within the body cannot, in principle, be truly linear, and one has to make a priori assumptions about the functional relation between conductivity and potential before experimental data can be interpreted. This problem becomes especially severe in multifluid phase Earth systems in which hydraulic conductivity and capacitance become very strong functions of capillary potential.

During the nineteenth century, materials were essentially divided into conductors, which had no ability to hold charge; and insulators, which offered infinite resistance and had the ability to hold charge. Because conductors by definition did not hold charge, *Maxwell* [1881] faulted Ohm for literally extending Fourier's equation to galvanic current by inventing an electrical capacitance term (γ in (2) above). However, no material can be either a perfect conductor or a perfect insulator. If so, is it reasonable to expect that conductors possess extremely small but finite capacitances? If they do, is it reasonable to infer that Ohm's transient equation is valid on extremely small timescales?

Looking at liquid diffusion, it is common practice in science and engineering to use Fick's law, essentially assuming that the liquid is stationary. However, the work of van't Hoff and Nernst suggests that in a solution, the solute is driven in one direction by differences in osmotic pressure, while the solvent is driven in the opposite direction by spatial variations in fluid potential. Thus transport in solutions should involve consideration of two migrating phases, salt and solvent. Under what conditions may we approximate this by restricting consideration only to the solute?

6.2.1. Capacitance, random walk, and error function. It seems rather remarkable that the macroscopic empirical view of heat conduction and the abstract notion of the random walk phenomenon should both lead to the same mathematical equation of diffusion. What may we learn from this similarity?

Consider the fundamental problem of an instantaneous plane source of heat [*Carslaw*, 1945] released at a point on a line extending to infinity on either side of the point. The homogeneous medium is initially in a thermostatic state. Because the specific heat of the medium is finite, temperature at the point of release (taken to be the origin) will instantly jump by a finite amount. Following this, the temperature will gradually decrease with time at the point, as heat moves away with equal facility in both directions. In order that heat may move away from the origin, heat must initially be taken into storage to raise the temperature, as governed by specific heat. Qualitatively, it is easy to see that the temperature profile should be symmetrical about and have a maximum value at the origin. Moreover, because of symmetry, the gradient of temperature is zero at the origin. Far away from the origin there will be no flow of heat and the temperature gradient will be zero. Consequently, at any instant in time, the symmetric temperature profile will have a bell shape. It turns out that the shape of this bell-shaped curve is describable [*Carslaw*, 1945] by,

$$T(x, t) = \frac{Q}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right), \tag{23}$$

where T is temperature and κ is thermal diffusivity, which is thermal conductivity divided by volumetric specific heat. Note that this solution is mathematically the same as that of Rayleigh (equation (8)) and Einstein (equation (20)). Yet in Rayleigh's problem the notions of resistance and capacitance are irrelevant, while in Einstein's model, the notion capacitance does not enter. While much has been written about diffusion being a manifestation of random processes, is there a rational for reversing the question and looking at random processes in terms of suitably contrived notions of conductivity, capacitance, and diffusivity?

6.2.2. Foundations of the diffusion equation. Clearly, Fourier's heat conduction equation continues to serve us well after nearly two centuries. Yet it has limitations, the chief one being that the linear heat conduction equation can be rigorously solved only for flow domains with simple geometry and heterogeneity. Systems with complicated geometry, heterogeneity, and material properties that are dependent on time can be solved only approximately. With the advent of the digital computer, these problems are now being integrated numerically to obtain approximate solutions. In numerically integrating Fourier's equation, the common practice is to either approximate the spatial and temporal gradients with finer and finer discretization of space and time, or evaluate the integrals with approximate weighting functions for the space and time domains. Current wisdom is to treat the differential equation as the truth and assume that the numerical solution will approximate the solution of the differential equation more and more closely as the discretization becomes finer and finer.

Be that as it may, it seems reasonable to pose a question from a different perspective. Is there an integral form of Fourier's differential equation? If so, can that integral be evaluated directly, as accurately as one may please? It turns out that for the special case of steady state diffusion, where the time derivative in Fourier's equation is zero, an integral statement of the problem does exist in the form of a variational principle. For the problem of steady state porous media flow, it is fairly straightforward to derive the variational integral by starting with the law of least action and postulating that under steady conditions of flow, the system will maximize work where the boundary potentials are prescribed or that it will minimize work to achieve a set of prescribed fluxes on the boundary.

However, for the transient diffusion problem for which Fourier's equation is valid, no physically realistic variational principle has as yet been formulated. In other words, no extension of a statement such as the principle of least action is available to describe why a transient diffusion system will choose a particular optimal way of evolving in time, given set of initial conditions and forcing functions. This issue of an integral statement of Fourier's equation is of fundamental importance because we cannot confidently integrate the equation over arbitrary domains of time and space unless we know what the exact form of the integral is.

While mathematicians may approach this important issue in one way, it is of value to speculate on this issue from an intuitive perspective. Note that in Fourier's equation, conductivity is defined as a parameter that relates flux and potential gradient under steady conditions of flow, when potentials are not changing with time. On the other hand, capacitance is defined as a parameter that relates change in quantity of the permeant over a domain, however small, and the corresponding change in potential, as the small domain jumps from jumps in time from one static state to another. The fact that we have used Fourier's equation successively for so long may suggest that in the infinitesimal limit the system can in fact be considered to simultaneously be steady and nonsteady. However, the intriguing question arises, what happens when we consider finite domains, which are not infinitesimal? Can the system be simultaneously steady and nonsteady?

As we try extend Fourier's equation to heterogeneous domains of complicated geometry and time-dependent material properties with the help of integral equations, it behooves us to examine the foundations of the heat conduction equation and try to understand intuitively as well as mathematically how we may formulate a physically meaningful and mathematically tractable integral statement of the diffusion process.

ACKNOWLEDGMENTS. I am grateful to the following for reading the first draft of this work: Grigory Barenblatt, Albert Bowker, David Brillinger, Lawrence Cathles, Darryl Chrzan, Frank Dalton, Tejal Desai, Harvey Doner, John Duey, Wilford Gardner, Andreas Glaeser, Ivor Grattan-Guinness, William Gray, Roger Hahn, Jeffrey Hanor, Wolfgang Kinzelbach, Marc Mangel, Ravi Narasimhan, Ivars Neretnieks, Jean-Yves Parlange, Karsten Pruess, Bruce Rickborn, Tim Sands, Genevieve Segol, Garrison Sposito, Fritz Stauffer, Stephen Stigler, Tetsu Tokunaga, Thomas Torgersen, and Chin Fu Tsang. An introduction to literature in fields beyond my sphere of knowledge has been essential for the success of this venture. Eugene Haller introduced me to literature pertaining to solid diffusion, and Tejal Desai unearthed important papers on osmosis. Stephen Stigler introduced me to the work of Edgeworth. Andrew Jackson and two anonymous reviewers gave constructive criticisms. Discussions with Art Lachenbruch, Dan McKenzie, and Lew Cohen and a careful reading of the manuscript by Ravi Narasimhan and Tom Torgersen helped greatly in the final revision process. This work was supported partly by the Director, Office of Energy Research, Office of Basic Energy Sciences of the U.S. Department of Energy, under contract DE-AC03-76SF00098 through the Earth Sciences Division of Ernest Orlando Lawrence Berkeley National Laboratory.

Thomas Torgersen was the Editor responsible for this paper. He thanks Jeff Hanor and two anonymous referees for their technical reviews, and an anonymous referee for the cross-disciplinary review.

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Chapter-01

History of Heat Equation

The heat equation is an important partial differential equation (PDE) which describes the distribution of heat (or variation in temperature) in a given region over time. For better understanding of this project, it is very important that we understand the difference between heat and temperature. Heat is a process of energy transfer as a result of temperature difference between the two points. Thus, the term 'heat' is used to describe the energy transferred through the heating process. Temperature, on the other hand, is a physical property of matter that describes the hotness or coldness of an object or environment. Therefore, no heat would be exchanged between bodies of the same temperature.

Suppose we have a function (x; y; z; t), which describes the temperature of a conducting material at a given location, (x; y; z), you can use this function to determine the temperature at any position on the material at a future time, t+1. The function U changes over time as heat spreads through-out the material and the heat equation is used to determine this change in the function U. The gradient of U describes which direction and at what rate is the temperature changing around a particular region of the material. Therefore, the gradient of temperature is the ow of heat through the material. This gradient will help us determine the ow of heat through various materials. This is analogous to the ow of water in a pipe.

The heat equation is a parabolic partial differential equation that describes the distribution of heat (or variation in temperature) in a given region over time.

For a function (x,y,z,t) of three spatial variables (x,y,z) (see Cartesian coordinates) and the time variable t, the heat equation is:

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

More generally in any coordinate system:

$$\frac{\partial u}{\partial t} - \alpha\nabla^2 u = 0$$

Where α is a positive constant, and Δ or ∇^2 denotes the Laplace operator. In the physical problem of temperature variation, u(x,y,z,t) is the temperature and α is the thermal diffusivity. For the mathematical treatment it is sufficient to consider the case $\alpha = 1$.

Note that the state equation, given by the first law of thermodynamics (i.e. conservation of energy), is written in the following form (assuming no mass transfer or radiation). This form is more general and particularly useful to recognize which property (e.g. c_p or ρ) influences which term.

$$\rho c_p \frac{\partial T}{\partial t} - \nabla.(k\nabla T) = \dot{q}v$$

Where $\dot{q}v$ is the volumetric heat flux.

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In probability theory, the heat equation is connected with the study of Brownian motion via the Fokker–Planck equation. In financial mathematics it is used to solve the Black–Scholes partial differential equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

The heat equation is used in probability and describes random walks. It is also applied in financial mathematics for this reason.

It is also important in Riemannian geometry and thus topology: it was adapted by Richard S. Hamilton when he defined the Ricci flow that was later used by Grigori Perelman to solve the topological Poincare conjecture.

The aim of this project is to be able to determine the ow of heat of various materials i.e. different thermal conductivities. Does the arrangement of conductors or insulators affect the rate at which the heat owes? Imagine a room with a wall that is made of different materials such as wood, metal or bricks arranged in different ways. The room is at room temperature, say 25° C and does not generate any heat (no air conditioner) and it is surrounded by the outside environment which has a temperature of 0° C. The room is so tiny relative to the outside environment therefore any heat ow from the room to the outside would not change the temperature outside. However, the temperature inside the room is prone to changes due to the surrounding temperature. How can we ensure that we maintain the room temperature for the longest possible time without the use of an air conditioner? If the walls of the room are bad insulators of heat, it is almost impossible to maintain the room temperature. This is when it is important that we maximize the materials and knowledge that we have to build a wall that would keep the room temperature constant. It is possible that one can just buy building materials with low thermal conductivity. However, the constraints are that we have a variety of bad and good thermal conductors and we are trying to build the best congruation with the materials that we have.

To answer these questions, I have created materials with different thermal conductivities arranged different ways. I am more interested in two cases: case 1) what happens to the heat ow when I reverse the order of thermal conductivities and case 2) what happens when I put the materials with high thermal conductivities on the edges or vice versa.

To test these arrangements, I will set the temperature on one end of the material to be at 0°C and the other end at 100°C. But before we get into that, let us have a look at the two kinds of conduction that are important to the understanding of this project.

To be able to solve the second-order partial differential heat equation in the spatial coordinates, we need to know the boundary conditions and the initial conditions. The boundary conditions specify how our system interacts with the outside surroundings. There are three general types of boundary conditions: Dirichlet, Neumann and Mixed boundary conditions.

The heat equation in one dimension is written as the following;

$$\frac{\partial u}{\partial t} = c\left(\frac{\partial^2 u}{\partial x^2}\right)$$

where U (x; t) is a function of temperature.

In this case we can think of a one dimensional rectangular thin wire with length x. Ignore the width and height dimensionality. The one end of the length of the wire is set at 0° C whereas the other end is set at 0° C. These are its boundary conditions. We also need to specify the temperature at every position on the wire at time; t_o; (the initial conditions). To solve this one-dimensional heat problem, we need to transform the above heat equation into an explicit method using the second-order central Finite Difference Method. Therefore, explicitly we can write the one dimensional heat equation as:

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \alpha(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2})$$

This equation can then be implemented and solved easily using Mat lab or other languages. The value should be much smaller than 1, other-wise you get unexpected errors. In this case, we assume that is a constant although later we are going to see that alpha could be de ned as a function that depends on space. As we can see, the heat equation in 1-D explicit form is straight forward because the right hand side has only one term.

In a metal rod with non-uniform temperature, heat (thermal energy) is transferred from regions of higher temperature to regions of lower temperature. Three physical principles are used here.

1. Heat (or thermal) energy of a body with uniform properties: Heat energy = cmu, where m is the body mass, u is the temperature, c is the specific heat, units [c] = L^2T −2U −1 (basic units are M mass, L length, T time, U temperature). c is the energy required to raise a unit mass of the substance 1 unit in temperature.

2. Fourier's law of heat transfer: rate of heat transfers proportional to negative temperature gradient, Rate of heat transfer ∂u = −K0 (1) area ∂x where K0 is the thermal conductivity, units [K0] = MLT −3U −1. In other words, heat is transferred from areas of high temp to low temp.

3. Conservation of energy: Consider a uniform rod of length l with non-uniform temperature lying on the x-axis from x = 0 to x = l. By uniform rod, we mean the density ρ, specific heat c, thermal conductivity K0, cross-sectional area A are ALL constant. Assume the sides of the rod are insulated and only the ends may be exposed. Also assume there is no heat source within the rod.

Chapter-02

Derivation of Heat Equation

2.1 From Fourier's Law

The heat equation is derived from Fourier's law and conservation of energy (Cannon 1984). By Fourier's law, the rate of flow of heat energy per unit area through a surface is proportional to the negative temperature gradient across the surface,

$$q = -k\nabla u$$

Where k is the thermal conductivity and u is the temperature. In one dimension, the gradient is an ordinary spatial derivative, and so Fourier's law is

$$q = -k\frac{\partial u}{\partial x}$$

In the absence of work done, a change in internal energy per unit volume in the material, ΔQ, is proportional to the change in temperature, Δu (in this section only, Δ is the ordinary difference operator with respect to time, not the Laplacian with respect to space). That is,

$$\Delta Q = c_p \rho \Delta u$$

Where c_p is the specific heat capacity and ρ is the mass density of the material. Choosing zero energy at absolute zero temperature, this can be rewritten as

$$Q = c_p \rho u$$

The increase in internal energy in a small spatial region of the material

$$x - \Delta x \le \xi \le x + \Delta x$$

over the time period

$$t - \Delta t \le \tau \le t + \Delta t$$

is given by

$$c_p\rho\int_{x-\Delta x}^{x+\Delta x}[u(\xi,t+\Delta t)-u(\xi,t-\Delta t)]\,d\xi = c_p\rho\int_{t-\Delta t}^{t+\Delta t}\int_{x-\Delta x}^{x+\Delta x}\frac{\partial u}{\partial \tau}d\xi d\tau$$

where the fundamental theorem of calculus was used. If no work is done and there are neither heat sources nor sinks, the change in internal energy in the interval [$x-\Delta x$, $x+\Delta x$] is accounted for entirely by the flux of heat across the boundaries. By Fourier's law, this is

$$k\int_{t-\Delta t}^{t+\Delta t}\left[\frac{\partial u}{\partial x}(x+\Delta x,\tau)-\frac{\partial u}{\partial x}(x-\Delta x,\tau)\right]d\tau = k\int_{t-\Delta t}^{t+\Delta t}\int_{x-\Delta x}^{x+\Delta x}\frac{\partial^2 u}{\partial \xi^2}d\xi d\tau$$

again by the fundamental theorem of calculus.[2] By conservation of energy,

$$\int_{t-\Delta t}^{t+\Delta t}\int_{x-\Delta x}^{x+\Delta x}[c_p\rho u_\tau - k u_{\xi\xi}]d\xi d\tau = 0$$

This is true for any rectangle [$t-\Delta t$, $t+\Delta t$] × [$x-\Delta x$, $x+\Delta x$]. By the fundamental lemma of the calculus of variations, the integrand must vanish identically:

$$c_p\rho u_t - k u_{xx} = 0$$

Which can be rewritten as:

$$u_t = \frac{k}{c_p\rho}u_{xx} \quad \text{or} \quad \frac{\partial u}{\partial t} = \frac{k}{c_p\rho}\frac{\partial^2 u}{\partial x^2}$$

which is the heat equation, where the coefficient (often denoted α), $\alpha = \frac{k}{c_p\rho}$ is called the thermal diffusivity.

An additional term may be introduced into the equation to account for radiative loss of heat, which depends upon the excess temperature $u = T - T_s$ at a given point compared with the surroundings. At low excess temperatures, the radiative loss is approximately μu, giving a one-dimensional heat-transfer equation of the form

$$\frac{\partial u}{\partial t} = \frac{k}{c_p\rho}\frac{\partial^2 u}{\partial x^2} - \mu u$$

At high excess temperatures, however, the Stefan–Boltzmann law gives a net radiative heat-loss proportional to $T^4 - T_s^4$, and the above equation is inaccurate. For large excess temperatures,$T^4 - T_s^4 \approx u^4$ giving a high-temperature heat-transfer equation of the form

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = \alpha\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}\right) - \mathrm{mu}^4$$

Where $m = \epsilon\sigma p/\rho A c_p$. Here, σ is Stefan's constant, ε is a characteristic constant of the material, *p* is the sectional perimeter of the bar and *A* is its cross-sectional area. However, using *T* instead of *u* gives a better approximation in this case.

2.2 Using a Rod Pipe

Heat is the energy transferred from one body to another due to a difference in temperature. (Better: heat is the kinetic energy of the molecules that compose the material.

Consider a long uniform tube surround by an insulating material like stir form along its length, so that heat can flow in and out only from its two ends:

There are two basic physical principle governing the motion of heat.

(i) The total heat energy H contained in a uniform, homogeneous body is related to its temperature T and mass in the following simple way

$$H = k_s MT$$

Where k_s is the specific heat capacity of the material (a measurable constant specific to the material from with the body is made). More generally, in a situation for which neither the temperature nor the density of the material is constant we have

$$H(t) = k_s \int_V \rho(x) T(x,t) dx \dots \dots \dots \dots (1)$$

(ii) The rate of heat transfer across a portion S of the boundary of a region R of the body is proportional directional derivative of T across the boundary and the area of contact

$$\text{Heat flux across } S = \sigma \int_S \nabla T . n dS \dots \dots (2)$$

Where n=n(x) is the direction normal to the surface of contact at the point x, and σ is another constant specific to the material from with the body is constructed. $\sigma$is called heat conductivity constant.

Applying Gauss's divergence theorem to (2) we have

$$\text{Heat flux entering or leaving a region} = \sigma \int_{\partial R} \nabla T . n dS = \sigma \int_R \nabla . \nabla T \, dx \dots \dots \dots (3)$$

This should be the total rate at which heat enters or leaves the region R, which in turn should correspond to the rate of change of the total amount of heat energy contained in the region:

$$\frac{dH}{dT} = k_s \int_R \rho \frac{\partial T}{\partial t} dx \dots\dots\dots\dots\dots\dots..(4)$$

Equating (3) and (4) we thus obtain

$$\sigma \int_R \nabla.\nabla T \, dx = k_s \int_R \rho \frac{\partial T}{\partial t} dx$$

Since the region R can be chosen arbitrarily, the two integrands must coincide at every point of the body. We thus obtain

(The Heat Equation) $\quad \nabla^2 T - \frac{\rho k_s}{\sigma}\frac{\partial T}{\partial t} = 0$

In such a situation we can assume that the temperature really only6 depends on the position x along the length of the heat pipe. Then

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \approx \frac{\partial^2 T}{\partial x^2}$$

And the heat equation reduces to a 2-dimensional PDE of the form

$$\frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} \dots\dots\dots\dots\dots\dots\dots\dots.(5)$$

Where $\alpha = \sqrt{\frac{\sigma}{\rho k_s}}$

(Replacing the ratio $\frac{\sigma}{\rho k_s}$ by α^2 will prove convenient later on.)

Or

We will now derive the heat equation with an external source,

$$u_t = \alpha^2 u_{xx} + F(x, t), 0 < x < L, t > 0,$$

where u is the temperature in a rod of length L, α^2 is a diffusion coefficient, and F(x,t) represents an external heat source. We begin with the following assumptions:

• The rod is made of a homogeneous material.

• The rod is laterally insulated, so that heat flows only in the x-direction.

• The rod is sufficiently thin so that the temperature within any particular cross-section is constant.

These last two assumptions are used to allow us to treat the problem as one-dimensional. As we will see, the first assumption is not absolutely necessary, but it does simplify certain solution techniques. From the principle of conservation of energy, it follows that the heat within a segment of the rod [x,x + Δx] satisfies the following:

Net change inside [x,x + Δx] = Net inward flux across boundaries +

Total heat generated inside [x,x + Δx]

The total amount of heat, in calories, in any segment [a,b] is given by

$$\int_a^b c\rho A u(s,t)ds$$

where c is the thermal capacity of the rod (also known as the specific heat), ρ is the density of the rod, and A is the cross-sectional area of the rod. In view of our assumptions, c, ρ and A are constants. Also, recall that the flux from left to right at x = a is given by $-ku_x(a,t)$, where k is the thermal conductivity of the rod.

Putting all of these facts together, we can translate the conservation relation into the equation

$$c\rho A\int_x^{x+\Delta x} u_t(s,t)ds = kA[u_x(x+\Delta x,t) - u_x(x,t) + A\int_x^{x+\Delta x} f(s,t)ds,$$

where f(x,t) is the amount of heat generated by the external source per unit of length per unit of time. Note that we must use inward flux, which is why the flux term at x = L must be negated. Applying the Fundamental Theorem of Calculus “in reverse”,

$f(b) - f(a) = \int_a^b f'(s)ds,$

we obtain, after dividing both sides by A,

$$c\rho \int_{x}^{x+\Delta x} u_t(s,t)ds = \int_{x}^{x+\Delta x} ku_{xx}(s,t) + f(s,t)ds.$$

Rearranging yields

$$\int_{x}^{x+\Delta x} u_t(s,t) - \alpha^2 u_{xx}(s,t) + F(s,t)ds = 0,$$

Where

$$\alpha^2 = \frac{k}{c\rho}, \qquad F(x,t) = \frac{1}{c\rho} f(x,t),$$

are the diffusivity of the rod and the heat source density, respectively.

Since this equation holds on an arbitrary segment of the rod, it follows that the integrand must vanish everywhere in the rod, which yields the equation

$$u_t = \alpha^2 u_{xx} + F(x,t)$$

It is worth noting that the diffusivity $\alpha^2 = k/c\rho$ is proportional to the conductivity, but inversely proportional to the thermal capacity and the density. Physically, this makes sense because the more an object tends to store heat, and the denser it is, the more difficult it should be for heat energy to diffuse through the object, whereas the better the ability of the material to conduct heat, the easier it should be for heat energy to move through the object and diffuse.

2.3 Heat equation properties

We would like to solve the heat (diffusion) equation,

$$u_t - k\Delta u = 0.$$

And obtain a solution formula depending on the given initial data, similar to the case of the wave equation. However, the methods that we used to arrive at d'Alambert's solution for the wave IVP do not yield much for the heat equation. To see this, recall that the heat equation is of parabolic type, and hence, it has it has only one family of characteristic lines. If we rewrite the equation in the form

$$ku_{xx} + \cdots = 0,$$

Where the dots stand for the lower order terms, then you can see that the coefficients of the leading order terms are

$$A = k, \qquad B = C = 0.$$

The slope of the characteristics lines will be given by,

$$\frac{dt}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = 0.$$

Consequently, the single family of characteristics lines will be given by

$$t = c$$

These characteristic lines are not very helpful, since they are parallel to the x axis. Thus, one cannot trace points in the xt plane along the characteristics to the x axis, along which the initial data is defined. Notice that there is also no way to factor the heat equation into first order equations, either, so the methods used for the wave equation do not shed any light on the solutions of the heat equation. Instead, we will study the properties of the heat equation, and use the gained knowledge to devise a way of reducing the heat equation to an ODE, as we have done for every PDE, as we have solved so far.

Chapter-03

Analytical Solution of Heat Equation

3.1 Method of Characteristics

In mathematics, the method of characteristics is a technique for solving partial differential equations. Typically, it applies to first-order equations, although more generally the method of characteristics is valid for any hyperbolic partial differential equation. The method is to reduce a partial differential equation to a family of ordinary differential equations along which the solution can be integrated from some initial data given on a suitable hyper surface. The equations in the problems we have investigated so far are all linear and the terms containing the unknown function and its derivatives have constants coefficients. The only exception is the type of problem when we need to make use of polar coordinates, but in such problems the polar radius is present in some of the coefficients in a very specific way, which does not disturb the solution scheme.

3.2 Solution of Heat Equation

Let us now consider the solution of the 1-dimensional heat equation

$$\frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

Subject to non-homogeneous boundary conditions

$$T(0,t) = T_1 \quad , T(L,t) = T_2 \quad , T(x,0) = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Which might correspond to a situation where a long rod with an initial temperature distribution f(x) has its two ends inserted into different heat baths that are maintained at different temperatures.

Since we expect that eventually as $t \to \infty$ the rod will eventually reach a steady state temperature distribution that is independent of time, we shall suppose that5 if

$$\text{for t sufficientlyb large } T(x,t) \approx T_{ss}(x)$$

Where $T_{ss}(x)$ is the (as yet undetermined) final steady state temperature distribution. Since even for large t, T(x,t) must still satisfy (i), (ii) , we have for sufficiently large t

$$0 = \frac{\partial T_{ss}}{\partial t} - \alpha^2 \frac{\partial^2 T_{ss}}{\partial x^2} => \frac{d^2 T_{ss}}{dx^2} = 0 \dots\dots\dots\dots (iii)$$

And

$$T_{ss}(0) = T_1 \quad , \qquad T_{ss}(L) = T_2 \dots\dots\dots\dots\dots (iv)$$

The differential equation $\frac{d^2T_{ss}}{dx^2} = 0$implies T_{ss} is a linear function of x,

$$T_{ss}(x) = Ax + B$$

And the boundary conditions (iv) require the constants A and B to be

$$B = T_1 \quad \text{and} \quad A = \frac{T_2 - T_1}{L}$$

Thus

$$T_{ss}(x) = \frac{T_2 - T_1}{L} x + T_1 \dots\dots\dots\dots\dots.. (v)$$

Let us now define an auxiliary function τ (x,t) by

$$T(x,t) = T_{ss}(x) + \tau(x,t) \dots\dots\dots\dots\dots. (vi)$$

Evidently, $\tau(x,t)$ represents the discrepancy between the actual solution and the final steady state solution. Plugging the right hand side of (vi) into equations (i) and (ii) we find (noting again $\frac{d^2T_{ss}}{dx^2} = 0 = \frac{\partial T_{ss}}{\partial t}$)

$$\frac{\partial \tau}{\partial t} - \alpha^2 \frac{\partial^2 \tau}{\partial x^2} = 0$$

And

$$T_1 = T(0,t) = T_{ss}(0) + \tau(0,t) = T_1 + \tau(0,t) => \tau(0,t) = 0$$

$$T_2 = T(L,t) = T_{ss}(L) + \tau(L,t) = T_2 + \tau(L,t) => \tau(L,t) = 0$$

$$f(x) = T(x,0) = T_{ss}(x) + \tau(x,0) = \frac{T_2 - T_1}{L} x + T_1 + \tau(x,0) => \tau(x,0)$$
$$= f(x) - \frac{T_2 - T_1}{L} x - T_1$$

$$\text{Thus } \tau(x,t) \text{satisfies} \qquad \frac{\partial \tau}{\partial t} - \alpha^2 \frac{\partial^2 \tau}{\partial x^2} = 0$$

$$\tau(0,t) = 0$$

$$\tau(L,t) = 0$$

$$\tau(x,0) = F(x) \ , \ \text{ where } F(x) = f(x) - \frac{T_2 - T_1}{L} x - T_1$$

In other words, a PDE / BVP of the form (v), (vi), (vii). We can thus conclude from the results of the last section that

$$\tau(x,t) = \sum_{n=0}^{\infty} c_n e^{-\left(\frac{\alpha n \pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

Where

$$c_n = \frac{2}{L} \int_0^L F(x) \sin\left(\frac{n\pi}{L} x\right)$$

Hence, the solution of equations (i) and (ii) is

$$T(x,t) = \frac{T_2 - T_1}{L} x + T_1 + \sum_{n=0}^{\infty} c_n e^{-\left(\frac{\alpha n \pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

Where

$c_n = \frac{2}{L} \int_0^L \left(f(x) - \frac{T_2 - T_1}{L} x - T_1\right) \sin\left(\frac{n\pi}{L} x\right) a$

3.3 Problem Solve

(a) Problem solve by separation of variable

Consider the initial boundary value problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$;0 < x < 1,u(0,t) = u(1,t) = 0,u(x,0) = Sin x

Solution:

Given, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ………………..(i)

u(0,t) = 0 ………………..(ii)

u(1,t) = 0 ………………..(iii)

u(x,0) = Sin x ………………..(iv)

Let, u(x,t) = X(x) T(t) ………………..(*) [Where X is a function of x and T is a function of t]

$\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial u}{\partial x} = X'T$

$$=> \frac{\partial^2 u}{\partial x^2} = X''T$$

From (i) we get,

$XT' = X''T$

$=> \frac{T'}{T} = \frac{X''}{X}$

Since T is a function of t and x is a function of x and since they are equal.

So they must be equal to constant.

$\therefore \frac{X''}{X} = \frac{T'}{T} = -\lambda^2$ (Say)

$\therefore \frac{X''}{X} = -\lambda^2$ and $\therefore \frac{T'}{T} = -\lambda^2$

$=> X'' + \lambda^2 X = 0$ ………………..(v)

$\Rightarrow T' + \lambda^2 T = 0$ …………….. (vi)

Solution of (v) is, $X(x) = c_1 \operatorname{Cos} \lambda x + c_2 \operatorname{Sin} \lambda x$

Solution of (vi) is $T(t) = c_3 e^{-\lambda^2 t}$

From (*) we get,

$u(x,t) = (c_1 \operatorname{Cos} \lambda x + c_2 \operatorname{Sin} \lambda x).\ c_3 e^{-\lambda^2 t}$

$= (A \operatorname{Cos} \lambda x + B \operatorname{Sin} \lambda x)\, e^{-\lambda^2 t}$ …………….. (vii) Where, $A = c_1 c_3$ and $B = c_2 c_3$

Now applying initial condition, we get,

$u(0,t) = e^{-\lambda^2 t}[A.1 + B.0]$

$\Rightarrow 0 = e^{-\lambda^2 t} . A$

$\Rightarrow A = 0 \quad [\because e^{-\lambda^2 t} \neq 0]$

Again,

$u(1,t) = e^{-\lambda^2 t} (A \operatorname{Cos} \lambda + B \operatorname{Sin} \lambda)$

$\Rightarrow 0 = e^{-\lambda^2 t} . B \operatorname{Sin} \lambda$

$\Rightarrow \operatorname{Sin} \lambda = 0$

$\Rightarrow \operatorname{Sin} \lambda = \operatorname{Sin} n\pi$

$\Rightarrow \lambda = n\pi$

Now putting the value of λ and A in equation (vii) we get,

$u(x,t) = e^{-n_1^2 \pi^2 t} B_1 \operatorname{Sin} n_1 \pi x$ …………….. (viii)

By using the initial condition,

$\Rightarrow u(x,0) = B_1 \operatorname{Sin} n_1 \pi x$

$\Rightarrow \operatorname{Sin} x = B_1 \operatorname{Sin} n_1 \pi x$

It is possible only if $B_1 = 1$ and $n_1 = \frac{1}{\pi}$

Now from the equation (viii) we have,

$u(x, t) = e^{-t} \text{Sin } x$; which is the required solution.

(b) Problem solve by Fourier Transformation

Consider the initial boundary value problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$; with u(0,t) = 0, u(1,t) = 0, u(x,0) = Sin x

Solution:

Given that,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \dots\dots\dots\dots\dots..(i)$$

Taking finite Sine, we get,

$$\therefore \int_0^1 \frac{\partial u}{\partial t} \text{Sin } \frac{n\pi x}{1} dx = \int_0^1 \frac{\partial^2 u}{\partial x^2} \text{Sin } \frac{n\pi x}{1} dx \quad \dots\dots\dots\dots\dots..(ii)$$

Let,

$$v = v(n,t) = \int_0^1 u(x,t)\ Sin\ n\pi x\ dx$$

$$= \int_0^1 \frac{\partial^2 u}{\partial x^2} \text{Sin } n\pi x \text{ dx}$$

$$= [\frac{\partial u}{\partial x} \text{ Sin } n\pi x]_0^1 - n\pi \int_0^1 \frac{\partial u}{\partial x}\ Cos\ n\pi x\ dx$$

$$= 0 - [n\pi . u(x,t)\text{Cos } n\pi x]_0^1 - n^2\pi^2 \int_0^1 u(x,t)\text{Sin } n\pi x \text{ dx}$$

$$= 0 - n^2\pi^2 . v(n,t) \qquad [\because u(0,t) = u(1,t) = 0]$$

$$=> \frac{dv}{dt} = -n^2\pi^2 . v(n,t)$$

$$=> \frac{dv}{v} = -n^2\pi^2 \text{ dt}$$

$$=> \ln v = -n^2\pi^2 t + \ln A$$

$$=> v = Ae^{-n^2\pi^2 t} \dots\dots\dots\dots\dots..(iii)$$

$$=> v(n,0) = A$$

$$=> \int_0^1 u(x,0)\text{Sin } n\pi x \text{ dx} = A$$

$$=> A = \int_0^1 Sin\ x\ .Sin\ n\pi x\ dx$$

$$= \frac{1}{2}\int_0^1 2.Sin\,x\,.Sin\,n\pi x\,dx$$

$$= \frac{1}{2}\int_0^1 [Cos(n\pi - 1)x - Cos(n\pi + 1)x]dx$$

$$= \frac{1}{2}\int_0^1 Cos(n\pi - 1)x\,dx - \frac{1}{2}\int_0^1 Cos(n\pi + 1)dx$$

$$= \frac{1}{2}[\frac{Sin(n\pi - 1)x}{(n\pi - 1)}]_0^1 - \frac{1}{2}[\frac{Sin(n\pi + 1)x}{(n\pi + 1)}]_0^1$$

$$= \frac{1}{2}\left[\frac{Sin(n\pi - 1)}{(n\pi - 1)}\right] - \frac{1}{2}\left[\frac{Sin(n\pi + 1)}{(n\pi + 1)}\right]$$

$$= \frac{1}{2}\left[\frac{Sin(n\pi - 1)}{(n\pi - 1)} - \frac{Sin(n\pi + 1)}{(n\pi + 1)}\right]$$

From equation (iii) we have,

$$\mathrm{v(n,t)} = e^{-n^2\pi^2 t}.\frac{1}{2}\left[\frac{\mathrm{Sin}(n\pi - 1)}{(n\pi - 1)} - \frac{\mathrm{Sin}(n\pi + 1)}{(n\pi + 1)}\right]$$

Taking inverse Fourier Sine Transform we get,

$$\mathrm{u(x,t)} = \frac{2}{1}\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}.\frac{1}{2}\left[\frac{\mathrm{Sin}(n\pi - 1)}{(n\pi - 1)} - \frac{\mathrm{Sin}(n\pi + 1)}{(n\pi + 1)}\right]\mathrm{Sin}\ n\pi x$$

$= \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}.\left[\frac{\mathrm{Sin}(n\pi-1)\mathrm{Sin}\ n\pi x}{(n\pi-1)} - \frac{\mathrm{Sin}(n\pi+1)\mathrm{Sin}\ n\pi x}{(n\pi+1)}\right]$; which is the required solution.

(c) Problem Solve by Laplace Transformation

Consider the initial boundary value problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$; with u(0,t) = 0, u(1,t) = 0, u(x,0) = Sin x

Solution:

Given that,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Now taking Laplace Transformation on both sides then,

$$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$=> su - u(x, o) = \frac{d^2 u}{dx^2}$$

$$=> \frac{d^2 u}{dx^2} - su = -u(x, 0)$$

$$=> \frac{d^2 u}{dx^2} - su = -\text{Sin x} \dots\dots\dots\dots\dots..(i)$$

C.E = $c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$

P.I = $-\frac{\text{Sin x}}{D^2 - s}$

$= \frac{Sin\ x}{S+1}$

∴ The general solution of (i) is,

$$u(x, s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{\text{Sin x}}{S+1} \dots\dots\dots\dots\dots..(ii)$$

Again,

u(0,t) = 0

=> L{u(0,t)} = 0

=> $u(0,s) = 0$

Now,

$u(1,t) = 0$

=> $L\{u(1,t)\} = 0$

=> $u(1,s) = 0$

Now using the 1st condition in (ii) then,

$u(0,s) = c_1 + c_2$

=> $c_1 + c_2 = 0$

Using the 2nd condition in (ii) then,

$$u(1,s) = c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{\text{Sin }1}{S+1}$$

$$\Rightarrow 0 = c_1 e^{\sqrt{s}} - c_1 e^{-\sqrt{s}} + \frac{\text{Sin }1}{S+1}$$

$$\Rightarrow -\frac{\text{Sin }1}{S+1} = -\, c_1 (e^{-\sqrt{s}} - e^{\sqrt{s}})$$

$$\Rightarrow \frac{\text{Sin }1}{(S+1)(e^{-\sqrt{s}} - e^{\sqrt{s}})} = c_1$$

$$\therefore c_2 = -\, c_1$$

$$= -\frac{\text{Sin }1}{(S+1)(e^{-\sqrt{s}} - e^{\sqrt{s}})}$$

So the equation (ii) becomes,

$$u(x,s) = \frac{\text{Sin }1\, .\, e^{\sqrt{s}x}}{(S+1)(e^{-\sqrt{s}} - e^{\sqrt{s}})} - \frac{\text{Sin }1\, .\, e^{-\sqrt{s}x}}{(S+1)(e^{-\sqrt{s}} - e^{\sqrt{s}})} + \frac{\text{Sin }x}{S+1}$$

Taking the inverse, we get,

$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} . \left[\frac{\text{Sin}(n\pi-1)\text{Sin } n\pi x}{(n\pi-1)} - \frac{\text{Sin}(n\pi+1)\text{Sin } n\pi x}{(n\pi+1)}\right]$; which is the required solution.

Experiments and results:

We develop a computer program (code) in Matlab programming of scientific computing and implement analytic solution for a heat equation. The main parts of the implementation of our analytic scheme are given as in the following algorithm:

Input:_nt and nx are the numbers of grid points of time and space respectively. k and h are the right end points of [0,k] and [0,h].

uo is as a initial condition and ua as a boundary condition.

Output: u(t,x) is the solution matrix.

Step 1: **Initialization:**

```
k=t(2)-t(1);

h=x(2)-x(1);
```

Step 2: **Calculation of Analytic solution:**

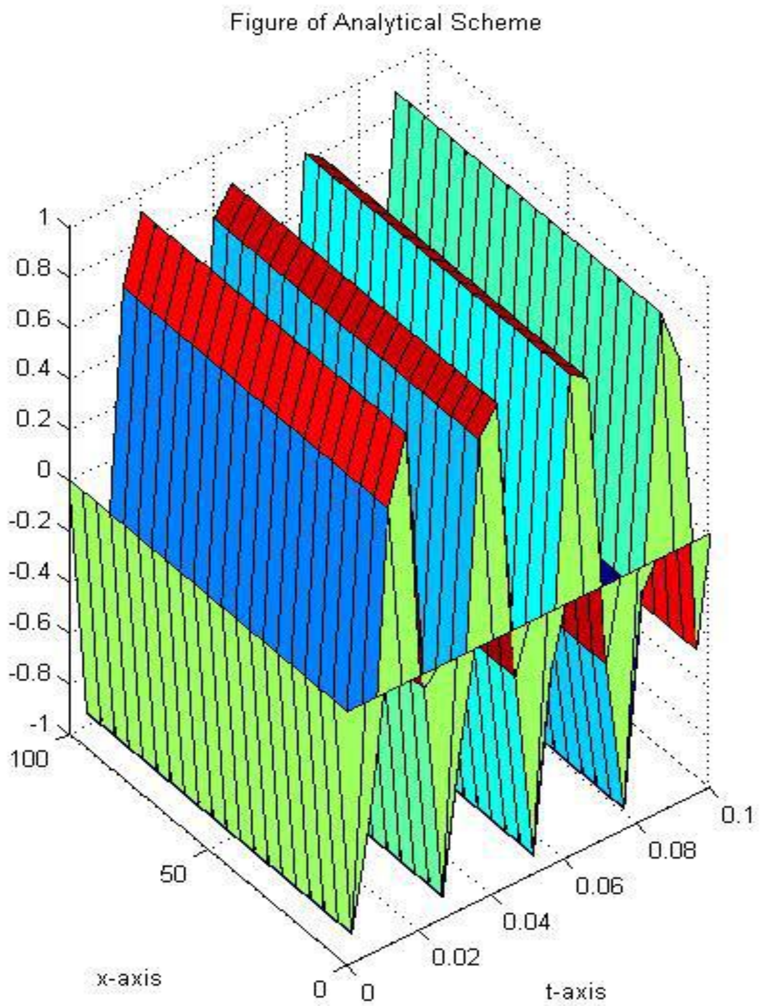
```
  for e=2:nt

     for f=2:nx

        z(e,f)=exp(-t(e))*sin(x(f));

     end

  end

end

surf(t,x,u)

title('Figure of Numerical Scheme');

xlabel('t-axis');
```

ylabel('x-axis');

Step 3: Print u(t,x)

Step 4: Stop

To test the accuracy of the implementation of the analytic scheme, we consider the heat equation. Now we show our results:



Conclusion:

In this project we have considered the second order heat equation. First we have shown that fundamentals of heat equation, analytical solution by using separation of variable method, Fourier transform method and Laplace transform method. Finally, we show the analytical solution in Matlab computer programming. In future work, we implement the numerical scheme and also compare in heat equation

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3. Heat equation

The heat equation is the prototypical parabolic PDE:

$$\boxed{u_t = \Delta u.} \tag{1}$$

This equation describes the isotropic diffusion of a quantity that might, for example, be heat in a solid or concentration of salt in a motionless body of water.

The history begins with the work of Joseph Fourier around 1807. In a remarkable memoir, Fourier invented both the equation (1) and the method of Fourier analysis for its solution. For definiteness, let us consider the one-dimensional problem $u_t = u_{xx}$ for $x \in \mathbb{R}$ with initial data $u_0(x)$. The Fourier transform decomposes u_0 into its components at various wave numbers k:

$$u_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}_0(k)\mathrm{d}k, \tag{2}$$

where $\hat{u}_0(k)$ is defined by the integral

$$\hat{u}_0(k) = \int_{-\infty}^{\infty} e^{-ikx}u_0(x)\mathrm{d}x. \tag{3}$$

The evolution of each component e^{ikx} under (1) is a trivial matter—it decays at the rate e^{-k^2t}. Superposition gives us the evolution of the general initial function u_0:

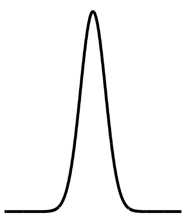
$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)\mathrm{d}k.$$

Replacing $\hat{u}_0(k)$ by its integral (3) and applying an identity for the Fourier transform of a Gaussian yields the formula

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4t}u_0(s)\mathrm{d}s. \tag{4}$$

In a bounded domain in $\mathbb{R}$ or $\mathbb{R}^n$, an analogous treatment of the heat equation would go by separation of variables, leading to solutions of the form $e^{\alpha_j t}\phi_j(x)$, where the functions $\phi_j(x)$ are eigenfunctions of the Laplacian operator for Ω (→ *ref*).

Fig. 1: Gaussian kernel of height $(4\pi t)^{-1/2}$, width $O(t^{1/2})$



Equation (4) asserts that the solution to (1) at time t is the convolution of the initial data u_0 with the *Gaussian kernel* $e^{-(x-s)^2/4t}/\sqrt{4\pi t}$, whose integral is 1. Heat is conserved ($\|u(t)\|_1 = \|u_0\|_1$ for all $t>0$), but it diffuses over a range of order $\sqrt{t}$, the width of the Gaussian. At time t, any structures of wavelengths shorter than $O(\sqrt{t})$ will have been smoothed away. Since the tail of the Gaussian is never zero, on the other hand, a small amount of information propagates unboundedly fast, in contrast to the situation for a hyperbolic PDE.

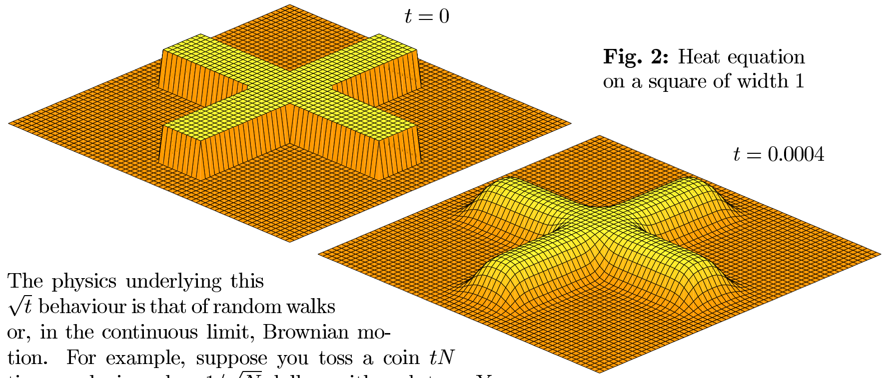


Fig. 2: Heat equation on a square of width 1

The physics underlying this $\sqrt{t}$ behaviour is that of random walks or, in the continuous limit, Brownian motion. For example, suppose you toss a coin tN times and win or lose $1/\sqrt{N}$ dollars with each toss. Your profit follows a binomial distribution that converges to the Gaussian $e^{-x^2/4t}/\sqrt{4\pi t}$ in the limit $N \to \infty$. Arbitrarily large profits are possible, but anything much bigger than $\sqrt{t}$ is very unlikely. This $\sqrt{t}$ effect is at the root of much of the field of statistics, and it was the basis of Einstein's epochal paper on Brownian motion in his *annus mirabilis* 1905.

It seems obvious that the solution to (1) should be unique, but in fact it is not. There are other solutions besides (4) in which an infinite amount of heat floods in from infinity just after $t=0$. For example, if $g(t) = \exp(-t^{-2})$, then the power series $\sum_{k=0}^{\infty}(\mathrm{d}^k g(t)/\mathrm{d}t^k)\,x^{2k}/(2k)!$ converges for each $t>0$ to an analytic function of x that satisfies (1) with initial data $u_0 = 0$. However, uniqueness for (1) is achieved if we require that $|u(x,t)|$ is bounded as $|x| \to \infty$. In fact, it is enough to require that $u(x,t)$ or $-u(x,t)$ is bounded, and thus, for example, if $u_0(x) \ge 0$, then (1) has a unique solution with $u(x,t) \ge 0$.

The heat equation is the canonical smoothing process, and as a application of this property we can prove the *Weierstrass approximation theorem:* a continuous function f on $[-1,1]$ can be approximated to within any error ε by a polynomial p. Take f, extended continuously to a function of compact support on $\mathbb{R}$, as initial data u_0 for (1). Let a sufficiently small time t elapse so that $|u(x,t) - u_0(x)| < \varepsilon/2$ everywhere. The function $u(x,t)$ is an entire function of x, i.e., analytic for all x. Now take $p(x)$ to be a truncation of the power series for $u(x,t)$ to enough terms so that $|p(x) - u(x,t)| < \varepsilon/2$ for all $x \in [-1,1]$.

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7.2 Energy for the heat equation

We next consider the (inhomogeneous) heat equation with some auxiliary conditions, and use the energy method to show that the solution satisfying those conditions must be unique. Consider the following mixed initial-boundary value problem, which is called the *Dirichlet problem for the heat equation*

$$\begin{cases} u_t - ku_{xx} = f(x,t) \qquad \text{for} \quad 0 \leq x \leq l, \quad t > 0 \\ u(x,0) = \phi(x), \\ u(0,t) = g(t), \quad u(l,t) = h(t), \end{cases} \tag{7.4}$$

for given functions f, ϕ, g, h.

Example 7.2. Show that there is at most one solution to the Dirichlet problem (7.4).

Just as in the case of the wave equation, we argue from the inverse by assuming that there are two functions, u, and v, that both solve the inhomogeneous heat equation and satisfy the initial and Dirichlet boundary conditions of (7.4). Then their difference, $w = u - v$, satisfies the homogeneous heat equation with zero initial-boundary conditions, i.e.

$$\begin{cases} w_t - kw_{xx} = 0 \qquad \text{for} \quad 0 \leq x \leq l, \quad t > 0 \\ w(x,0) = 0, \\ u(0,t) = 0, \quad u(l,t) = 0, \end{cases} \tag{7.5}$$

Now define the following "energy"

$$E[w](t) = \frac{1}{2}\int_0^l [w(x,t)]^2\,dx, \tag{7.6}$$

which is always positive, and decreasing, if w solves the heat equation. Indeed, differentiating the energy with respect to time, and using the heat equation we get

$$\frac{d}{dt}E = \int_0^l ww_t\,dx = k\int_0^l ww_{xx}\,dx.$$

Integrating by parts in the last integral gives

$$\frac{d}{dt}E = kww_x\Big|_0^l - \int_0^l w_x^2\,dx \leq 0,$$

since the boundary terms vanish due to the boundary conditions in (7.5), and the integrand in the last term is nonnegative.

Due to the initial condition in (7.5), the energy at time $t = 0$ is zero. But then using the fact that the energy is a nonnegative decreasing quantity, we get

$$0 \leq E[w](t) \leq E[w](0) = 0.$$

Hence,

$$\frac{1}{2}\int_0^l [w(x,t)]^2\,dx = 0, \quad \text{for all } t \geq 0,$$

which implies that the nonnegative continuous integrand must be identically zero over the integration interval, i.e $w \equiv 0$, for all $x \in [0,l], t > 0$. Hence also

$$u_1 \equiv u_2,$$

which finishes the proof of uniqueness. □

The energy (7.6) arises if one multiplies the heat equation by w and integrates in x over the interval $[0,l]$. Then realizing that the first term will be the time derivative of the energy, and performing the same integration by parts on the second term as above, we can reprove that this energy is decreasing.

Notice that all of the above arguments hold for the case of the infinite interval $-\infty < x < \infty$ as well. In this case one ignores the effect of the infinitely far endpoints and considers an IVP.

7.3 Conclusion

Using the energy motivated by the vibrating string model behind the wave equation, we derived a conserved quantity, which corresponds to the total energy of motion for the infinite string. This positive definite quantity was then used to prove uniqueness of solution to the wave IVP via the energy method, which essentially asserts that zero initial total energy precludes any (nonzero) dynamics. A similar approach was used to prove uniqueness for the heat IBVP, concluding that zero initial heat implies steady zero temperatures at later times.

Problem Set 4

1. (#2.1.1 in [Str]) Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

2. (#2.1.5 in [Str]) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c, a/c, 3a/2c, 2a/c$, and $5a/c$. [*Hint:* Calculate

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2c}\{\text{length of } (x-ct, x+ct) \cap (-a,a)\}.$$

Then $u(x, a/2c) = (1/2c)\{\text{length of } (x - a/2, x + a/2) \cap (-a.a)\}$. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]

3. (#2.1.7 in [Str]) If both ϕ and ψ are odd functions of x, show that the solution $u(x,t)$ of the wave equation is also odd in x for all t.

4. (#2.1.9 in [Str]) Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (*Hint:* Factor the operator as the textbook does for the wave equation.)

5. (#2.2.3 in [Str]) Show that the wave equation has the following invariance properties.
   (a) Any translate $u(x-y,t)$, where y is fixed, is also a solution.
   (b) Any derivative, say u_x, of a solution is also a solution.
   (c) The dilated function $u(ax,at)$ is also a solution, for any constant a.

6. (#2.2.4 in [Str]) If $u(x,t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

$$u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h)$$

for all x, t, h and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.

7. (#2.2.5 in [Str]) For the damped string, $u_{tt} - c^2 u_{xx} + r u_t = 0, r > 0$, show that the energy decreases.

8 Heat equation: properties

We would like to solve the heat (diffusion) equation

$$u_t - ku_{xx} = 0, \tag{8.1}$$

and obtain a solution formula depending on the given initial data, similar to the case of the wave equation. However the methods that we used to arrive at d'Alambert's solution for the wave IVP do not yield much for the heat equation. To see this, recall that the heat equation is of parabolic type, and hence, it has only one family of characteristic lines. If we rewrite the equation in the form

$$ku_{xx} + \cdots = 0,$$

where the dots stand for the lower order terms, then you can see that the coefficients of the leading order terms are

$$A = k, \quad B = C = 0.$$

The slope of the characteristic lines are then

$$\frac{dt}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{B}{2A} = 0.$$

Consequently, the single family of characteristic lines will be given by

$$t = c.$$

These characteristic lines are not very helpful, since they are parallel to the x axis. Thus, one cannot trace points in the xt plane along the characteristics to the x axis, along which the initial data is defined. Notice that there is also no way to factor the heat equation into first order equations, either, so the methods used for the wave equation do not shed any light on the solutions of the heat equation.

Instead, we will study the properties of the heat equation, and use the gained knowledge to devise a way of reducing the heat equation to an ODE, as we have done for every PDE we have solved so far.

8.1 The maximum principle

The first properties that we need to make sure of, are the uniqueness and stability for the solution of the problem with certain auxiliary conditions. This would guarantee that the problem is wellpossed, and the chosen auxiliary conditions do not break the physicality of the problem. We begin by establishing the following property, that will be later used to prove uniqueness and stability.

Maximum Principle. If $u(x,t)$ satisfies the heat equation (8.1) in the rectangle $R = \{0 \leq x \leq l, 0 \leq t \leq T\}$ in space-time, then the maximum value of $u(x,t)$ over the rectangle is assumed either initially ($t = 0$), or on the lateral sides ($x = 0$, or $x = l$).

Mathematically, the maximum principle asserts that the maximum of $u(x,t)$ over the three sides must be equal to the maximum of the $u(x,t)$ over the entire rectangle. If we denote the set of points comprising the three sides by $\Gamma = \{(x,t) \in R \mid t = 0 \quad \text{or} \quad x = 0 \quad \text{or} \quad x = l\}$, then the maximum principle can be written as

$$\max_{(x,t)\in\Gamma} \{u(x,t)\} = \max_{(x,t)\in R} \{u(x,t)\}. \tag{8.2}$$

If you think of the heat conduction phenomena in a thin rod, then the maximum principle makes physical sense, since the initial temperature, as well as the temperature at the endpoints will dissipate through conduction of heat, and at no point the temperature can rise above the highest initial or endpoint temperature. In fact, a stronger version of the maximum principle holds, which asserts that the maximum over the rectangle R can not be attained at a point not belonging to Γ, unless $u \equiv$ constant, i.e. for nonconstant solutions the following strict inequality holds

$$\max_{(x,t)\in R\setminus\Gamma} \{u(x,t)\} < \max_{(x,t)\in R} \{u(x,t)\},$$

where $R \setminus \Gamma$ is the set of all points of R that are not in Γ (difference of sets). This makes physical sense as well, since the heat from the point of highest initial or boundary temperature will necessarily transfer to points of lower temperature, thus decreasing the highest temperature of the rod.

We finally note, that the maximum principle also implies a *minimum principle*, since one can apply it to the function $-u(x,t)$, which also solves the heat equation, and make use of the fact that

$$\min\{u(x,t)\} = -\max\{-u(x,t)\}.$$

Thus, the minima points of the function $u(x,t)$ will exactly coincide with the maxima points of $-u(x,t)$, of which, by the maximum principle, there must necessarily be in Γ.

Proof of the maximum principle. If the maximum of the function $u(x,t)$ over the rectangle R is assumed at an internal point (x_0,t_0), then the gradient of u must vanish at that point, i.e. $u_t(x_0,t_0) = u_x(x_0,t_0) = 0$. If in addition we had the strict inequality $u_{xx}(x_0,t_0) < 0$, then one would get a contradiction by plugging the point (x_0,t_0) into the heat equation. Indeed, we would have

$$u_t(x_0,t_0) - ku_{xx}(x_0,t_0) = -ku_{xx}(x_0,t_0) > 0.$$

This contradicts the heat equation (8.1), which must hold for all values of x and t. Thus, the contradiction would imply that the maximum point (x_0,t_0) cannot be an internal point. However, from the second derivative test we have the weaker inequality $u_{xx}(x_0,t_0) \leq 0$ (the point would not be a maximum if $u_{xx}(x_0,t_0) > 0$), which is not enough for this argument to go through.

The way out, is to recycle the above argument with a slight modification to the function u. Define a new function

$$v(x,t) = u(x,t) + \epsilon x^2, \tag{8.3}$$

where $\epsilon > 0$ is a constant that can be taken as small as one wants. Now let M be the maximum value of u over the three sides, which we denoted by Γ above. That is

$$M = \max_{(x,t)\in\Gamma} \{u(x,t)\}.$$

To prove the maximum principle, we need to establish (8.2). The maximum over Γ is always less than or equal to the maximum over R, since $\Gamma \subset R$. So we only need to show the opposite inequality, which would follow from showing that

$$u(x,t) \leq M, \qquad \text{for all the points } (x,t) \in R. \tag{8.4}$$

Notice that from the definition of v, we have that at the points of Γ, $v(x,t) \leq M + \epsilon l^2$, since the maximum value of ϵx^2 on Γ is ϵl^2. Then, instead of proving inequality (8.4), we will prove that

$$v(x,t) \leq M + \epsilon l^2, \qquad \text{for all the points } (x,t) \in R, \tag{8.5}$$

which implies (8.4). Indeed, from the definition of v in (8.3), we have that in the rectangle R

$$u(x,t) \leq v(x,t) - \epsilon x^2 \leq M + \epsilon(l^2 - x^2),$$

where we used (8.5) to bound $v(x,t)$. Now, since the point (x,t) is taken from the rectangle R, we have that $0 \leq x \leq l$, and the difference $l^2 - x^2$ is bounded. But then the right hand side of the above inequality can be made as close to M as possible by taking ϵ small enough, which implies the bound (8.4).

If we formally apply the heat operator to the function v, and use the definition (8.3), we will get

$$v_t - kv_{xx} = u_t - k(u_{xx} + 2\epsilon) = (u_t - ku_{xx}) - 2k\epsilon < 0,$$

since both $k, \epsilon > 0$, and u satisfies the heat equation (8.1) on R. Thus, v satisfies the *heat inequality* in R

$$v_t - kv_{xx} < 0. \tag{8.6}$$

We now recycle the above argument, which barely failed for u, applying it to v instead. Suppose $v(x,t)$ attains its maximum value at an internal point (x_0,t_0). Then necessarily $v_t(x_0,t_0)=0$, and $v_{xx}(x_0,t_0)\le 0$. Hence, at this point we have

$$v_t(x_0,t_0)-kv_{xx}(x_0,t_0)=-kv_{xx}(x_0,t_0)\ge 0,$$

which contradicts the heat inequality (8.6). Thus, v cannot have an internal maximum point in R.

Similarly, suppose that $v(x,t)$ attains its maximum value at a point (x_0,t_0) on the fourth side of the rectangle R, i.e. when $t_0=T$. Then we still have that $v_x(x_0,t_0)=0$, and $v_{xx}(x_0,t_0)\le 0$, but $v_t(x_0,t_0)$ does not have to be zero, since $t_0=T$ is an endpoint in the t direction. However, from the definition of the derivative, and our assumption that (x_0,t_0) is a point of maximum, we have

$$v_t(x_0,t_0)=\lim_{\delta\to 0+}\frac{v(x_0,t_0)-v(x_0,t_0-\delta)}{\delta}\ge 0.$$

So at this point we still have

$$v_t(x_0,t_0)-kv_{xx}(x_0,t_0)\ge 0,$$

which again contradicts the heat inequality (8.6).

Now, since the continuous function $v(x,t)$ must attain its maximum value somewhere in the closed rectangle R, this must happen on one of the remaining three sides, which comprise the set Γ. Hence,

$$v(x,t)\le \max_{(x,t)\in R}\{v(x,t)\}=\max_{(x,t)\in\Gamma}\{v(x,t)\}\le M+\epsilon l^2,$$

which finishes the proof of (8.5). □

8.2 Uniqueness

Consider the Dirichlet problem for the heat equation,

$$\begin{cases} u_t-ku_{xx}=f(x,t) \qquad \text{for} \quad 0\le x\le l, \quad t>0\\ u(x,0)=\phi(x),\\ u(0,t)=g(t), \quad u(l,t)=h(t), \end{cases} \tag{8.7}$$

for given functions f,ϕ,g,h. We will use the maximum principle to show uniqueness and stability for the solutions of this problem (recall that last time we used the energy method to prove uniqueness for this problem).

Uniqueness of solutions. There is at most one solution to the Dirichlet problem (8.7).

Indeed, arguing from the inverse, suppose that there are two functions, u, and v, that both solve the inhomogeneous heat equation and satisfy the initial and Dirichlet boundary conditions of (8.7). Then their difference, $w=u-v$, satisfies the homogeneous heat equation with zero initial-boundary conditions, i.e.

$$\begin{cases} w_t-kw_{xx}=0 \qquad \text{for} \quad 0\le x\le l, \quad t>0\\ w(x,0)=0,\\ u(0,t)=0, \quad u(l,t)=0, \end{cases} \tag{8.8}$$

But from the maximum principle, we know that w assumes its maximum and minimum values on one of the three sides $t=0$, $x=0$, and $x=l$. And since $w=0$ on all of these three sides from the initial and boundary conditions in (8.8), we have that for $x\in[0,l], t>0$

$$0\le w\le 0 \quad\Rightarrow\quad w(x,t)\equiv 0.$$

Hence,

$$u-v=w\equiv 0, \qquad \text{or} \qquad u\equiv v,$$

and the solution must indeed be unique.

Notice again that all of the above arguments hold for the case of the infinite interval $-\infty<x<\infty$ as well. In this case one ignores the effect of the infinitely far endpoints and considers an IVP. And the maximum principle simply asserts that the maximum of the solutions must be attained initially. We will use this in the next lecture when deriving the solution for the IVP for the heat equation on the entire real line $x\in\mathbb{R}$.

8.3 Stability

Stability of solutions with respect to the auxiliary conditions is the third ingredient of well-posedness, after existence and uniqueness. It asserts that *close* auxiliary conditions lead to *close* solutions. One may have different ways of measuring the closeness of the solutions, and the initial and boundary data. Consider two solutions, u_1, u_2, of the heat equation (8.1) for $x \in [0, l], t > 0$, which satisfy the following initial-boundary conditions

$$\begin{cases} u_1(x,0) = \phi_1(x), \\ u_1(0,t) = g_1(t), \quad u_1(l,t) = h_1(t), \end{cases} \qquad \begin{cases} u_2(x,0) = \phi_2(x), \\ u_2(0,t) = g_2(t), \quad u_2(l,t) = h_2(t). \end{cases} \tag{8.9}$$

Stability of solutions means that *closeness* of ϕ_1 to ϕ_2, g_1 to g_2 and h_1 to h_2 implies the closeness of u_1 to u_2. Notice that the difference $w = u_1 - u_2$ solves the heat equation as well, and satisfies the following initial-boundary conditions

$$\begin{cases} w_1(x,0) = \phi_1(x) - \phi_2(x), \\ w(0,t) = g_1(t) - g_2(t), \quad w(l,t) = h_1(t) - h_2(t). \end{cases}$$

But then the maximum and minimum principles imply

$$-\max_{(x,t)\in\Gamma}\{|w(x,t)|\} \le \max_{\substack{0\le x\le l \\ t\ge 0}}\{w(x,t)\} \le \max_{(x,t)\in\Gamma}\{|w(x,t)|\},$$

and hence, the absolute value of the difference $u_1 - u_2$ will be bounded by

$$\begin{aligned} \max_{\substack{0\le x\le l \\ t\ge 0}}\{|u_1(x,t) - u_2(x,t)|\} = \max_{\substack{0\le x\le l \\ t\ge 0}}\{|w(x,t)|\} &\le \max_{(x,t)\in\Gamma}\{|w(x,t)|\} \\ &= \max_{\substack{0\le x\le l \\ t\ge 0}}\{|\phi_1(x) - \phi_2(x)|, |g_1(t) - g_2(t)|, |h_1(t) - h_2(t)|\}. \end{aligned}$$

Thus, the smallness of the maximum of the differences $|\phi_1 - \phi_2|$, $|g_1 - g_2|$ and $|h_1 - h_2|$ implies the smallness of the maximum of the difference of solutions $|u_1 - u_2|$. In this case the stability is said to be in the *uniform* sense, i.e. smallness is understood to hold uniformly in the (x, t) variables.

An alternate way of showing the stability is provided by the energy method. Suppose u_1 and u_2 solve the heat equation with initial data ϕ_1 and ϕ_2 respectively, and zero boundary conditions. This would be the case for the problem over the entire real line $x \in \mathbb{R}$, or if $g_1 = g_2 = h_1 = h_2 = 0$ in (8.9). In this case the energy method for the difference $w = u_1 - u_2$ implies that $E[w](t) \le E[w](0)$ for all $t \ge 0$, or

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2\,dx \le \int_0^l [\phi_1(x) - \phi_2(x)]^2\,dx, \qquad \text{for all} \quad t \ge 0.$$

Thus the closeness of ϕ_1 to ϕ_2 in the sense of the square integral of the difference implies the closeness of the respective solutions in the same sense. This is called stability in the *square integral* (L^2) sense.

8.4 Conclusion

As expected, the method of characteristics is inefficient for solving the heat equation. We then need to find an alternative method of reducing the equation to an ODE. But before embarking on this path, we first study the properties of the heat equation, which will serve as beacons in the later reduction to an ODE. Today we established the maximum principle for the heat equation, which immediately implied the uniqueness and stability for the solution. Next time we will look at the invariance properties of the equation and derive the solution using these properties.

9 Heat equation: solution

Equipped with the uniqueness property for the solutions of the heat equation with appropriate auxiliary conditions, we will next present a way of deriving the solution to the heat equation

$$u_t - ku_{xx} = 0. \tag{9.1}$$

Considering the equation on the entire real line $x \in \mathbb{R}$ simplifies the problem by eliminating the effect of the boundaries, we will first concentrate on this case, which corresponds to the dynamics of the temperature in a rod of infinite length. We want to solve the IVP

$$\begin{cases} u_t - ku_{xx} = 0 & (-\infty < x < \infty, 0 < t < \infty), \\ u(x,0) = \phi(x). \end{cases} \tag{9.2}$$

Since the solution to the above IVP is not easy to derive directly, unlike the case of the wave IVP, we will first derive a particular solution for a special simple initial data, and try to produce solutions satisfying all other initial conditions by exploiting the invariance properties of the heat equation.

9.1 Invariance properties of the heat equation

The heat equation (9.1) is invariant under the following transformations

(a) Spatial translations: If $u(x,t)$ is a solution of (9.1), then so is the function $u(x-y,t)$ for any fixed y.

(b) Differentiation: If u is a solution of (9.1), then so are u_x, u_t, u_{xx} and so on.

(c) Linear combinations: If $u_1, u_2, \ldots, u_n$ are solutions of (9.1), then so is $u = c_1u_1 + c_2u_2 + \ldots c_nu_n$ for any constants $c_1, c_2, \ldots, c_n$.

(d) Integration: If $S(x,t)$ is a solution of (9.1), then so is the integral

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

for any function $g(y)$, as long as the improper integral converges (we will ignore the issue of the convergence for the time being).

(e) Dilation (scaling): If $u(x,t)$ is a solution of (9.1), then so is the dilated function $v(x,t) = u(\sqrt{a}x, at)$ for any constant $a > 0$ (compare this to the scaling property of the wave equation, which is invariant under the dilation $u(x,t) \mapsto u(ax,at)$ for all $a \in \mathbb{R}$).

Properties (a), (b) and (c) are trivial (check by substitution), while property (d) is the limiting case of property (c). Indeed, if we use the notation $u^y(x,t) = S(x-y,t)$, and $c^y = g(y)\Delta y$, then u^y is also a solution by property (a), and we have the formal limit

$$\int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy = \lim_{\Delta y \to 0} \sum_y c^y u^y.$$

To make this precise, we need to consider a finite interval of integration, which is partitioned by points $\{y_i\}_{i=1}^n$ into subintervals of length Δy, and use the definition of the integral as the limit of the corresponding Riemann sum to write

$$\int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy = \lim_{b\to\infty} \int_{-b}^{b} S(x-y,t)g(y)\,dy = \lim_{b\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} S(x-y_i)g(y_i)\Delta y,$$

where $-b = y_1 < y_2 < \ldots y_n = b$ is a partition of the interval $[-b,b]$.

Finally, property (e) can be checked by direct substitution as well. Notice that we cannot formally reverse the time by dilating with the factor $a = -1$, as was the case for the wave equation, since the $\sqrt{a}$ factor in front of the x argument would make the dilated function complex, which is not allowed in the theory of real PDEs (what is the meaning of complex valued temperature?!). We will later see that the heat equation is indeed time irreversible.

9.2 Solving a particular IVP

As a special initial data we take the following function

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{9.3}$$

which is called the Heaviside step function. We consider the IVP

$$\begin{cases} Q_t - kQ_{xx} = 0 \quad (-\infty < x < \infty, 0 < t < \infty), \\ Q(x,0) = H(x). \end{cases} \tag{9.4}$$

We will solve this IVP in successive steps.

Step 1: Reduction to an ODE. Notice that the Heaviside function (9.3) is invariant under the dilation $x \mapsto \sqrt{a}x$, i.e. $H(\sqrt{a}x) = H(x)$. From the dilation property of the heat equation, we know that $Q(\sqrt{a}x, at)$ also solves the heat equation. But $Q(\sqrt{a}x, 0) = H(\sqrt{a}x) = H(x)$, thus $Q(\sqrt{a}x, at)$ and $Q(x,t)$ both solve the IVP (9.4). The uniqueness of solutions then implies that $Q(\sqrt{a}x, at) = Q(x,t)$ for all $x \in \mathbb{R}, t > 0$, so Q is invariant under the dilation $(x,t) \mapsto (\sqrt{a}x, at)$ as well.

Due to this invariance, Q can depend only on the ratio $\frac{x}{\sqrt{t}}$, that is $Q(x,t) = q(\frac{x}{\sqrt{t}})$. To see this, define the function q in the following way $q(z) = Q(z,1)$. But then for fixed (x,t), we have

$$Q(x,t) = Q\left(\frac{1}{\sqrt{t}}x, \frac{1}{t}t\right) = Q\left(\frac{x}{\sqrt{t}}, 1\right) = q\left(\frac{x}{\sqrt{t}}\right).$$

Thus Q is completely determined by the function of one variable q.

For convenience of future calculations we pass to the function $g(z) = q(\sqrt{4k}z)$, so that

$$Q(x,t) = q\left(\frac{x}{\sqrt{t}}\right) = g\left(\frac{x}{\sqrt{4kt}}\right) = g(p),$$

where we used the notation $p = x/\sqrt{4kt}$. We next compute the derivatives of Q in terms of g, and substitute them into the heat equation in order to obtain an ODE for g. Using the chain rule, one gets

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{4k}{2}\frac{x}{(\sqrt{4kt})^3}g'(p) = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p), \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p), \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial t} = \frac{1}{4kt}g''(p). \end{aligned}$$

The heat equation then implies

$$0 = Q_t - kQ_{xx} = \frac{1}{4t}[-2pg'(p) - g''(p)],$$

which gives the following equation for g

$$g'' + 2pg' = 0. \tag{9.5}$$

Step 2: Solving the ODE. Using the integrating factor $\exp(\int 2p\,dp) = e^{p^2}$, the ODE (9.5) reduces to

$$[e^{p^2}g'(p)]' = 0.$$

Thus, we have

$$e^{p^2} g'(p) = c_1.$$

Solving for $g'(p)$, and integrating, we obtain

$$g(p) = c_1 \int e^{-p^2}\, dp + c_2.$$

Step 3: Checking the initial condition. Recalling that $Q(x,t) = g(p)$, where $p = x/\sqrt{4kt}$, we obtain the following explicit formula for Q

$$Q(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp + c_2. \tag{9.6}$$

Notice that we chose a particular antiderivative, which we are free to do due to the presence of the arbitrary constants. Also note that the above formula is only valid for $t > 0$, so to check the initial condition, we need to take the limit $t \to 0+$. Recalling the initial condition from (9.4), we have that,

$$\begin{aligned}
&\text{if} \quad x > 0, \qquad 1 = \lim_{t\to 0+} Q(x,t) = c_1 \int_0^{+\infty} e^{-p^2}\, dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2,\\
&\text{if} \quad x < 0, \qquad 0 = \lim_{t\to 0+} Q(x,t) = c_1 \int_0^{-\infty} e^{-p^2}\, dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2,
\end{aligned}$$

where we used the fact that $\int_0^\infty e^{-p^2}\, dp = \sqrt{\pi}/2$ to compute the improper integrals. The above identities give

$$c_1 \frac{\sqrt{\pi}}{2} + c_2 = 1, \qquad -c_1 \frac{\sqrt{\pi}}{2} + c_2 = 0.$$

Solving for c_1 and c_2, we get $c_1 = 1/\sqrt{\pi}$ and $c_2 = 1/2$. Substituting these into (9.6) gives the unique solution of the IVP (9.4),

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp, \qquad \text{for} \quad t > 0. \tag{9.7}$$

9.3 Solving the general IVP

Returning to the general IVP (9.2), we would like to derive a solution formula, which will express the solution to the IVP in terms of the initial data (similar to d'Alambert's solution for the wave equation).

We first define the function

$$S(x,t) = \frac{\partial Q}{\partial x}(x,t), \tag{9.8}$$

where $Q(x,t)$ is the solution to the particular IVP (9.4), and is given by (9.7). Then, by the invariance properties of the heat equation, $S(x,t)$ also solves the heat equation, and so does

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy, \qquad \text{for} \quad t > 0. \tag{9.9}$$

We claim that this u is the unique solution of the IVP (9.2). To verify this claim one only needs to check the initial condition of (9.2). Notice that using $S = Q_x$, we can rewrite u as follows

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\, dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)\, dy$$

Integrating by parts in the last integral, we get

$$u(x,t) = -Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy.$$

We assume that the boundary terms vanish, which can be guaranteed for example by assuming that $\phi(y)$ vanishes for large $|y|$ (this is not strictly necessary, since $S(x-y,t)$ decays rapidly as $|y-x|$ becomes large, as we will shortly see). Now plugging in $t=0$, and using that Q has the Heaviside function (9.3) as its initial data, we have

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,dy = \int_{-\infty}^{x} \phi'(y)\,dy = \phi(y)\Big|_{y=-\infty}^{y=x} = \phi(x).$$

So $u(x,t)$ indeed satisfies the initial condition of (9.2).

We can compute $S(x,t)$ from (9.8), which will give

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}, \tag{9.10}$$

Using this expression of $S(x,t)$, we can now rewrite the solution given by (9.9) in the following explicit form

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy, \qquad \text{for} \quad t>0. \tag{9.11}$$

The function $S(x,t)$ is known as the *heat kernel*, *fundamental solution*, *source function*, *Green's function*, or *propagator* of the heat equation. Notice that it gives a way of *propagating* the initial data ϕ to later times, giving the solution at any time $t>0$.

It is clear that formula (9.11) does not make sense for $t=0$, although one can compute the limit of $u(t,x)$ as $t \to 0+$ in that formula, which will give an alternate way of checking the initial condition of (9.2).

9.4 Conclusion

We derived the solution to the heat equation by first looking at a particular initial data, which was invariant under dilation. This guaranteed that the solution corresponding to this initial data is also dilation invariant, which reduced the heat equation to an ODE. After solving this ODE, and obtaining the solution, we saw that the solution to the general heat IVP can be written in an integral form using this particular solution. Next time we will explore the solution given by formula (9.11), and will study its qualitative behavior.

Problem Set 5

1. (#2.3.1 in [Str]) Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1, 0 \le t \le T\}$.

2. (#2.3.5 in [Str]) The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.

   (a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2, 0 \le t \le 1\}$.

   (b) Where precisely does the proof of the maximum principle break down for this equation?

3. (#2.3.6 in [Str]) Prove the comparison principle for the diffusion equation: If u and v are two solutions, and if $u \le v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \le v$ for $0 \le t < \infty, 0 \le x \le l$.

4. (#2.3.8 in [Str]) Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0,t) - a_0 u(0,t) = 0$, and $u_x(l,t) + a_l u(l,t) = 0$. If $a_0 > 0$, and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")

5. (#2.4.1 in [Str]) Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

   Write your answer in terms of $\mathcal{E}\text{rf}(x)$.

6. (#2.4.6 in [Str]) Compute $\int_0^\infty e^{-x^2}\,dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)

7. (#2.4.9 in [Str]) Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x,0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.

10 Heat equation: interpretation of the solution

Last time we considered the IVP for the heat equation on the whole line

$$\begin{cases} u_t - ku_{xx} = 0 \quad (-\infty < x < \infty, 0 < t < \infty), \\ u(x,0) = \phi(x), \end{cases} \tag{10.1}$$

and derived the solution formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy, \qquad \text{for} \quad t > 0, \tag{10.2}$$

where $S(x,t)$ is the heat kernel,

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}. \tag{10.3}$$

Substituting this expression into (10.2), we can rewrite the solution as

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy, \qquad \text{for} \quad t > 0. \tag{10.4}$$

Recall that to derive the solution formula we first considered the heat IVP with the following particular initial data

$$Q(x,0) = H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{10.5}$$

Then using dilation invariance of the Heaviside step function $H(x)$, and the uniqueness of solutions to the heat IVP on the whole line, we deduced that Q depends only on the ratio $x/\sqrt{t}$, which lead to a reduction of the heat equation to an ODE. Solving the ODE and checking the initial condition (10.5), we arrived at the following explicit solution

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp, \qquad \text{for} \quad t > 0. \tag{10.6}$$

The heat kernel $S(x,t)$ was then defined as the spatial derivative of this particular solution $Q(x,t)$, i.e.

$$S(x,t) = \frac{\partial Q}{\partial x}(x,t), \tag{10.7}$$

and hence it also solves the heat equation by the differentiation property.

The key to understanding the solution formula (10.2) is to understand the behavior of the heat kernel $S(x,t)$. To this end some technical machinery is needed, which we develop next.

10.1 Dirac delta function

Notice that, due to the discontinuity in the initial data of Q, the derivative $Q_x(x,t)$, which we used in the definition of the function S in (10.7), is not defined in the traditional sense when $t = 0$. So how can one make sense of this derivative, and what is the initial data for $S(x,t)$?

It is not difficult to see that the problem is at the point $x = 0$. Indeed, using that $Q(x,0) = H(x)$ is constant for any $x \neq 0$, we will have $S(x,0) = 0$ for all x different from zero. However, $H(x)$ has a jump discontinuity at $x = 0$, as is seen in Figure 10.1, and one can imagine that at this point the rate of growth of H is infinite. Then the "derivative"

$$\delta(x) = H'(x) \tag{10.8}$$

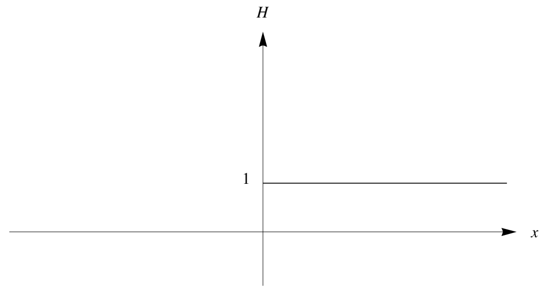


Figure 10.1: The graph of the Heaviside step function.

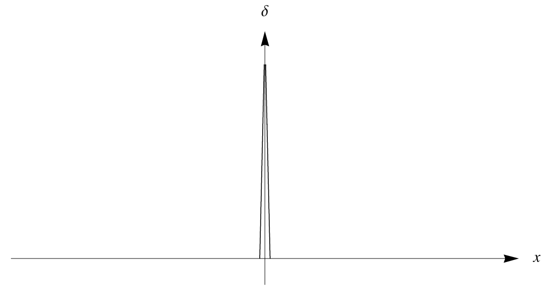


Figure 10.2: The sketch of the Dirac δ function.

is zero everywhere, except at $x = 0$, where it has a spike of zero width and infinite height. Refer to Figure 10.2 below for an intuitive sketch of the graph of δ. Of course, δ is not a function in the traditional sense, but is rather a *generalized function*, or *distribution*. Unlike regular functions, which are characterized by their finite values at every point in their domains, distributions are characterized by how they act on regular functions.

To make this rigorous, we define the set of *test functions* $\mathcal{D} = C_c^\infty$, the elements of which are smooth functions with compact support. So $\phi \in \mathcal{D}$, if and only if ϕ has continuous derivatives of any order $k \in \mathbb{N}$, and the closure of the support of ϕ,

$$\mathrm{supp}(\phi) = \{x \in \mathbb{R} \mid \phi(x) \neq 0\},$$

is compact. Recall that compact sets in $\mathbb{R}$ are those that are closed and bounded. In particular for any test function ϕ there is a rectangle $[-R, R]$, outside of which ϕ vanishes. Notice that derivatives of test functions are also test functions, as are sums, scalar multiples and products of test functions.

Distributions are continuous linear *functionals* on $\mathcal{D}$, that is, they are continuous linear maps from $\mathcal{D}$ to the real numbers $\mathbb{R}$. Notice that for any regular function f, we can define the functional

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx, \tag{10.9}$$

which makes f into a distribution, since to every $\phi \in \mathcal{D}$ it assigns the number $\int_{-\infty}^{\infty} f(x)\phi(x)\,dx$. This integral will converge under very weak conditions on f ($f \in L^1_{loc}$), due to the compact support of ϕ. In particular, f can certainly have jump discontinuities. Notice that we committed an abuse of notation to identify the distribution associated with f by the same letter f. The particular notion in which we use the function will be clear from the context.

One can also define the *distributional derivative* of f to be the distribution, which acts on the test functions as follows

$$f'[\phi] = -\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx.$$

Notice that integration by parts and the compact support of test functions makes this definition consistent with the regular derivative for differentiable functions (check that the distribution formed as in (10.9) by the derivative of f coincides with the distributional derivative of f).

We can also apply the notion of the distributional derivative to the Heaviside step function $H(x)$, and think of the definition (10.8) in the sense of distributional derivatives. Let us now compute how δ, called the *Dirac delta function*, acts on test functions. By the definition of the distributional derivative,

$$\delta[\phi] = -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx.$$

Recalling the definition of $H(x)$ in (10.5), we have that

$$\delta[\phi] = -\int_0^{\infty} \phi'(x)\,dx = -\phi(x)\Big|_0^{\infty} = \phi(0). \tag{10.10}$$

Thus, the Dirac delta function maps test functions to their values at $x = 0$. We can make a translation in the x variable, and define $\delta(x-y) = H'(x-y)$, i.e. $\delta(x-y)$ is the distributional derivative of the distribution formed by the function $H(x-y)$. Then it is not difficult to see that $\delta(x-y)[\phi] = \phi(y)$. That is, $\delta(x-y)$ maps test functions to their values at y. We will make the abuse of notation mentioned above, and write this as

$$\int_{-\infty}^{\infty} \delta(x-y)\phi(x)\,dx = \phi(y).$$

We also note that $\delta(x-y) = \delta(y-x)$, since δ is even, if we think of it as a regular function with a centered spike (one can prove this from the definition of δ as a distribution).

Using these new notions, we can make sense of the initial data for $S(x,y)$. Indeed,

$$S(x,0) = \delta(x). \tag{10.11}$$

Since the initial data is a distribution, one then thinks of the equation to be in the sense of distributions as well, that is, treat the derivatives appearing in the equation as distributional derivatives. This requires the generalization of the idea of a distribution to two dimensions. We call this type of solutions *weak solutions* (recall the solutions of the wave equation with discontinuous data). Thus $S(x,t)$ is a week solution of the heat equation, if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x,t)[\phi_t(x,t) - k\phi_{xx}(x,t)]\,dxdt = 0,$$

for any test function ϕ of two variables. This means that the distribution $(\partial_t - k\partial_x^2)S$, with the derivatives taken in the distributional sense, is the zero distribution. Notice that the weak solution $S(x,t)$ arising from the initial data (10.11) has the form (10.3), which is an infinitely differentiable function of x and t. This is in stark contrast to the case of the wave equation, where, as we have seen in the examples, the discontinuity of the initial data is preserved in time.

Having the δ function in our arsenal of tools, we can now give an alternate proof that (10.2) satisfies the initial conditions of (10.1). Directly plugging in $t = 0$ into (10.2), which we are now allowed to do by treating it as a distribution, and using (10.11), we get

$$u(x,0) = \int_{-\infty}^{\infty} \delta(x-y)\phi(y)\,dy = \phi(x).$$

10.2 Interpretation of the solution

Let us look at the solution (10.4) in detail, and try to understand how the heat kernel $S(x,t)$ propagates the initial data $\phi(x)$. Notice that $S(x,t)$, given by (10.3), is a well-defined function of (x,t) for any $t > 0$. Moreover, $S(x,t)$ is positive, is even in the x variable, and for a fixed t has a bell-shaped graph. In general, the function

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

is called *Gaussian function* or *Gaussian*. In the probability theory, it gives the density of the normal distribution with mean μ and standard deviation σ. The graph of the Gaussian is a bell-curve with its peak of height $1/\sqrt{2\pi\sigma^2}$ at $x = \mu$ and the width of the bell at mid-height roughly equal to 2σ. Thus, for some fixed time t the height of $S(x,t)$ at its peak $x = 0$ is $\frac{1}{\sqrt{4\pi kt}}$, which decays as t grows.

Notice that as $t \to 0+$, the height of the peak becomes arbitrarily large, and the width of the bell-curve, $\sqrt{2kt}$ goes to zero. This, of course, is expected, since $S(x,t)$ has the initial data (10.11). One can think of $S(x,t)$ as the temperature distribution at time t that arises from the initial distribution given by the Dirac delta function. With passing time the highest temperature at $x = 0$ gets gradually transferred to the other points of the rod. It also makes sense, that points closer to $x = 0$ will have higher temperature than those farther away. Graphs of $S(x,t)$ for three different times are sketched in Figure 10.3 below.

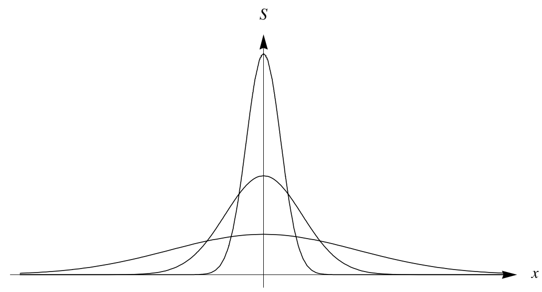


Figure 10.3: The graphs of the heat kernel at different times.

From the initial condition (10.11), we see that initially the temperature at every point $x \neq 0$ is zero, but $S(x,t) > 0$ for any x and $t > 0$. This means that heat is instantaneously transferred to all points of the rod (closer points get more heat), so the speed of heat conduction is infinite. Compare this to the finite speed of propagation for the wave equation. One can also compute the area below the graph of $S(x,t)$ at any time $t > 0$ to get

$$\int_{-\infty}^{\infty} S(x,t)\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp = 1,$$

where we used the change of variables $p = x/\sqrt{4kt}$. At $t = 0$, we have

$$\int_{-\infty}^{\infty} S(x,0)\,dx = \int_{-\infty}^{\infty} \delta(x)\,dx = 1,$$

where we think of the last integral as the δ distribution applied to the constant function 1 (more precisely, a test function that is equal to 1 in some open interval around $x = 0$). This shows that the area below the graph of $S(x,t)$ is preserved in time and is equal to 1, so for any fixed time $t \geq 0$, $S(x,t)$ can be thought of as a probability density function. At time $t = 0$ its the probability density that assigns probability 1 to the point $x = 0$, as was seen in (10.10), and for times $t > 0$ it is a normal distribution with mean $x = 0$ and standard deviation $\sigma = \sqrt{2kt}$ that grows with time. As we mentioned earlier, $S(x,t)$ is smooth, in spite of having a discontinuous initial data. We will see in the next lecture that this is true for any solution of the heat IVP (10.1) with general initial data.

We now look at the solution (10.4) with general data $\phi(x)$. First, notice that the integrand in (10.2),

$$S(x-y,t)\phi(y),$$

measures the effect of $\phi(y)$ (the initial temperature at the point y) felt at the point x at some later time t. The source function $S(x-y,t)$, which has its peak precisely at y, *weights* the contribution of $\phi(y)$ according to the distance of y from x and the elapsed time t.

Since the value of $u(x,t)$ (temperature at the point x at time t) is the total sum of contributions from the initial temperature at all points y, we have the formal sum

$$u(x,t) \approx \sum_{y} S(x-y,t)\phi(y),$$

which in the limit gives formula (10.2). So, the heat kernel $S(x,t)$ gives a way of propagating the initial data ϕ to later times. Of course the contribution from a point y_1 closer to x has a bigger weight $S(x-y_1,t)$, than the contribution from a point y_2 farther away, which gets weighted by $S(x-y_2,t)$.

The function $S(x,t)$ appears in various other physical situations. For example in the random (Brownian) motion of a particle in one dimension. If the probability of finding the particle at position x initially is given by the density function $\phi(x)$, then the density defining the probability of finding the particle at position x at time t is given by the same formula (10.2).

Example 10.1. Solve the heat equation with the initial condition $u(x,0) = e^x$.

Using the solution formula (10.4), we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^y \, dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{[-x^2+2xy-y^2+4kty]/4kt} \, dy$$

We can complete the squares in the numerator of the exponent, writing it as

$$\begin{aligned} \frac{-x^2+2xy-y^2+4kty}{4kt} &= \frac{-x^2+2(x+2kt)y-y^2}{4kt} \\ &= \frac{-(y-2kt-x)^2+4ktx+4k^2t^2}{4kt} = -\left(\frac{y-2kt-x}{\sqrt{4kt}}\right)^2 + x + kt. \end{aligned}$$

We then have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{x+kt} e^{[(y-2kt-x)/\sqrt{4kt}]^2} \, dy = \frac{1}{\sqrt{\pi}} e^{x+kt} \int_{-\infty}^{\infty} e^{-p^2} \, dp = e^{kt+x}.$$

Notice that $u(x,t)$ grows with time, which may seem to be in contradiction with the maximum principle. However, thinking in terms of heat conduction, we see that the initial temperature $u(x,0) = e^x$ is itself infinitely large at the far right end of the rod $x = +\infty$. So the temperature does not grow out of nowhere, but rather gets transferred from right to left with the "speed" k. Thus the initial exponential distribution of the temperature "travels" from right to left with the speed k as t grows. Compare this to the example in Strauss, where the initial temperature $u(x,0) = e^{-x}$ "travels" from left to right, since the initial temperature peaks at the far left end $x = -\infty$. $\square$

In the above example we were able to compute the solution explicitly, however, the integral in (10.4) may be impossible to evaluate completely in terms of elementary functions for general initial data $\phi(x)$. Due to this, the answers for particular problems are usually written in terms of the error function in statistics,

$$\mathcal{E}\mathrm{rf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} \, dp.$$

Notice that $\mathcal{E}\mathrm{rf}(0) = 0$, and $\lim_{x\to\infty} \mathcal{E}\mathrm{rf}(x) = 1$. Using this function, we can rewrite the function $Q(x,t)$ given by (10.6), which solves the heat IVP with Heaviside initial data, as follows

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right).$$

10.3 Conclusion

Using the notions of distribution and distributional derivative, we can make sense of the heat kernel $S(x,t)$ that has the Dirac δ function as its initial data. Comparing the expression of the heat kernel (10.3) with the density function of the normal (Gaussian) distribution, we saw that the solution formula (10.2) essentially weights the initial data by the bell-shaped curve $S(x,t)$, thus giving the contribution from the initial heat at different points towards the temperature at point x at time t.

In the last several lectures we solved the initial value problems associated with the wave and heat equations on the whole line $x \in \mathbb{R}$. We would like to summarize the properties of the obtained solutions, and compare the propagation of waves to conduction of heat.

Recall that the solution to the wave IVP on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases} \tag{11.1}$$

is given by d'Alambert's formula

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds. \tag{11.2}$$

Most of the properties of this solution can be deduced from the solution formula, which can be understood fairly well, if one thinks in terms of the characteristic coordinates. This is how we arrived at the properties of finite speed of propagation, propagation of discontinuities of the data along the characteristics, and others.

On the other hand, the solution to the heat IVP on the whole line

$$\begin{cases} u_t - k u_{xx} = 0, \\ u(x,0) = \phi(x), \end{cases} \tag{11.3}$$

is given by the formula

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy. \tag{11.4}$$

We saw some of the properties of the solutions to the heat IVP, for example the smoothing property, in the case of the fundamental solution or the heat kernel

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt}, \tag{11.5}$$

which had the Dirac delta function as its initial data. The solution u given by (11.4) can be written in terms of the heat kernel, and we use this to prove the properties for solutions to the general IVP (11.3). In terms of the heat kernel the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy = \int_{-\infty}^{\infty} S(z,t)\phi(x-z)\, dz,$$

where we made the change of variables $z = x - y$ to arrive at the last integral. Making a further change of variables $p = z/\sqrt{kt}$, the above can be written as

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp. \tag{11.6}$$

This last form of the solution will be handy when proving the smoothing property of the heat equation, the precise statement of which is contained in the following.

Theorem 11.1. *Let $\phi(x)$ be a bounded continuous function for $-\infty < x < \infty$. Then* (11.4) *defines an infinitely differentiable function $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$, which satisfies the heat equation, and* $\lim_{t\to 0+} u(x,t) = \phi(x)$, $\forall x \in \mathbb{R}$.

The proof is rather straightforward, and amounts to pushing the derivatives of $u(x,t)$ onto the heat kernel inside the integral. All one needs to guarantee for this procedure to go through is the uniform convergence of the resulting improper integrals. Let us first take a look at the solution itself given by (11.4). Notice that using the form in (11.6), we have

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}\left|e^{-p^2/4}\phi(x-p\sqrt{kt})\right| dp \leq \frac{1}{\sqrt{4\pi}}(\max|\phi|)\int_{-\infty}^{\infty} e^{-p^2/4}\,dp = \max|\phi|,$$

which shows that u, given by the improper integral, is well-defined, since ϕ is bounded. One can also see the maximum principle in the above inequality. We will use similar logic to show that the improper integrals appearing in the derivatives of u converge uniformly in x and t.

Notice that formally

$$\frac{\partial u}{\partial x} = \int_{-\infty}^{\infty}\frac{\partial S}{\partial x}(x-y,t)\phi(y)\,dy. \tag{11.7}$$

To make this rigorous, one must prove the uniform convergence of the integral. For this, we use expression (11.5) for the heat kernel to write

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\partial S}{\partial x}(x-y,t)\phi(y)\,dy &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[e^{-(x-y)^2/4kt}\right]\phi(y)\,dy\\ &= -\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\frac{x-y}{2kt}e^{-(x-y)^2/4kt}\phi(y)\,dy.\end{aligned}$$

Making the change of variables $p=(x-y)/\sqrt{kt}$ in the above integral, we get

$$\int_{-\infty}^{\infty}\frac{\partial S}{\partial x}(x-y,t)\phi(y)\,dy = \frac{1}{4\sqrt{\pi kt}}\int_{-\infty}^{\infty}pe^{-p^2/4}\phi(x-p\sqrt{kt})\,dp \leq \frac{c}{\sqrt{t}}(\max|\phi|)\int_{-\infty}^{\infty}|p|e^{-p^2/4}\,dp,$$

where $c=1/(4\sqrt{\pi k})$ is a constant. The last integral is finite, so the integral in the formal derivative (11.7) converges uniformly and absolutely for all $x\in\mathbb{R}$ and $t>\epsilon>0$, where ϵ can be taken arbitrarily small. So the derivative $u_x=\partial u/\partial x$ exists and is given by (11.7).

The above argument works for the t derivative, and all the higher order derivatives as well, since for the n^{th} order derivatives one will end up with the integral $\int_{-\infty}^{\infty}|p|^n e^{-p^2/4}\,dp$, which is finite for all $n\in\mathbb{N}$. This proves the infinite differentiability of the solution, even though the initial data is only continuous.

We have already seen that u given by (11.4) solves the heat equation, due to the invariance properties. It then only remains to prove that $\lim_{t\to 0+}u(x,t)=\phi(x)$, $\forall x$. Recall that our previous proofs of this used the derivative of $\phi(x)$, or the language of distributions to employ the Dirac δ, where we assumed that ϕ is a test function, i.e. infinitely differentiable with compact support. To prove that $u(x,t)$ satisfies the initial condition in (11.3) in the case of continuous initial data ϕ as well, one can either use a *density argument*, in which $\phi(x)$ is uniformly approximated by smooth functions, and make a use of our earlier proofs, or provide a direct proof. The basic idea behind the direct proof is given next. We need to show that the difference $u(x,t)-\phi(x)$ becomes arbitrarily small when $t\to 0+$. First notice that

$$u(x,t)-\phi(x) = \int_{-\infty}^{\infty}S(x-y,t)[\phi(y)-\phi(x)]\,dy = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}e^{-p^2/4}[\phi(x-p\sqrt{kt})-\phi(x)]\,dp, \tag{11.8}$$

where we used the same change of variables as before, $p=(x-y)/\sqrt{kt}$. To see that the last integral becomes arbitrarily small as t goes to zero, notice that if $p\sqrt{kt}$ is small, then $|\phi(x-p\sqrt{kt})-\phi(x)|$ is small due to the continuity of ϕ, and the rest of the integral is finite. Otherwise, when $p\sqrt{kt}$ is large, then p is large, and the exponential in the integral becomes arbitrarily small, while the ϕ term is bounded. Thus, one estimates the above integral by breaking it into the following two integrals

$$\begin{aligned}&\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}e^{-p^2/4}[\phi(x-p\sqrt{kt})-\phi(x)]\,dp\\ &\quad= \frac{1}{\sqrt{4\pi}}\int_{|p|<\delta/\sqrt{kt}}e^{-p^2/4}[\phi(x-p\sqrt{kt})-\phi(x)]\,dp + \frac{1}{\sqrt{4\pi}}\int_{|p|\geq\delta/\sqrt{kt}}e^{-p^2/4}[\phi(x-p\sqrt{kt})-\phi(x)]\,dp.\end{aligned}$$

For some small δ, the first integral is small due to the continuity of ϕ, while for arbitrarily small t the second integral is the tail of a converging improper integral, and is hence small. You should try to fill in the rigorous details. This completes the proof of Theorem 11.1.

It turns out, that the result in the above theorem can be proved even if the assumption of continuity of ϕ is relaxed to piecewise continuity. One then has the following.

Theorem 11.2. *Let $\phi(x)$ be a bounded piecewise-continuous function for $-\infty < x < \infty$. Then* (11.4) *defines an infinitely differentiable function $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$, which satisfies the heat equation, and*

$$\lim_{t\to 0+} u(x,t) = \frac{1}{2}[\phi(x+) + \phi(x-)], \qquad \text{for all } x \in \mathbb{R}, \tag{11.9}$$

where $\phi(x+)$ and $\phi(x-)$ stand for the right hand side and left hand side limits of ϕ at x.

Of course the fact that ϕ has jump discontinuities will not effect the convergence of the improper integrals encountered in the proof of Theorem 11.1. To see why (11.9) holds, notice that the integral in the right hand side of (11.6) can be broken into integrals over positive and negative half-lines

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{0} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp + \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp.$$

Then, since $p < 0$ in the first integral, $\phi(x - p\sqrt{kt})$ goes to $\phi(x+)$ as $t \to 0+$, while it goes to $\phi(x-)$ in the second integral, due to p being positive. So one can make the obvious changes in the proof of the previous theorem to show (11.9). This curious fact is one of the reasons why some people prefer to define the value of the Heaviside step function at $x = 0$ to be $H(0) = \frac{1}{2}$. Then one has

$$\lim_{t\to 0+} Q(x,t) = H(x) \qquad \text{for all } x \in \mathbb{R} \quad (\text{including } x = 0!),$$

where $Q(x,t)$ was the solution arising from the initial data given by $H(x)$.

11.1 Comparison of wave to heat

We now summarize and compare the fundamental properties of the wave and heat equations in the table below. Brief discussion of each of the properties will follow.

	Property	**Wave** ($u_{tt} - c^2 u_{xx} = 0$)	**Heat** ($u_t - k u_{xx} = 0$)
(i)	Speed of propagation	Finite (speed $\leq c$)	Infinite
(ii)	Singularities for $t > 0$	Transported along characteristics (speed $= c$)	Lost immediately
(iii)	Well-posed for $t > 0$	Yes	Yes (for bounded solutions)
(iv)	Well-posed for $t < 0$	Yes	No
(v)	Maximum principle	No	Yes
(vi)	Behaviour as $t \to \infty$	Does not decay	Decays to zero (if ϕ is integrable)
(vii)	Information	Transported	Lost gradually

Let us now recall why each of the properties listed in the table holds or does not for each equation.

(i) Finite speed of propagation for the wave equation is immediately seen from d'Alambert's formula (11.2).

The infinite speed of propagation for the heat equation was seen in the example of the heat kernel, which is strictly positive for all $x \in \mathbb{R}$ for $t > 0$, but has Dirac δ function as its initial data, and hence is zero for all $x \neq 0$ initially.

(ii) We saw in the box-wave (initial displacement in the form of a box, no initial velocity) and the "hammer blow" (no initial displacement, initial box-shaped velocity) that singularities are preserved and are transported along the characteristics. The same is seen from (11.2).

For the heat equation we saw in the last section that the solution (11.4) is infinitely differentiable even for piecewise continuous initial data (this is true for even weaker conditions on ϕ).

(iii) Well-posedness for the wave IVP is seen immediately from d'Alambert's formula.

In the case of the heat equation, we proved uniqueness and stability using either the maximum principle, or alternatively, the energy method. Existence follows from our construction of the explicit solution (11.4).

(iv) For the wave equation, this follows from the invariance under time reversion. Indeed, if $u(x,t)$ is a solution, then so is $u(x,-t)$, which has data $(\phi(x), -\psi(x))$.

If we reverse the time in the heat equation, we get $u_t + ku_{xx} = 0, t > 0$. One can solve this equation in much the same way as the heat equation, and due to the symmetry in t, will get the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{(x-y)^2/4kt} \phi(y)\, dy,$$

which diverges for all $x \in \mathbb{R}$ (unless ϕ decays to zero faster than e^{-p^2}). So the heat equation is not well-posed *backward in time*. This makes physical sense as well, since the processes described by the heat equation, namely diffusion, heat flow and random motion are irreversible processes.

(v) The fact that there is no maximum principle for the wave equation is apparent from the "hammer blow" example, where the solution was everywhere zero initially, but due to the nonzero initial velocity, had nonzero displacement for any time $t > 0$. For the heat equation, the maximum principle was proved rigorously in a previous lecture.

(vi) We saw that the energy is conserved for the wave equation, so the solutions do not decay. We also saw this in the box-wave example, in which the initial box-shaped data split into two box-shaped waves of half the height that traveled in opposite directions without changing the shape.

For the heat equation, the decay is seen from formula (11.4), since $S(x-y,t) \to 0$ as $t \to \infty$, and the integral will be bounded if ϕ is integrable. Notice that in the example we considered in the last lecture, with $\phi(x) = e^x$, the solution did not decay, but rather "traveled" from right to left. This was due to ϕ being non-integrable.

(vii) The fact that information is transported by the solutions of the wave equation is seen from the fact that the initial data is propagated along the characteristics. So the information will travel along the characteristics as well.

In the case of the heat equation, the information is gradually lost, which can be seen from the graph of a typical solution (think of the heat kernel). The heat from the higher temperatures gets dissipated and after a while it is not clear what the original temperatures were.

11.2 Conclusion

Although the wave and heat equations are both second order linear constant coefficient PDEs, their respective solutions posses very different properties. By now we have learned how to solve the initial value problems on the whole line for both of these equations, and understood these solutions in terms of the physics behind the corresponding problems. We also saw that the properties of the solutions of the respective equations correspond to our intuition for each of the physical phenomena described by the equations.

Problem Set 6

1. (#2.4.4 in [Str]) Solve the heat equation if $\phi(x) = e^{-x}$ for $x > 0$, and $\phi(x) = 0$ for $x < 0$.

2. (#2.4.11 (a) in [Str]) Consider the heat equation on the whole line with the usual initial condition $u(x, 0) = \phi(x)$. If $\phi(x)$ is an *odd* function, show that the solution $u(x, t)$ is also an *odd* function of x. (*Hint:* Consider $u(-x, t) + u(x, t)$ and use the uniqueness.)

3. (#2.4.15 in [Str]) Prove the uniqueness of the heat problem with Neumann boundary conditions:
$$\begin{cases} u_t - ku_{xx} = f(x,t) \quad \text{for } 0 < x < l, t > 0, \\ u(x,0) = \phi(x), \\ u_x(0,t) = g(t), \quad u_x(l,t) = h(t), \end{cases}$$
by the energy method.

4. (#2.4.16 in [Str]) Solve the initial value problem for the diffusion equation with constant dissipation:
$$\begin{cases} u_t - ku_{xx} + bu = 0 \quad \text{for } -\infty < x < \infty, \\ u(x,0) = \phi(x), \end{cases}$$
where $b > 0$ is a constant. (*Hint:* Make the change of variables $u(x, t) = e^{-bt}v(x, t)$.)

5. (#2.5.4 in [Str]) Here is a direct relationship between the wave and diffusion equations. Let $u(x, t)$ solve the wave equation on the whole line with bounded second derivatives. Let
$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\, ds.$$
   (a) Show that $v(x, t)$ solves the diffusion equation!
   (b) Show that $\lim_{t\to 0} v(x, t) = u(x, 0)$.

   (*Hint:* (a) Write the formula as $v(x, t) = \int_{-\infty}^{\infty} H(s, t)u(x, s)\, ds$, where $H(x, t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x, t)$, assuming that you can freely differentiate inside the integral. (b) Use the fact that $H(s, t)$ is essentially the diffusion (heat) kernel with the spatial variable s. You can use the fact that the diffusion kernel has the Dirac delta function as its initial data.)

6. Solve the heat equation with the initial data $\phi(x) = \delta(x - 2) + 3\delta(x)$.

7. Show that the distributional derivative of the Dirac delta function acts on test functions as
$$\delta'[f] = -f'(0).$$

12 Heat conduction on the half-line

In previous lectures we completely solved the initial value problem for the heat equation on the whole line, i.e. in the absence of boundaries. Next, we turn to problems with physically relevant boundary conditions. Let us first add a boundary consisting of a single endpoint, and consider the heat equation on the half-line $D = (0, \infty)$. The following initial/boundary value problem, or IBVP, contains a Dirichlet boundary condition at the endpoint $x = 0$.

$$\begin{cases} v_t - kv_{xx} = 0, \quad 0 < x < \infty, 0 < t < \infty, \\ v(x,0) = \phi(x), \quad x > 0, \\ v(0,t) = 0, \quad t > 0. \end{cases} \tag{12.1}$$

If the solution to the above mixed initial/boundary value problem exists, then we know that it must be unique from an application of the maximum principle. In terms of the heat conduction, one can think of v in (12.1) as the temperature in an infinite rod, one end of which is kept at a constant zero temperature. The initial temperature of the rod is then given by $\phi(x)$.

Our goal is to solve the IBVP (12.1), and derive a solution formula, much like what we did for the heat IVP on the whole line. But instead of constructing the solution from scratch, it makes sense to try to reduce this problem to the IVP on the whole line, for which we already have a solution formula. This is achieved by extending the initial data $\phi(x)$ to the whole line. We have a choice of how exactly to extend the data to the negative half-line, and one should try to do this in such a fashion that the boundary condition of (12.1) is automatically satisfied by the solution to the IVP on the whole line that arises from the extended data. This is the case, if one chooses the *odd extension* of $\phi(x)$, which we describe next.

By the definition a function $\psi(x)$ is odd, if $\psi(-x) = -\psi(x)$. But then plugging in $x = 0$ into this definition, one gets $\psi(0) = 0$ for any odd function. Recall also that the solution $u(x,t)$ to the heat IVP with odd initial data is itself odd in the x variable. This follows from the fact that the sum $[u(x,t) + u(-x,t)]$ solves the heat equation and has zero initial data, hence, it is the identically zero function by the uniqueness of solutions. Then, by our above observation for odd functions, we would have that $u(0,t) = 0$ for any $t > 0$, which is exactly the boundary condition of (12.1).

This shows that if one extends $\phi(x)$ to an odd function on the whole line, then the solution with the extended initial data automatically satisfies the boundary condition of (12.1). Let us then define

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ -\phi(-x) & \text{for } x < 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{12.2}$$

It is clear that ϕ_{odd} is an odd function, since we defined it for negative x by reflecting the $\phi(x)$ with respect to the vertical axis, and then with respect to the horizontal axis. This procedure produces a function whose graph is symmetric with respect to the origin, and thus it is odd. One can also verify this directly from the definition of odd functions. Now, let $u(x,t)$ be the solution of the following IVP on the whole line

$$\begin{cases} u_t - ku_{xx} = 0, \quad -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{odd}}(x). \end{cases} \tag{12.3}$$

From previous lectures we know that the solution to (12.3) is given by the formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,dy, \qquad t > 0. \tag{12.4}$$

Restricting the x variable to only the positive half-line produces the function

$$v(x,t) = u(x,t)\big|_{x \geq 0}. \tag{12.5}$$

We claim that this $v(x,t)$ is the unique solution of IBVP (12.1). Indeed, $v(x,t)$ solves the heat equation on the positive half-line, since so does $u(x,t)$. Furthermore,

$$v(x,0) = u(x,0)\Big|_{x>0} = \phi_{\text{odd}}(x)\Big|_{x>0} = \phi(x),$$

and $v(0,t) = u(0,t) = 0$, since $u(x,t)$ is an odd function of x. So $v(x,t)$ satisfies the initial and boundary conditions of (12.1).

Returning to formula (12.4), we substitute the expressions for ϕ_{odd} from (12.2) and write

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t)\phi_{\text{odd}}(y)\,dy + \int_{-\infty}^0 S(x-y,t)\phi_{\text{odd}}(y)\,dy \\ &= \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_{-\infty}^0 S(x-y,t)\phi(-y)\,dy. \end{aligned}$$

Making the change of variables $y \mapsto -y$ in the second integral on the right, and flipping the integration limits gives

$$u(x,t) = \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_0^\infty S(x+y,t)\phi(y)\,dy.$$

Using (12.5) and the above expression for $u(x,t)$, as well as the expression of the heat kernel $S(x,t)$, we can write the solution formula for the IBVP (12.1) as follows

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}\right] \phi(y)\,dy. \tag{12.6}$$

The method used to arrive at this solution formula is called the *method of odd extensions* or the *reflection method*. We can make a physical sense of formula (12.6) by interpreting the integrand as the contribution from the point y minus the heat loss from this point due to the constant zero temperature at the endpoint.

Example 12.1. Solve the IBVP (12.1) with the initial data $\phi(x) = e^x$.

Using the solution formula (12.6), we have

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt}e^y - e^{-(x+y)^2/4kt}e^y\right] dy. \tag{12.7}$$

Combining the exponential factors of the first product under the integral, we will get an exponential with the following exponent

$$\frac{-[y^2 - 2(x+2kt)y + x^2]}{4kt} = -\left(\frac{y-(x+2kt)}{\sqrt{4kt}}\right)^2 + kt - x = -p^2 + kt + x,$$

where we made the obvious notation

$$p = \frac{y - x - 2kt}{\sqrt{4kt}}.$$

Similarly, the exponent of the combined exponential from the second product under integral (12.7) is

$$\frac{-[y^2 + 2(x-2kt)y + x^2]}{4kt} = -\left(\frac{y+x-2kt}{\sqrt{4kt}}\right)^2 + kt - x = -q^2 + kt - x,$$

with

$$q = \frac{y + x - 2kt}{\sqrt{4kt}}.$$

Braking integral (12.7) into a difference of two integrals, and making the changes of variables $y \mapsto p$, and $y \mapsto q$ in the respective integrals, we will get

$$v(x,t) = e^{kt+x}\frac{1}{\sqrt{\pi}}\int_{\frac{-x-2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\,dp - e^{kt-x}\frac{1}{\sqrt{\pi}}\int_{\frac{x-2kt}{\sqrt{4kt}}}^{\infty} e^{-q^2}\,dq. \tag{12.8}$$

Notice that

$$\frac{1}{\sqrt{\pi}}\int_{\frac{-x-2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\,dp = \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-p^2}\,dp + \frac{1}{\sqrt{\pi}}\int_{\frac{-x-2kt}{\sqrt{4kt}}}^{0} e^{-p^2}\,dp = \frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right),$$

and similarly for the second integral. Putting this back into (12.8), we will arrive at the solution

$$v(x,t) = e^{kt+x}\left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right)\right] - e^{kt-x}\left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x-2kt}{\sqrt{4kt}}\right)\right].$$

□

12.1 Neumann boundary condition

Let us now turn to the Neumann problem on the half-line,

$$\begin{cases} w_t - kw_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ w(x,0) = \phi(x), & x > 0 \\ w_x(0,t) = 0, & t > 0. \end{cases} \tag{12.9}$$

To find the solution of (12.9), we employ a similar idea used in the case of the Dirichlet problem. That is, we seek to reduce the IBVP to an IVP on the whole line by extending the initial data $\phi(x)$ to the negative half-axis in such a fashion that the boundary condition is automatically satisfied.

Notice that if $\psi(s)$ is an even function, i.e. $\psi(-x) = \psi(x)$, then its derivative function will be odd. Indeed, differentiating in the definition of the even function, we get $-\psi'(-x) = \psi'(x)$, which is the same as $\psi'(-x) = -\psi'(x)$. Hence, for an arbitrary even function $\psi(x)$, $\psi'(0) = 0$. It is now clear that extending the initial data so that the resulting function is even will produce solutions to the IVP on the whole line that automatically satisfy the Neumann condition of (12.9).

We define the even extension of $\phi(x)$,

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\ \phi(-x) & \text{for } x \leq 0, \end{cases} \tag{12.10}$$

and consider the following IVP on the whole line

$$\begin{cases} u_t - ku_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{even}}(x). \end{cases} \tag{12.11}$$

It is clear that the solution $u(x,t)$ of the IVP (12.11) will be even in x, since the difference $[u(-x,t) - u(x,t)]$ solves the heat equation and has zero initial data. We then use the solution formula for the IVP on the whole line to write

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)\,dy, \qquad t > 0, \tag{12.12}$$

and take

$$w(x,t) = u(x,t)\big|_{x\geq 0},$$

similar to the case of the Dirichlet problem. One can show that this $w(x,t)$ solves the IBVP (12.9), and use the expression for the heat kernel, as well as the definition (12.10), to write the solution formula as follows

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] \phi(y)\, dy. \tag{12.13}$$

Notice that the formulas (12.6) and (12.13) differ only by the sign between the two exponential terms inside the integral.

In terms of heat conduction, the Neumann condition in (12.9) means that there is no heat exchange between the rod and the environment (recall that the heat flux is proportional to the spatial derivative of the temperature). The physical interpretation of formula (12.13) is that the integrand is the contribution of $\phi(y)$ plus an additional contribution, which comes from the lack of heat transfer to the points of the rod with negative coordinates.

Example 12.2. Solve the IBVP (12.9) with the initial data $\phi(x) \equiv 1$.

Using the formula (12.13), we can write the solution as

$$\begin{aligned} w(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^\infty e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-q^2}\, dq, \end{aligned}$$

where we made the changes of variables

$$p = \frac{y-x}{\sqrt{4kt}}, \qquad \text{and} \qquad q = \frac{y+x}{\sqrt{4kt}}.$$

Using the same idea as in the previous example, we can write the solution in terms of the $\mathcal{E}\mathrm{rf}$ function as follows

$$w(x,t) = \left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}} \right) \right] + \left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}} \right) \right] \equiv 1.$$

So the solution is identically 1, which is clear if one thinks in terms of heat conduction. Indeed, problem (12.9) describes the temperature dynamics with identically 1 initial temperature, and no heat loss at the endpoint. Obviously there is no heat transfer between points of equal temperature, so the temperatures remain steady along the entire rod. $\square$.

12.2 Conclusion

We derived the solution to the heat equation on the half-line by reducing the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In the case of zero Dirichlet boundary condition the odd extension of the initial data automatically guarantees that the solution will satisfy the boundary condition. While for the case of zero Neumann boundary condition the appropriate choice is the even extension. This reflection method relies on the fact that the solution to the heat equation on the whole line with odd initial data is odd, while the solution with even initial data is even.

13 Waves on the half-line

Similar to the last lecture on the heat equation on the half-line, we will use the reflection method to solve the boundary value problems associated with the wave equation on the half-line $0 < x < \infty$. Let us start with the Dirichlet boundary condition first, and consider the initial boundary value problem

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), & x > 0, \\ v(0,t) = 0, & t > 0. \end{cases} \tag{13.1}$$

For the vibrating string, the boundary condition of (13.1) means that the end of the string at $x = 0$ is held fixed. We reduce the Dirichlet problem (13.1) to the whole line by the reflection method. The idea is again to extend the initial data, in this case ϕ, ψ, to the whole line, so that the boundary condition is automatically satisfied for the solutions of the IVP on the whole line with the extended initial data. Since the boundary condition is in the Dirichlet form, one must take the odd extensions

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\phi(-x) & \text{for } x < 0. \end{cases} \qquad \psi_{\text{odd}}(x) = \begin{cases} \psi(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\psi(-x) & \text{for } x < 0. \end{cases} \tag{13.2}$$

Consider the IVP on the whole line with the extended initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{odd}}(x), u_t(x,0) = \psi_{\text{odd}}(x). \end{cases} \tag{13.3}$$

Since the initial data of the above IVP are odd, we know from a homework problem that the solution of the IVP, $u(x,t)$, will also be odd in the x variable, and hence $u(0,t) = 0$ for all $t > 0$. Then defining the restriction of $u(x,t)$ to the positive half-line $x \geq 0$,

$$v(x,t) = u(x,t)\big|_{x \geq 0}, \tag{13.4}$$

we automatically have that $v(0,t) = u(0,t) = 0$. So the boundary condition of the Dirichlet problem (13.1) is satisfied for v. Obviously the initial conditions are satisfied as well, since the restrictions of $\phi_{\text{odd}}(x)$ and $\psi_{\text{odd}}(x)$ to the positive half-line are $\phi(x)$ and $\psi(x)$ respectively. Finally, $v(x,t)$ solves the wave equation for $x > 0$, since $u(x,t)$ satisfies the wave equation for all $x \in \mathbb{R}$, and in particular for $x > 0$. Thus, $v(x,t)$ defined by (13.4) is a solution of the Dirichlet problem (13.1). It is clear that the solution must be unique, since the odd extension of the solution will solve IVP (13.3), and therefore must be unique.

Using d'Alambert's formula for the solution of (13.3), and taking the restriction (13.4), we have that for $x \geq 0$,

$$v(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s)\,ds. \tag{13.5}$$

Notice that if $x \geq 0$ and $t > 0$, then $x+ct > 0$, and $\phi_{\text{odd}}(x+ct) = \phi(x+ct)$. If in addition $x-ct > 0$, then $\phi_{\text{odd}}(x-ct) = \phi(x-ct)$, and over the interval $s \in [x-ct, x+ct]$, $\psi_{odd}(s) = \psi(s)$. Thus, for $x > ct$, we have

$$v(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds, \tag{13.6}$$

which is exactly d'Alambert's formula.

For $0 < x < ct$, the argument $x - ct < 0$, and using (13.2) we can rewrite the solution (13.5) as

$$v(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\left[\int_{x-ct}^{0} -\psi(-s)\,ds + \int_{0}^{x+ct} \psi(s)\,ds\right].$$

Making the change of variables $s \mapsto -s$ in the first integral on the right, we get

$$v(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\left[\int_{ct-x}^{0} \psi(s)\,ds + \int_{0}^{x+ct} \psi(s)\,ds\right]$$
$$= \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds.$$

One could also use the fact that the integral of the odd function $\psi_{\text{odd}}(s)$ over the symmetric interval $[x-ct, ct-x]$ is zero, thus $\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s)\,ds = \int_{ct-x}^{x+ct} \psi(s)\,ds$.

The two different cases giving different expressions are illustrated in Figures 13.1 and 13.2 below. Notice how one of the characteristics from a point with $x_0 < ct_0$ gets reflected from the "wall" at $x = 0$ in Figure 13.2.

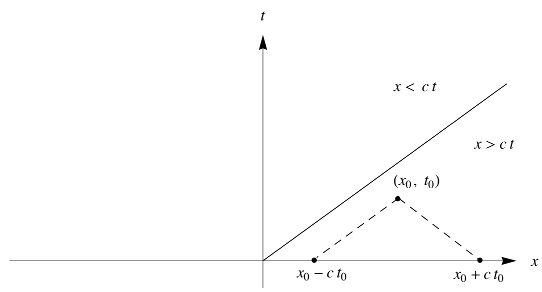


Figure 13.1: The case with $x_0 > ct_0$.

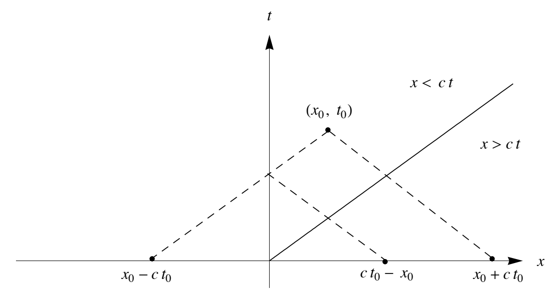


Figure 13.2: The case with $x_0 < ct_0$.

Combining the two expressions for $v(t,x)$ over the two regions, we arrive at the solution

$$v(x,t) = \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds, & \text{for } x > ct \\ \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds, & \text{for } 0 < x < ct. \end{cases} \tag{13.7}$$

The minus sign in front of $\phi(ct-x)$ in the second expression above, as well as the reduction of the integral of ψ to the smaller interval are due to the cancellation stemming from the *reflected wave*. The next example illustrates this phenomenon.

Example 13.1. Solve the Dirichlet problem (13.1) with the following initial data

$$\phi(x) = \begin{cases} h & \text{for } a < x < 2a, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \psi(x) \equiv 0.$$

The initial data defines a box-like displacement over the interval $(a, 2a)$ and zero initial velocity for the string. It is clear that for some small time the wave will propagate just like in the case of the IVP on the whole line, i.e. without "seeing" the boundary. This is due to the finite speed of propagation property of the wave equation, according to which it takes some time, specifically a/c time for the initial displacement to reach the boundary. Thus, we expect that the box-like wave will break into two box-waves, each with half the height of the initial box-like displacement, which travel with speed c in opposite directions. The box-wave traveling in the right direction will never hit the boundary at $x = 0$, and will continue traveling unaltered for all time. However, the left box-wave hits the wall (the fixed end of the vibrating string), and gets reflected in an odd fashion, that is the displacement gets the minus sign in the second expression of (13.7).

To find the values of the solution at any point (x_0, t_0), we draw the backward characteristics from that point and trace the point back to the x axis, where the initial data is defined. Since the initial data is

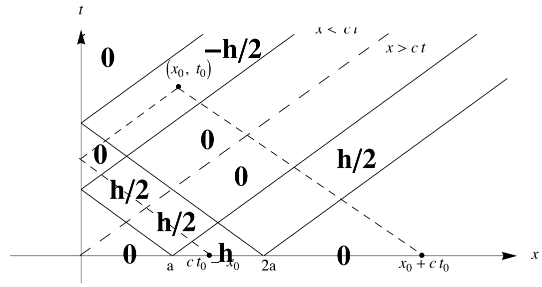


Figure 13.3: The values of $v(x,t)$ carried forward in time by the characteristics.

nonzero only over the interval $(a, 2a)$, only those characteristics that hit the x axis between the points a and $2a$ will carry nonzero values forward in time. Notice also that if a characteristic hits the interval $(a, 2a)$ after being reflected from the wall $x = 0$, then the value it gets must be taken with a minus sign.

The different values determined by this method are illustrated in Figure 13.3 above. Notice how the left box-wave with height $h/2$ flips after hitting the boundary, and travels in the opposite direction with negative height $-h/2$.

One can then use the values from Figure 13.3 to draw the profile of the string at any time t. □

13.1 Neumann boundary condition

For the Neumann problem on the half-line,

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, \quad 0 < x < \infty, 0 < t < \infty, \\ w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), \quad x > 0, \\ w_x(0,t) = 0, \quad t > 0, \end{cases} \tag{13.8}$$

we use the reflection method with even extensions to reduce the problem to an IVP on the whole line. Define the even extensions of the initial data

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\ \phi(-x) & \text{for } x \leq 0, \end{cases} \qquad \psi_{\text{even}} = \begin{cases} \psi(x) & \text{for } x \geq 0, \\ \psi(-x) & \text{for } x \leq 0. \end{cases} \tag{13.9}$$

and consider the following IVP on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{even}}(x), \quad u_t(x,0) = \psi_{\text{even}}(x). \end{cases} \tag{13.10}$$

Clearly, the solution $u(x,t)$ to the IVP (13.10) will be even in x, and since the derivative of an even function is odd, $u_x(x,t)$ will be odd in x, and hence $u_x(0,t) = 0$ for all $t > 0$. Similar to the case of the Dirichlet problem, the restriction

$$w(x,t) = u(x,t)\big|_{x \geq 0}$$

will be the unique solution of the Neumann problem (13.8).

Using d'Alambert's formula for the solution $u(x,t)$ of (13.10), and taking the restriction to $x \geq 0$, we get

$$w(x,t) = \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\,ds. \tag{13.11}$$

One again needs to consider the two cases $x > ct$ and $0 < x < ct$ separately. Notice that with the even extensions we will get additions, rather than cancellations. Using the definitions (13.9), the solution (13.11) can be written as

$$w(x,t) = \begin{cases} \dfrac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \psi(s)\,ds, & \text{for } x > ct \\ \dfrac{1}{2}[\phi(x+ct) + \phi(ct-x)] & \\ \qquad + \dfrac{1}{2c}\left[\displaystyle\int_0^{ct-x} \psi(s)\,ds + \int_0^{x+ct} \psi(s)\,ds\right], & \text{for } 0 < x < ct. \end{cases}$$

The Neumann boundary condition corresponds to a vibrating string with a free end at $x = 0$, since the string tension, which is proportional to the derivative $v_x(x,t)$, vanishes at $x = 0$. In this case the reflected wave adds to the original wave, rather than canceling it.

Example 13.2. Solve the Neumann problem (13.8) with the following initial data

$$\phi(x) = \begin{cases} h & \text{for } a < x < 2a, \\ 0 & \text{otherwise,} \end{cases} \qquad \psi(x) \equiv 0.$$

The initial data is exactly the same as in the previous example for the Dirichlet problem. The only difference from the Dirichlet case is that the free end reflects the wave with a plus sign. The values carried forward in time by the characteristics are determined in the same way as before. Figure 13.4 illustrates this method. Notice that the reflected wave has the same (positive) height $h/2$ as the wave right before the reflection.

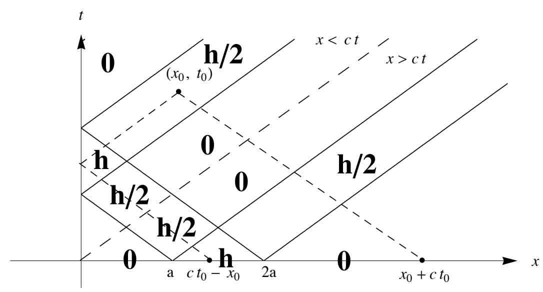


Figure 13.4: The values of $v(x,t)$ carried forward in time by the characteristics.

One can again use the values from Figure 13.4 to draw the profile of the string at any time t. □

13.2 Conclusion

We derived the solution to the wave equation on the half-line in much the same way as was done for the heat equation. That is, we reduced the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In this case the characteristics nicely illustrate the reflection phenomenon. We saw that the characteristics that hit the initial data after reflection from the boundary wall $x = 0$ carry the values of the initial data with a minus sign in the case of the Dirichlet boundary condition, and with a plus sign in the case of the Neumann boundary condition. This corresponds to our intuition of reflected waves from a fixed end, and free end respectively.

In the last lecture we used the reflection method to solve the boundary value problem for the wave equation on the half-line. We would like to apply the same method to the boundary value problems on the finite interval, which correspond to the physically realistic case of a finite string. Consider the Dirichlet wave problem on the finite line

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, \quad 0 < x < l, 0 < t < \infty, \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), \quad x > 0, \\ v(0,t) = v(l,t) = 0, \quad t > 0. \end{cases} \tag{14.1}$$

The homogeneous Dirichlet conditions correspond to the vibrating string having fixed ends, as is the case for musical instruments. Using our intuition from the half-line problems, where the wave reflects from the fixed end, we can imagine that in the case of the finite interval the wave bounces back and forth infinitely many times between the endpoints. In spite of this, we can still use the reflection method to find the value of the solution to problem (14.1) at any point (x,t).

Recall that the idea of the reflection method is to extend the initial data to the whole line in such a way, that the boundary conditions are automatically satisfied. For the homogeneous Dirichlet data the appropriate choice is the odd extension. In this case, we need to extend the initial data ϕ, ψ, which are defined only on the interval $0 < x < l$, in such a way that the resulting extensions are odd with respect to both $x = 0$, and $x = l$. That is, the extensions must satisfy

$$f(-x) = -f(x) \qquad \text{and} \qquad f(l-x) = -f(l+x). \tag{14.2}$$

Notice that for such a function $f(0) = -f(0)$ from the first condition, and $f(l) = -f(l)$ from the second condition, hence, $f(0) = f(l) = 0$. Subsequently, the solution to the IVP with such data will be odd with respect to both $x = 0$ and $x = l$, and the boundary conditions will be automatically satisfied. Notice that the conditions (14.2) imply that functions that are odd with respect to both $x = 0$ and $x = l$ satisfy $f(2l + x) = -f(-x) = f(x)$, which means that such functions must be $2l-$periodic. Using this we can define the extensions of the initial data ϕ, ψ as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < l, \\ -\phi(-x) & \text{for } -l < x < 0, \\ \text{extended to be } 2l-\text{periodic}, \end{cases} \qquad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{for } 0 < x < l, \\ -\psi(-x) & \text{for } -l < x < 0, \\ \text{extended to be } 2l-\text{periodic}. \end{cases} \tag{14.3}$$

Now, consider the IVP on the whole line with the extended initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{ext}}(x), u_t(x,0) = \psi_{\text{ext}}(x). \end{cases} \tag{14.4}$$

For the solution of this IVP we automatically have $u(0,t) = u(l,t) = 0$, and the restriction

$$v(x,t) = u(x,t)\big|_{0 \le x \le l},$$

will solve the boundary value problem (14.1). By d'Alambert's formula, the solution will be given as

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds \tag{14.5}$$

for $0 < x < l$. Although formula (14.5) contains all the information about our solution, we would like to have an expression in terms of the original initial data, so that the values of the solution can be directly computed using the given functions $\phi(x)$ and $\psi(x)$. For this, we need to "bring" the points $x - ct$ and $x + ct$ into the interval $(0, l)$ using the periodicity and oddity of the extended data. To illustrate how this is done, let us fix a point (x,t) and try to find the value of the solution at this point by tracing it

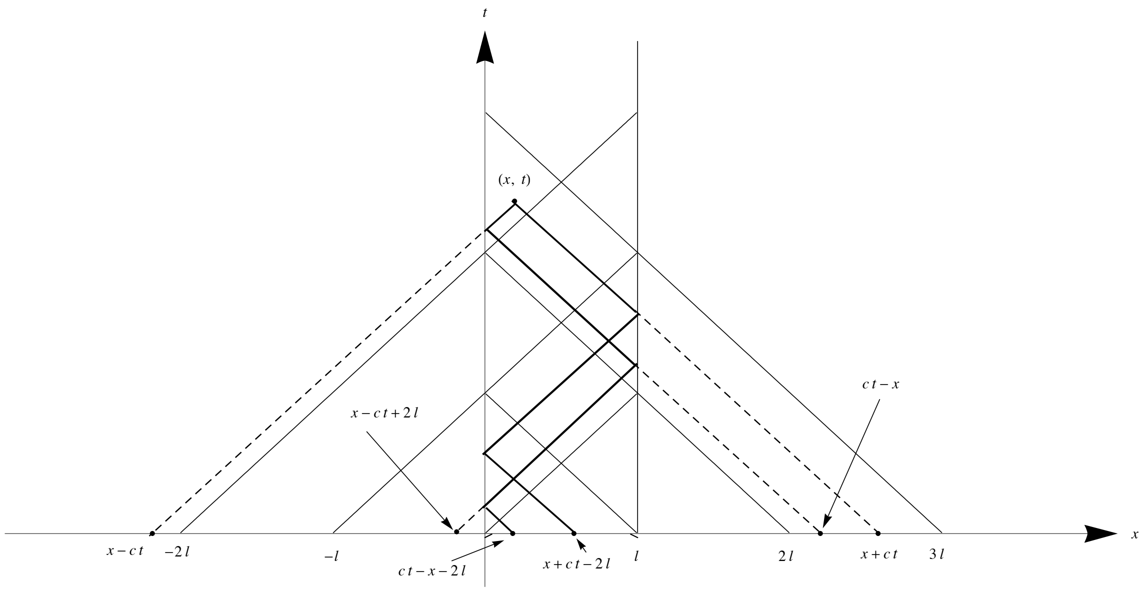


Figure 14.1: The backwards characteristics from the point (x, t).

back in time along the characteristics to the initial data. The sketch of the backwards characteristics from this point appears in Figure 14.1 above.

In general, the points $x+ct$ and $x-ct$ will end up either in the interval $(0, l)$ or $(-l, 0)$ after finitely many translations by the period $2l$. If the point ends up in $(0, l)$ (even number of reflections), then the value of the initial data picked up by the reflected characteristic will be taken with a positive sign. If, however, the point ends up in the interval $(-l, 0)$ (odd number of reflections), then we need to reflect this point with respect to $x = 0$, and the corresponding value of the initial data will be taken with a negative sign.

From Figure 14.1 we see that $x + ct$ goes into the interval $(0, l)$ (2 reflections) after translating it to the left by one period $2l$, but the point $x - ct$ goes into the interval $(-l, 0)$ (3 reflections) after a right translation by $2l$, so we need to reflect the resulting point $x - ct + 2l$ to arrive at the point $ct - x - 2l$ in the interval $(0, l)$. The solution will then be

$$u(x, t) = \frac{1}{2}[\phi(x + ct - 2l) - \phi(ct - x - 2l)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds.$$

For the integral term, we can break it into two integrals as follows

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds = \frac{1}{2c}\int_{x-ct}^{ct-x} \psi_{\text{ext}}(s)\, ds + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi_{\text{ext}}(s)\, ds.$$

Notice that from the oddity of ψ_{ext}, the integral over the interval $[x - ct, ct - x]$ will be zero, while by periodicity, we can bring the interval $[ct - x, x + ct]$ into the interval $(0, l)$ by subtracting one period $2l$. Thus, the solution can be written as

$$u(x, t) = \frac{1}{2}[\phi(x + ct - 2l) - \phi(ct - x - 2l)] + \frac{1}{2c}\int_{ct-x-2l}^{x+ct-2l} \psi(s)\, ds. \tag{14.6}$$

Clearly, the derivation of the above expression for the solution depends on the chosen point, which in turn determines how many reflections the backward characteristics undergo before arriving at the x axis. Hence, the solution will be given by different expressions, depending on the region from which the point is taken. These different regions are depicted in Figure 14.2, where the labels (m, n) show how many times each of the two backward characteristics gets reflected before reaching the x axis. Expression (14.6) will be valid for all the points in the region $(3, 2)$.

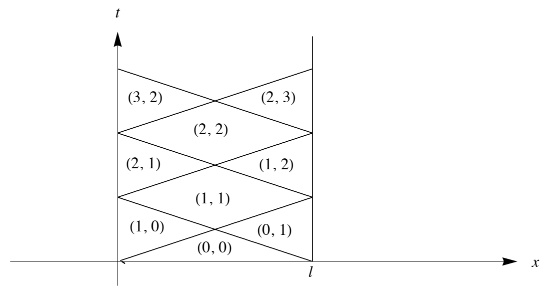


Figure 14.2: Regions of $(x,t) \in (0,l) \times (0,\infty)$ with the different number of reflections.

The method used to arrive at the expression (14.6) can be used to find the value of the solution at any point (x,t), although it is quite impractical to derive the expression for each of the regions depicted in Figure 14.2. Furthermore, it does not generalize to higher dimensions, nor does it apply to the heat equation (no characteristics to trace back). Later we will study another method, which allows for a more general way of approaching boundary value problems on finite intervals.

Example 14.1. Consider the Dirichlet wave problem on the finite interval

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad \text{for } 0 < x < 1, \\ u(x,0) = x(1-x), u_t(x,0) = x^2, \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Find the value of the solution at the point $(\frac{3}{4}, \frac{5}{2})$.

Notice that in this problem $c = 1$, and $l = 1$, so the period of the extended data will be $2l = 2$. The sketch of the backward characteristics from the point $(x,t) = (\frac{3}{4}, \frac{5}{2})$ appears in the figure below.

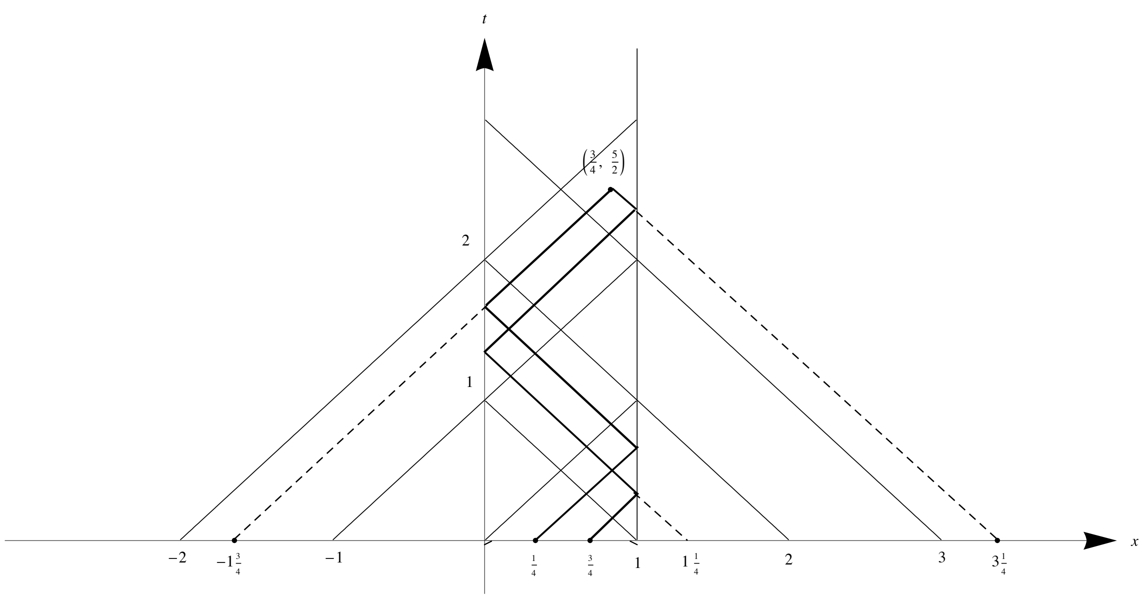


Figure 14.3: The backwards characteristics from the point $(\frac{3}{4}, \frac{5}{2})$.

The characteristics intersect the x axis at the points

$$x - t = \tfrac{3}{4} - \tfrac{5}{2} = -1\tfrac{3}{4} \qquad \text{and} \qquad x + t = \tfrac{3}{4} + \tfrac{5}{2} = 3\tfrac{1}{4}.$$

The point $-1\frac{3}{4}$ goes to the point $\frac{1}{4}$ after a right translation by one period, while the point $3\frac{1}{4}$ goes to the point $1\frac{1}{4}$ after a left translation by one period. After a reflection with respect to $x = 1$, this point will end up at $\frac{3}{4}$, thus, the value of the initial data must be taken with a negative sign at this point. Also, the integral over the interval $[\frac{3}{4}, 1\frac{1}{4}]$ of ψ_{ext} will be zero due to its oddity with respect to $x = 1$. The value of the solution is then

$$u(\tfrac{3}{4}, \tfrac{5}{2}) = \frac{1}{2}[-\phi(\tfrac{3}{4}) + \phi(\tfrac{1}{4})] + \frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}} \psi(s)\, ds = \frac{1}{2}\left[-\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{3}{4}\right] + \frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}} x^2\, dx$$

$$= \frac{x^3}{6}\bigg|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{1}{6}\left(\frac{27}{64} - \frac{1}{64}\right) = \frac{13}{192}.$$

□

14.1 The parallelogram rule

Recall from a homework problem, that for the vertices of a characteristic parallelogram A, B, C and D as for example in Figure 14.4, the values of the solution of the wave equation are related as follows

$$u(A) + u(C) = u(B) + u(D).$$

Hence, we can find the value at the vertex A from the values at the three other vertices.

$$u(A) = u(B) + u(D) - u(C).$$

Notice that the values at the vertices B and C in Figure 14.4 can be found from the expression of the solution for the region $(0, 0)$, while the value at D comes from the boundary data. Thus we reduced finding the value at a point in the region $(1, 0)$ to finding values in the region $(0, 0)$. One can always follow this procedure to evaluate the solution in the regions $(m + 1, n)$ and $(m, n + 1)$ via the values in the region (m, n), provided the boundary condition is in the Dirichlet form.

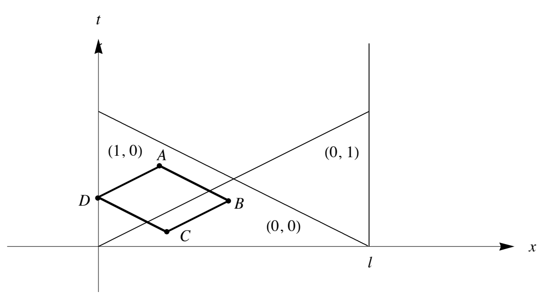


Figure 14.4: The parallelogram rule.

14.2 Conclusion

We applied the reflection method to derive expressions for the solution to the Dirichlet wave problem on the finite interval. However, the method yields infinitely many expressions for different regions in $(x, t) \in (0, l) \times (0, \infty)$, depending on the number of times the backward characteristics from a point get reflected before reaching the x axis, where the initial data is defined. This makes the method impractical in applications, and is not generalizable to higher dimensions and other PDEs. An alternative method (separation of variables) of solving boundary value problems on the finite interval will be described later in the course.

Problem Set 7

1. (#3.1.1 in [Str]) Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.

2. (#3.1.2 in [Str]) Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half-line $0 < x < \infty$.

3. Solve the following Neumann problem for the heat equation on the half-line

$$\begin{cases} u_t - ku_{xx} = 0 & \text{in } 0 < x < \infty, t > 0, \\ u(x,0) = \delta(x-2), \\ u_x(0,t) = 0, \end{cases}$$

   with δ being the Dirac delta function. Explain your solution in terms of heat propagation.

4. (#3.2.5 in [Str]) Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0,t) = 0$, $u(x,0) \equiv 1$, $u_t(x,0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.

5. Consider the Dirichlet problem for the "hammer blow" on the half line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } 0 < x < \infty, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), \\ u(0,t) = 0, \end{cases} \qquad \text{with} \qquad \begin{array}{l} \phi(x) \equiv 0, \\ \psi(x) = \begin{cases} h, & a < x < 2a, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

   Sketch the regions in the xt plane corresponding to the different expressions for the solution. Draw the profile of the string at several times $a/2c$, a/c, $3a/2c$, $5a/2c$ to understand how the wave gets reflected from the boundary.

6. (#3.2.9 in [Str])

   (a) Find $u(\frac{2}{3}, 2)$ if $u_{tt} = u_{xx}$ in $0 < x < 1$, $u(x,0) = x^2(1-x)$, $u_t(x,0) = (1-x)^2$, $u(0,t) = u(1,t) = 0$.

   (b) Find $u(\frac{1}{4}, \frac{7}{2})$.

7. (#3.2.10 in [Str]) Solve the initial boundary value problem on the finite interval (Neumann condition at the left endpoint, Dirichlet condition at the right endpoint)

$$\begin{cases} u_{tt} = 9u_{xx} & \text{in } 0 < x < \pi/2, \\ u(x,0) = \cos x, u_t(x,0) = 0, \\ u_x(0,t) = 0, u(\pi/2,t) = 0. \end{cases}$$

15 Heat with a source

So far we considered homogeneous wave and heat equations and the associated initial value problems on the whole line, as well as the boundary value problems on the half-line and the finite line (for wave only). The next step is to extend our study to the inhomogeneous problems, where an external heat source, in the case of heat conduction in a rod, or an external force, in the case of vibrations of a string, are also accounted for. We first consider the inhomogeneous heat equation on the whole line,

$$\begin{cases} u_t - ku_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), \end{cases} \tag{15.1}$$

where $f(x,t)$ and $\phi(x)$ are arbitrary given functions. The right hand side of the equation, $f(x,t)$ is called the *source* term, and measures the physical effect of an external heat source. It has units of heat flux (left hand side of the equation has the units of u_t, i.e. change in temperature per unit time), thus it gives the instantaneous temperature change due to an external heat source.

From the superposition principle, we know that the solution of the inhomogeneous equation can be written as the sum of the solution of the homogeneous equation, and a particular solution of the inhomogeneous equation. We can thus break problem (15.1) into the following two problems

$$\begin{cases} u^h_t - ku^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \end{cases} \tag{15.2}$$

and

$$\begin{cases} u^p_t - ku^p_{xx} = f(x,t), \\ u^p(x,0) = 0. \end{cases} \tag{15.3}$$

Obviously, $u = u^h + u^p$ will solve the original problem (15.1).

Notice that we solve for the general solution of the homogeneous equation with arbitrary initial data in (15.2), while in the second problem (15.3) we solve for a particular solution of the inhomogeneous equation, namely the solution with zero initial data. This reduction of the original problem to two simpler problems (homogeneous, and inhomogeneous with zero data) using the superposition principle is a standard practice in the theory of linear PDEs.

We have solved problem (15.2) before, and arrived at the solution

$$u^h(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy, \tag{15.4}$$

where $S(x,t)$ is the heat kernel. Notice that the physical meaning of expression (15.4) is that the heat kernel averages out the initial temperature distribution along the entire rod.

Since $f(x,t)$ plays the role of an external heat source, it is clear that this heat contribution must be averaged out, too. But in this case one needs to average not only over the entire rod, but over time as well, since the heat contribution at an earlier time will effect the temperatures at all later times. We claim that the solution to (15.3) is given by

$$u^p(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds. \tag{15.5}$$

Notice that the time integration is only over the interval $[0,t]$, since the heat contribution at later times can not effect the temperature at time t. Combining (15.4) and (15.5) we obtain the following solution to the IVP (15.1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds, \tag{15.6}$$

or, substituting the expression of the heat kernel,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4k(t-s)}}{\sqrt{4\pi k(t-s)}} f(y,s)\, dy\, ds.$$

One can draw parallels between formula (15.6) and the solution to the inhomogeneous ODE analogous to the heat equation. Indeed, consider the IVP for the following ODE.

$$\begin{cases} \frac{d}{dt} u(t) - Au(t) = f(t), \\ u(0) = \phi, \end{cases} \tag{15.7}$$

where A is a constant (more generally, for vector valued u, the equation will be a system of ODEs for the components of u, and A will be a matrix with constant entries). Using an integrating factor e^{-At}, the ODE in (15.7) yields

$$\frac{d}{dt}\left(e^{-At}u\right) = e^{-At}\frac{du}{dt} - Ae^{-At}u = e^{-At}(u' - Au) = e^{-At} f(t).$$

But then

$$e^{-At}u = \int_0^t e^{-As} f(s)\, ds + e^{-A\cdot 0} u(0),$$

and multiplying both sides by e^{At} gives

$$u(t) = e^{At}\phi + \int_0^t e^{A(t-s)} f(s)\, ds. \tag{15.8}$$

The operator $\mathscr{S}(t)\phi = e^{At}\phi$, called the *propagator* operator, maps the initial value ϕ to the solution of the homogeneous equation at later times. In terms of this operator, we can rewrite solution (15.8) as

$$u(t) = \mathscr{S}(t)\phi + \int_0^t \mathscr{S}(t-s) f(s)\, ds. \tag{15.9}$$

In the case of the heat equation, the heat propagator operator is

$$\mathscr{S}(t)\phi = \int_{-\infty}^{\infty} S(x-y,t)\phi(t)\, dy,$$

which again maps the initial data ϕ to the solution of the homogeneous equation at later times. Using the heat propagator, we can rewrite formula (15.6) in exactly the same form as (15.9).

We now show that (15.6) indeed solves the problem (15.1) by a direct substitution. Since we have solved the homogeneous equation before, it suffices to show that u^p given by (15.5) solves problem (15.3). Differentiating (15.5) with respect to t gives

$$\partial_t u^p = \int_{-\infty}^{\infty} S(x-y,0) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s) f(y,s)\, dy\, ds.$$

Recall that the heat kernel solves the heat equation and has the Dirac delta function as its initial data, i.e. $S_t = kS_{xx}$, and $S(x-y,0) = \delta(x-y)$. Hence,

$$\begin{aligned} \partial_t u^p &= \int_{-\infty}^{\infty} \delta(x-y) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s) f(y,s)\, dy\, ds \\ &= f(x,t)\, dy + k\frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\, dy\, ds = f(x,t) + ku^p_{xx}, \end{aligned}$$

which shows that u^p solves the inhomogeneous heat equation. It is also clear that

$$\lim_{t\to 0} u^p(x,t) = \lim_{t\to 0} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds = 0.$$

Thus, u^p given by (15.5) indeed solves problem (15.3), which finishes the proof that (15.6) solves the original IVP (15.1).

Example 15.1. Find the solution of the inhomogeneous heat equation with the source $f(x,t) = \delta(x-2)\delta(t-1)$ and zero initial data.

Using formula (15.6), and substituting the expression for $f(x,t)$, and $\phi(x) = 0$, we get

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\delta(y-2)\delta(s-1)\,dy\,ds = \int_0^t S(x-2,t-s)\delta(s-1)\,ds.$$

For the last integral, notice that if $t < 1$, then $\delta(s-1) = 0$ for all $s \in [0,t]$, and if $t > 1$, then the delta function will act on the heat kernel by assigning its value at $s = 1$. Hence,

$$u(x,t) = \begin{cases} 0 & \text{for } 0 < t < 1, \\ S(x-2,t-1) & \text{for } t > 1. \end{cases}$$

This, of course, coincides with our intuition of heat conduction, since the external heat source in this case gives an instantaneous temperature boost to the point $x = 1$ at time $t = 1$. Henceforth, the temperature in the rod will remain zero till the time $t = 1$, and afterward the heat will transfer exactly as in the case of the homogeneous heat equation with data given at time $t = 1$ as $u(x,1) = \delta(x-2)$. □

15.1 Source on the half-line

We will use the reflection method to solve the inhomogeneous heat equation on the half-line. Consider the Dirichlet heat problem

$$\begin{cases} v_t - kv_{xx} = f(x,t), & \text{for } 0 < x < \infty, \\ v(x,0) = \phi(x), \\ v(0,t) = h(t). \end{cases} \tag{15.10}$$

Notice that in the above problem not only the equation is inhomogeneous, but the boundary data is given by an arbitrary function $h(t)$. In this case the Dirichlet condition is called inhomogeneous. We can reduce the above problem to one with zero initial data by the following subtraction method. Defining the new quantity

$$V(x,t) = v(x,t) - h(t), \tag{15.11}$$

we have that

$$\begin{aligned} &V_t - kV_{xx} = v_t - h'(t) - kv_{xx} = f(x,t) - h'(t), \\ &V(x,0) = v(x,0) - h(0) = \phi(x) - h(0), \\ &V(0,t) = v(0,t) - h(t) = h(t) - h(t) = 0. \end{aligned}$$

Thus, $v(x,t)$ solves problem (15.10) if and only if $V(x,t)$ solves the Dirichlet problem

$$\begin{cases} V_t - kV_{xx} = f(x,t) - h'(t), & \text{for } 0 < x < \infty, \\ V(x,0) = \phi(x) - h(0), \\ V(0,t) = 0. \end{cases} \tag{15.12}$$

With this procedure, we essentially combined the heat source given as the boundary data at the endpoint $x = 0$ with the external heat source $f(x,t)$. Notice that $h(t)$ has units of temperature, so its derivative will have units of heat flux, which matches the units of $f(x,t)$. We will denote the combined source in the last problem by $F(x,t)=f(x,t)-h'(t)$, and the initial data by $\Phi(x)=\phi(x)-h(0)$. Since the Dirichlet boundary condition for V is homogeneous, we can extend $F(x,t)$ and $\Phi(x,t)$ to the whole line in an odd fashion, and use the reflection method to solve (15.12). The extensions are

$$\Phi_{\text{odd}}(x)=\begin{cases}\phi(x)-h(0) & \text{for } x>0,\\ 0 & \text{for } x=0,\\ -\phi(-x)+h(0) & \text{for } x<0,\end{cases}\qquad F_{\text{odd}}(x,t)=\begin{cases}f(x,t)-h'(t) & \text{for } x>0,\\ 0 & \text{for } x=0,\\ -f(-x,t)+h'(t) & \text{for } x<0.\end{cases}$$

Clearly, the solution to the problem

$$\begin{cases}U_t-kU_{xx}=F_{\text{odd}}(x,t), & \text{for } -\infty<x<\infty,\\ U(x,0)=\Phi_{\text{odd}}(x),\end{cases}$$

is odd, since $U(x,t)+U(-x,t)$ will solve the homogeneous heat equation with zero initial data. Then $U(0,t)=0$, and the restriction to $x\geq 0$ will solve the Dirichlet problem (15.12) on the half-line. Thus, for $x>0$,

$$V(x,t)=\int_{-\infty}^{\infty}S(x-y,t)\Phi_{\text{odd}}(y)\,dy+\int_0^t\int_{-\infty}^{\infty}S(x-y,t-s)F_{\text{odd}}(y,s)\,dy\,ds.$$

Proceeding exactly as in the case of the (homogeneous) heat equation on the half-line, we will get

$$\begin{aligned}V(x,t)=&\int_0^{\infty}[S(x-y,t)-S(x+y,t)](\phi(y)-h(0))\,dy\\ &+\int_0^t\int_0^{\infty}[S(x-y,t-s)-S(x+y,t-s)](f(y,s)-h'(s))\,dy\,ds.\end{aligned}$$

Finally, using that $v(x,t)=V(x,t)+h(t)$, we have

$$\begin{aligned}v(x,t)=h(t)+&\int_0^{\infty}[S(x-y,t)-S(x+y,t)](\phi(y)-h(0))\,dy\\ &+\int_0^t\int_0^{\infty}[S(x-y,t-s)-S(x+y,t-s)](f(y,s)-h'(s))\,dy\,ds.\end{aligned}$$

15.2 Conclusion

Using our intuition of heat conduction as an averaging process with the weight given by the heat kernel, we guessed formula (15.6) for the solution of the inhomogeneous heat equation, treating the inhomogeneity as an external heat source. Employing the propagator operator, this formula coincided exactly with the solution formula for the analogous inhomogeneous ODE, which further hinted at the correctness of the formula. However, to obtain a rigorous proof that formula (15.6) indeed gives the unique solution, we verified that the function given by the formula satisfies the equation and the initial condition by a direct substitution. One can then use this formula along with the reflection method to also find the solution for the inhomogeneous heat equation on the half-line.

16 Waves with a source

Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{16.1}$$

where $f(x,t)$, $\phi(x)$ and $\psi(x)$ are arbitrary given functions. Similar to the inhomogeneous heat equation, the right hand side of the equation, $f(x,t)$, is called the *source* term. In the case of the string vibrations this term measures the external force (per unit mass) applied on the string, and the equation again arises from Newton's second law, in which one now also has a nonzero external force.

As was done for the inhomogeneous heat equation, we can use the superposition principle to break problem (16.1) into two simpler ones:

$$\begin{cases} u^h_{tt} - c^2 u^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \quad u^h_t(x,t) = \psi(x), \end{cases} \tag{16.2}$$

and

$$\begin{cases} u^p_{tt} - c^2 u^p_{xx} = f(x,t), \\ u^p(x,0) = 0, \quad u^p_t(x,t) = 0. \end{cases} \tag{16.3}$$

Obviously, $u = u^h + u^p$ will solve the original problem (16.1). u^h solves the homogeneous equation, so it is given by d'Alambert's formula. Thus, we only need to solve the inhomogeneous equation with zero data, i.e. problem (16.3). We will show that the solution to the original IVP (16.1) is

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{16.4}$$

The first two terms in the above formula come from d'Alambert's formula for the homogeneous solution u^h, so to prove formula (16.4), it suffices to show that the solution to the IVP (16.3) is

$$u^p(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{16.5}$$

For simplicity, we will seize specifying the superscript and write $u = u^p$ (this corresponds to the assumption $\phi(x) \equiv \psi(x) \equiv 0$, which is the only remaining case to solve).

Recall that we have already solved inhomogeneous hyperbolic equations by the method of characteristics, which we will apply to the inhomogeneous wave equation as well. The change of variables into the characteristic variables and back are given by the following formulas

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct, \end{cases} \qquad \begin{cases} t = \frac{\xi-\eta}{2c}, \\ x = \frac{\xi+\eta}{2}. \end{cases} \tag{16.6}$$

To write the equation in the characteristic variables, we compute u_{tt} and u_{xx} in terms of (ξ, η) using the chain rule.

$$\begin{aligned} u_t &= cu_\xi - cu_\eta, & u_x &= u_\xi + u_\eta, \\ u_{tt} &= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}, & u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \end{aligned}$$

so

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}. \tag{16.7}$$

Notice that $u(x,t) = u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$, and we made an abuse of notation above to identify u with the function $U(\xi,\eta) = u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$. In the same way, we will identify f with the function $F(\xi,\eta) = f(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$, and will implicitly understand that the functions in terms of (ξ,η) depend on $(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$.

Using (16.7), we can rewrite the inhomogeneous wave equation in terms of the characteristic variables as

$$u_{\xi\eta} = -\frac{1}{4c^2} f(\xi, \eta). \tag{16.8}$$

To solve this equation, we need to successively integrate in terms of η and then ξ. Recall that in previous examples of inhomogeneous hyperbolic equations we performed these integrations explicitly, then changed the variables back to (x, t), and determined the integration constants from the initial conditions. In our present case, however, we would like to obtain a formula for the general function f, so explicit integration is not an option. Thus, to determine the constants of integration, we need to rewrite the initial conditions in terms of the characteristic variables.

Notice that from (16.6), $t = 0$ is equivalent to $(\xi - \eta)/2c = 0$, or $\xi = \eta$. The initial conditions of (16.3) then imply

$$\begin{aligned} &u(\xi, \xi) = 0, \\ &cu_\xi(\xi, \xi) - cu_\eta(\xi, \xi) = 0, \\ &u_\xi(\xi, \xi) + u_\eta(\xi, \xi) = 0, \end{aligned}$$

where the last identity is equivalent to the identity $u_x(x, 0) = 0$, which can be obtained by differentiating the first initial condition of (16.3). From the last two conditions above, it is clear that $u_\xi(\xi, \xi) = u_\eta(\xi, \xi) = 0$, so the initial conditions in terms of the characteristic variables are

$$u(\xi, \xi) = u_\xi(\xi, \xi) = u_\eta(\xi, \xi) = 0. \tag{16.9}$$

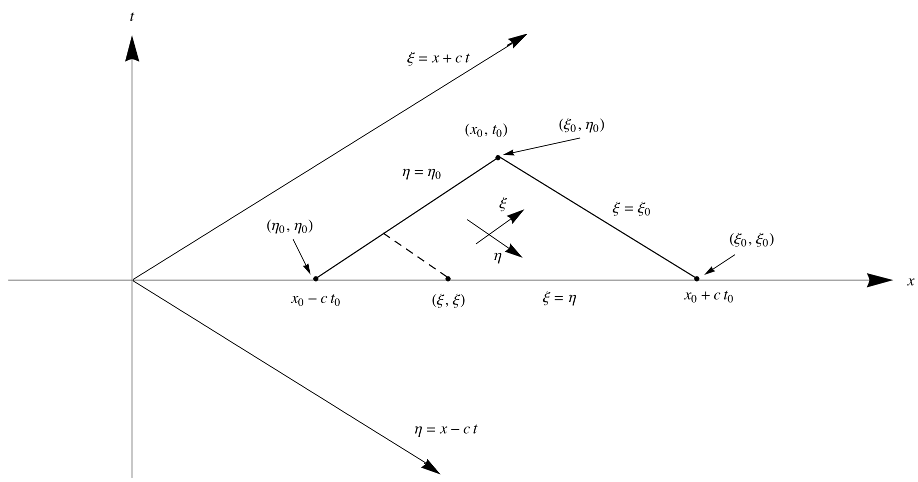


Figure 16.1: The triangle of dependence of the point (x_0, t_0).

Now fix a point (x_0, t_0) for which we will show formula (16.5). This point has the coordinates (ξ_0, η_0) in the characteristic variables. To find the value of the solution at this point, we first integrate equation (16.8) in terms of η from ξ to η_0

$$\int_\xi^{\eta_0} u_{\xi\eta}\, d\eta = -\frac{1}{4c^2} \int_\xi^{\eta_0} f(\xi, \eta)\, d\eta.$$

But

$$\int_\xi^{\eta_0} u_{\xi\eta}\, d\eta = u_\xi(\xi, \eta_0) - u_\xi(\xi, \xi) = u_\xi(\xi, \eta_0)$$

due to (16.9) (this is precisely the reason for the choice of the lower limit), so we have

$$u_\xi(\xi, \eta_0) = \frac{1}{4c^2} \int_{\eta_0}^{\xi} f(\xi, \eta)\, d\eta.$$

Integrating this identity with respect to ξ from η_0 to ξ_0 gives

$$\int_{\eta_0}^{\xi_0} u_\xi(\xi,\eta_0)\,d\xi = \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f(\xi,\eta)\,d\eta\,d\xi.$$

Similar to the previous integral,

$$\int_{\eta_0}^{\xi_0} u_\xi(\xi,\eta_0)\,d\xi = u(\xi_0,\eta_0) - u_\xi(\eta_0,\eta_0) = u(\xi_0,\eta_0)$$

due to (16.9). We then have

$$u(\xi_0,\eta_0) = \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f(\xi,\eta)\,d\eta\,d\xi = \frac{1}{4c^2}\iint_\triangle f(\xi,\eta)\,d\xi\,d\eta, \tag{16.10}$$

where the double integral is taken over the triangle of dependence of the point (x_0,t_0), as depicted in Figure 16.1. Using the change of variables (16.6), and computing the Jacobian,

$$J = \frac{\partial(\xi,\eta)}{\partial(x,t)} = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -2c,$$

we can transform the double integral in (16.10) to a double integral in terms of the (x,t) variables to get

$$u(x_0,t_0) = \frac{1}{4c^2}\iint_\triangle f(x,t)|J|\,dx\,dt = \frac{1}{2c}\iint_\triangle f(x,t)\,dx\,dt.$$

Finally, rewriting the last double integral as an iterated integral, we will arrive at formula (16.5). This finishes the proof that (16.4) is the unique solution of the IVP (16.1). One can alternatively show that formula (16.4) gives the solution by directly substituting it into (16.1), which is left as a homework problem.

Example 16.1. Solve the inhomogeneous wave IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^x, \\ u(x,0) = u_t(x,0) = 0. \end{cases}$$

Using formula (16.4) with $\phi = \psi = 0$, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^y\,dy\,ds = \frac{1}{2c}\int_0^t [e^{x+c(t-s)} - e^{x-c(t-s)}]\,ds \\ &= \frac{e^x}{2c}\left(-\frac{1}{c}e^{c(t-s)}\Big|_0^t - \frac{1}{c}e^{-c(t-s)}\Big|_0^t\right) = \frac{e^x}{2c^2}(e^{ct} + e^{-ct} - 2). \end{aligned}$$

□

16.1 Source on the half-line

Consider the following inhomogeneous Dirichlet wave problem on the half-line

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t), \quad \text{for } 0 < x < \infty, t > 0, \\ v(x,0) = \phi(x), \quad v_t(x,t) = \psi(x), \\ v(0,t) = h(t). \end{cases} \tag{16.11}$$

One can employ the subtraction method that we used for the heat equation to reduce the problem to one with zero Dirichlet data, and then use the reflection method to derive a solution formula for the reduced problem. An alternative simple way, however, is to derive the solution from scratch as follows. Since we know how to find the solution for zero Dirichlet data (use the standard reflection method), we treat the complementary case, that is, assume that the boundary data is nonzero, while $f(x,t) \equiv \phi(x) \equiv \psi(x) \equiv 0$.

From the method of characteristics, we know that the solution can be written as

$$v(x,t) = j(x+ct) + g(x-ct). \tag{16.12}$$

The zero initial conditions then give

$$\begin{aligned} v(x,0) &= j(x) + g(x) = 0, \\ v_t(x,0) &= cj'(x) - cg'(x) = 0, \end{aligned}$$

for $x > 0$. Differentiating the first identity, and dividing the second identity by c, we arrive at the following system for j' and g'

$$\begin{cases} j'(x) + g'(x) = 0, \\ j'(x) - g'(x) = 0, \end{cases} \qquad \Rightarrow \qquad j'(x) = g'(x) = 0.$$

This means that for $s > 0$,

$$j(s) = -g(s) = a$$

for some constant a. On the other hand, the boundary condition for $v(x,t)$ implies

$$v(0,t) = j(ct) + g(-ct) = h(t).$$

But since $ct > 0$, we have $j(ct) = a$, and

$$g(-ct) = h(t) - a, \qquad \text{or} \qquad g(s) = h(-s/c) - a$$

for $s < 0$. Returning to (16.12), notice that the argument of the j term is always positive, so

$$v(x,t) = \begin{cases} a - a & \text{for } x > ct, \\ a + h\left(t - \frac{x}{c}\right) - a & \text{for } x < ct. \end{cases} = \begin{cases} 0 & \text{for } x > ct, \\ h\left(t - \frac{x}{c}\right) & \text{for } x < ct. \end{cases}$$

Thus, for $x > ct$ the solution of (16.11) will be given by (16.4), while for $x < ct$ we have

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,dy + h\left(t - \frac{x}{c}\right) + \frac{1}{2c}\iint_D f(y,s)\,dy\,ds,$$

where D is the domain of dependence of the point (x,t).

16.2 Conclusion

The superposition principle was again used to write the solution to the IVP for the inhomogeneous wave equation as a sum of the general homogeneous solution, and the inhomogeneous solution with zero initial data. The inhomogeneous solution was obtained by the method of characteristics through a successive integration in terms of the characteristic variables. One can also derive the solution formula for the inhomogeneous wave equation by simply integrating the equation over the domain of dependence, and using Green's theorem to compute the integral of the left hand side. Yet another way is to approach the solution of the inhomogeneous equation by studying the propagator operator of the wave equation, similar to what we did for the heat equation. These methods are discussed in the appendix.

In the previous lecture we used the method of characteristics to solve the initial value problem for the inhomogeneous wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{17.1}$$

and obtained the formula

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{17.2}$$

Another way to derive the above solution formula is to integrate both sides of the inhomogeneous wave equation over the triangle of dependence and use Green's theorem.

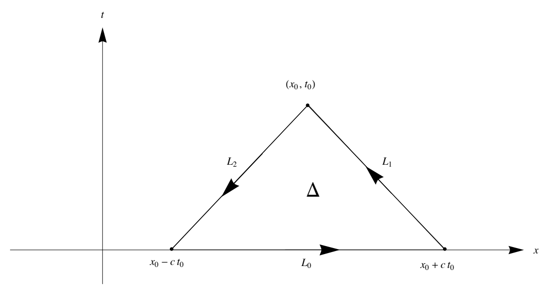


Figure 17.1: The triangle of dependence of the point (x_0, t_0).

Fix a point (x_0, t_0), and integrate both sides of the equation in (17.1) over the triangle of dependence for this point.

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\,dx\,dt = \iint_\triangle f(x,t)\,dx\,dt. \tag{17.3}$$

Recall that by Green's theorem

$$\iint_D (Q_x - P_t)\,dx\,dt = \oint_{\partial D} P\,dx + Q\,dt,$$

where ∂D is the boundary of the region D with counterclockwise orientation. We thus have

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\,dx\,dt = \iint_\triangle (-c^2 u_x)_x - (-u_t)_t\,dx\,dt = \oint_{\partial\triangle} -u_t\,dx - c^2 u_x\,dt.$$

The boundary of the triangle of dependence consists of three sides, $\partial\triangle = L_0 + L_1 + L_2$, as can be seen in Figure 17.1, so

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\,dx\,dt = \int_{L_0 + L_1 + L_2} -u_t\,dx - c^2 u_x\,dt,$$

and we have the following relations on each of the sides

$$\begin{aligned} L_0 &: \quad dt = 0 \\ L_1 &: \quad dx = -cdt \\ L_2 &: \quad dx = cdt \end{aligned}$$

Using these, we get

$$\int_{L_0} -c^2 u_x\,dt - u_t\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx,$$
$$\int_{L_1} -c^2 u_x\,dt - u_t\,dx = c\int_{L_1} du = c[u(x_0,t_0) - u(0,x_0+ct_0)] = cu(x_0,t_0) - c\phi(x_0+ct_0),$$
$$\int_{L_2} -c^2 u_x\,dt - u_t\,dx = -c\int_{L_2} du = -c[u(0,x_0-ct_0) - u(x_0,t_0)] = cu(x_0,t_0) - c\phi(x_0-ct_0).$$

Putting all the sides together gives

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\,dx\,dt = 2cu(x_0,t_0) - c[\phi(x_0+ct_0) + \phi(x_0-ct_0)] - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx,$$

and using (17.3), we obtain

$$u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0) + \phi(x_0-ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx + \frac{1}{2c}\iint_\triangle f(x,t)\,dx\,dt,$$

which is equivalent to formula (17.2).

17.1 The operator method

For the inhomogeneous heat equation we interpreted the solution formula in terms of the heat propagator, which also showed the parallels between the heat equation and the analogous ODE. We would like to obtain such a description for the solution formula (17.2) as well. For this, consider the ODE analog of the wave equation with the associated initial conditions

$$\begin{cases} \frac{d^2u}{dt^2} + A^2 u = f(t), \\ u(0) = \phi, \quad u'(0) = \psi, \end{cases} \tag{17.4}$$

where A is a constant (a matrix, if we allow u to be vector valued). To find the solution of the inhomogeneous ODE, we need to first solve the homogeneous equation, and then use variation of parameters to find a particular solution of the inhomogeneous equation. The solution of the homogeneous equation is

$$u^h(t) = c_1\cos(At) + c_2\sin(At),$$

and the initial conditions imply that $c_1 = \phi$, and $c_2 = A^{-1}\psi$. To obtain a particular solution of the inhomogeneous equation we assume that c_1 and c_2 depend on t,

$$u^p(t) = c_1(t)\cos(At) + c_2(t)\sin(At),$$

and substitute u^p into the equation to solve for $c_1(t)$ and $c_2(t)$. This procedure leads to

$$c_1(t) = -\int_0^t A^{-1}\sin(As)f(s)\,dt, \qquad c_2(t) = \int_0^t A^{-1}\cos(As)f(s)\,ds.$$

Putting everything together, the solution to (17.4) will be

$$u(t) = \cos(At)\phi + A^{-1}\sin(At)\psi + \int_0^t A^{-1}\sin(A(t-s))f(s)\,ds.$$

If we now define the propagator

$$\mathscr{S}(t)\psi = A^{-1}\sin(At)\psi,$$

then the solution to (17.4) can be written as

$$u(t) = \mathscr{S}'(t)\phi + \mathscr{S}(t)\psi + \int_0^t \mathscr{S}(t-s)f(s)\,ds. \tag{17.5}$$

For the wave equation, similarly denoting the operator acting on ψ from d'Alambert's formula by

$$\mathscr{S}(t)\psi = \int_{x-ct}^{x+ct} \psi(y)\,dy,$$

we can rewrite formula (17.2) in exactly the same form as (17.4). The moral of this story is that having solved the homogeneous equation and found the propagator, we have effectively derived the solution of the inhomogeneous equation as well. The rigorous connection between the solution of the homogeneous equation and that of the inhomogeneous wave equation is contained in the following statement.

Duhamel's Principle. Consider the following 1-parameter family of wave IVPs

$$\begin{cases} u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s) = 0, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), \end{cases} \tag{17.6}$$

then the function

$$v(x,t) = \int_0^t u(x,t;s)\,ds$$

solves the inhomogeneous wave equation with vanishing data, i.e.

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = f(x,t), \\ v(x,0) = 0, \quad v_t(x,0) = 0. \end{cases}$$

Note that the initial conditions of the s-IVP (17.6) are given at time $t = s$, and the initial velocity is $\psi(x;s) = f(x,s)$. Duhamel's principle has the physical description of replacing the external force by its effect on the velocity. From Newton's second law, the force is responsible for acceleration, or change in velocity per unit time. So if we can account for the effect of the external force on the instantaneous velocity, then the the solution of the equation with the external force can be found by solving the homogeneous equations with the effected velocities, namely (17.6), and "summing" these solutions over the instances $t = s$.

We prove Duhamel's principle by direct substitution. The derivatives of v are

$$\begin{aligned}
v_t(x,t) &= u(x,t;t) + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds, \\
v_{tt}(x,t) &= u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds = f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds, \\
v_{xx}(x,t) &= \int_0^t u_{xx}(x,t;s)\,ds,
\end{aligned}$$

where we used the initial conditions of (17.6). Substituting this into the wave equation gives

$$(\partial_t^2 - c^2\partial_x^2)v = \int_0^t [u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s)]\,ds + f(x,t) = f(x,t),$$

and v indeed solves the inhomogeneous wave equation. It is also clear that v has vanishing initial data.

Duhamel's principle gives an alternative way of proving that (17.2) solves the inhomogeneous wave equation. Indeed, from d'Alambert's formula for (17.6) and a time shift $t \mapsto t - s$, we have

$$u(x, t; s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s)\, dy.$$

Thus, the solution of the inhomogeneous wave equation with zero initial data is

$$v(x, t) = \int_0^t u(x, t; s)\, ds = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s)\, dy\, ds.$$

17.2 Conclusion

We defined the wave propagator as the operator that maps the initial velocity to the solution of the homogeneous wave equation with zero initial displacement. Using this operator, the solution of the inhomogeneous wave equation can be written in exactly the same form as the solution of the analogous inhomogeneous ODE in terms of its propagator. The significance of this observation is in the connection between the solution of the homogeneous and that of the inhomogeneous wave equations, which is the substance of Duhamel's principle. Hence, to solve the inhomogeneous wave equation, all one needs is to find the propagator operator for the homogeneous equation.

Problem Set 8

1. Solve the initial value problem for the following inhomogeneous heat equation

$$\begin{cases} u_t - \frac{1}{4}u_{xx} = e^{-t} & \text{in } -\infty < x < \infty, t > 0, \\ u(x,0) = x^2. \end{cases}$$

2. Solve the following Dirichlet problem for the inhomogeneous heat equation on the half-line

$$\begin{cases} v_t - kv_{xx} = \delta(t-1), & \text{for } 0 < x < \infty, 0 < t < \infty, \\ v(0,t) = 0; \quad v(x,0) = \delta(x-2). \end{cases}$$

Explain in terms of heat conduction how the external heat source effects the temperature in the rod.

3. (#3.3.3 in [Str]) Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{cases} w_t - kw_{xx} = 0, & \text{for } 0 < x < \infty, 0 < t < \infty, \\ w_x(0,t) = h(t); & w(x,0) = \phi(x), \end{cases}$$

by the subtraction method indicated in the text.

4. (#3.4.1 in [Str]) Solve $u_{tt} = c^2u_{xx} + xt, \quad u(x,0) = 0, \quad u_t(x,0) = 0.$

5. (#3.4.3 in [Str]) Solve $u_{tt} = c^2u_{xx} + \cos x, \quad u(x,0) = \sin x, \quad u_t(x,0) = 1 + x.$

6. (#3.4.5 in [Str]) Let $f(x,t)$ be any function and let $u(x,t) = (1/2c)\iint_\triangle f$, where $\triangle$ is the triangle of dependence. Written as an iterated integral,

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\,dy\,ds.$$

Verify directly by differentiation that

$$u_{tt} = c^2u_{xx} + f(x,t) \quad \text{and} \quad u(x,0) \equiv u_t(x,0) \equiv 0.$$

7. (#3.4.13 in [Str]) Solve the Dirichlet wave problem on the half-line

$$\begin{cases} u_{tt} = c^2u_{xx} & \text{for } 0 < x < \infty, \\ u(0,t) = t^2; & u(x,0) = x, \quad u_t(x,0) = 0. \end{cases}$$

Earlier in the course we solved the Dirichlet problem for the wave equation on the finite interval $0 < x < l$ using the reflection method. This required separating the domain $(x,t) \in (0,l) \times (0,\infty)$ into different regions according to the number of reflections that the backward characteristic originating in the regions undergo before reaching the x axis. In each of these regions the solution was given by a different expression, which is impractical in applications, and the method does not generalize to higher dimensions or other equations. We now study a different method of solving the boundary value problems on the finite interval, called *separation of variables*.

18.1 Wave equation

Let us start by considering the wave equation on the finite interval with homogeneous Dirichlet conditions.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < l, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \\ u(0,t) = u(l,t) = 0. \end{cases} \tag{18.1}$$

The idea of the separation of variables method is to find the solution of the boundary value problem as a linear combination of simpler solutions (compare this to finding the simpler solution $S(x,t)$ of the heat equation, and then expressing any other solution in terms of the heat kernel). The building blocks in this case will be the *separated solutions*, which are the solutions that can be written as a product of two functions, one of which depends only on x, and the other only on t, i.e.

$$u(x,t) = X(x)T(t). \tag{18.2}$$

Let us try to find all the separated solutions of the wave equation. Substituting (18.2) into the equation gives

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Dividing both sides of these identity by $-c^2 X(x)T(t)$, we get

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{c^2 T(t)} = \lambda. \tag{18.3}$$

Clearly λ is a constant, since it is independent of x from $\lambda = -T''/(c^2 T)$, and is independent of t from $\lambda = -X''/X$. We will shortly see that the boundary conditions force λ to be positive, so let $\lambda = \beta^2$, for some $\beta > 0$. One can then rewrite (18.3) as a pair of separate ODEs for $X(x)$ and $T(t)$

$$T'' + c^2\beta^2 T = 0, \qquad \text{and} \qquad X'' + \beta^2 X = 0.$$

The solutions of these ODEs are

$$T(t) = A\cos\beta ct + B\sin\beta ct, \qquad \text{and} \qquad X(x) = C\cos\beta x + D\sin\beta x, \tag{18.4}$$

where A, B, C and D are arbitrary constants. From the boundary conditions in (18.1), we have

$$X(0)T(t) = X(l)T(t) = 0, \quad \forall t \qquad \Rightarrow \qquad X(0) = X(l) = 0,$$

since $T(t) \equiv 0$ would result in the trivial solution $u(x,t) \equiv 0$ (our goal is to find all separated solutions). With this boundary condition for $X(x)$, we have from (18.4)

$$X(0) = C = 0, \qquad \text{and} \qquad X(l) = D\sin\beta l = 0.$$

The solution with $D = 0$ will again lead to the trivial zero solution, so we consider the case when $\sin\beta l = 0$. But this implies that $\beta l = n\pi$ for $n = 1, 2, \ldots$, and

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad X_n(x) = \sin\frac{n\pi x}{l} \quad \text{for } n = 1, 2, \ldots$$

These formulas give distinct solutions for $X(x)$, and multiplying these by the $T(t)$ corresponding to λ_n, we find infinitely many separated solutions

$$u_n(x,t) = \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l} \quad \text{for } n = 1, 2, \ldots,$$

where A_n, B_n are arbitrary constants as before. Since a linear combination of solutions of the wave equation is also a solution, any finite sum

$$u(x,t) = \sum_n \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}, \tag{18.5}$$

will also solve the wave equation.

Returning to our boundary value problem (18.1), we would like to find the solution as a linear combination of separated solutions. However, finite sums in the form (18.5) are very special, since not every function is a finite sum of sines and cosines. Checking the initial conditions, we have

$$\begin{aligned} \phi(x) &= \sum_n A_n \sin\frac{n\pi x}{l}, \\ \psi(x) &= \sum_n \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l}. \end{aligned} \tag{18.6}$$

Obviously, not all initial data ϕ, ψ can be written as finite sums of sine functions. So instead of restricting ourselves to finite sums, we allow infinite sums, and ask the question whether any functions ϕ, ψ can be written as infinite sums of sine functions. This question was first studied by Fourier, and these infinite sums have the name of *Fourier series* (Fourier sine series in this case). It turns out that practically any function defined on $0 < x < l$ can be expressed in the form (18.6). Leaving the question of convergence of such sums, we see that if the initial data can be expressed in the form (18.6), then the solution is given by (18.5).

The coefficients of t inside the series (18.5), $\frac{n\pi c}{l}$, are called the frequencies. For a violin string of length l, we had $c^2 = \frac{T}{\rho}$, so the frequencies are

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \quad n = 1, 2, \ldots$$

The smallest frequency, $\frac{\pi\sqrt{T}}{l\sqrt{\rho}}$, is the fundamental note, while the double, triple, and so on of the fundamental note are the overtones. Notice that by shortening the length l of the vibrating portion of the string with a finger, a violinist produces notes of higher frequency.

18.2 Heat equation

For the Dirichlet heat problem on the finite interval,

$$\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < l, \\ u(x,0) = \phi(x), \\ u(0,t) = u(l,t) = 0, \end{cases} \tag{18.7}$$

we similarly search for all the separated solutions in the form $u(x,t) = X(x)T(t)$. In this case the equation gives

$$-\frac{X''}{X} = -\frac{T'}{kt} = \beta^2,$$

and the resulting ODEs are

$$T' = -\beta^2 kT, \qquad \text{and} \qquad X'' + \beta^2 X = 0.$$

The solution for the T equation is then $T(t) = Ae^{-\beta^2 kt}$, while the function $X(x)$ satisfies the same equation and boundary conditions as before. This yields the same values $\beta_n = n\pi/l$. We thus have that

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin\frac{n\pi x}{l} \tag{18.8}$$

is the solution to problem (18.7), provided that the initial data is given as

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}. \tag{18.9}$$

Notice that as t grows, all the terms in the series (18.8) decay exponentially, so the solution itself will decay, which makes sense in terms of heat conduction, since in the absence of a heat source, the temperatures in the rod will equalize with the zero temperature of the environment.

Example 18.1. Solve the following Dirichlet problem for the heat equation by separation of variables.

$$\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < \pi/2, \\ u(x,0) = 3\sin 4x, \\ u(0,t) = u(\frac{\pi}{2},t) = 0, \end{cases}$$

In this problem $l = \pi/2$, so $\beta_n = 2n$. We can write the initial data in the form (18.9),

$$3\sin 4x = \sum_{n=1}^{\infty} A_n \sin 2nx,$$

which implies that $A_2 = 3$, and $A_n = 0$ for $n \neq 2$. But then from (18.8) the solution will be

$$u(x,t) = 3e^{-16kt}\sin 4x.$$

□

18.3 Eigenvalues

The numbers $\lambda = \left(\frac{n\pi}{l}\right)^2$ are called eigenvalues, and the functions $X_n(x) = \sin\frac{n\pi x}{l}$ are called eigenfunctions. Notice that we can think of the equation $-X'' = \lambda X$ as an eigenvalue problem for the operator $-\frac{d^2}{dx^2}$ in the space of functions that satisfy the Dirichlet conditions $X(0) = X(l) = 0$. An eigenfunction is then a solution of the equation which is not identically zero, i.e. $X(x) \not\equiv 0$.

However, unlike the operators in linear algebra, which have finitely many eigenvalues, in our case we have an infinite number of eigenvalues. This is due to the fact that the space of functions is infinite dimensional.

We return to the question of the sign of the eigenvalues. Suppose $\lambda = 0$, then we would have $X'' = 0$, which leads to $X(x) = C + Dx$. The boundary conditions then imply that $C = 0$, and $Dl = 0$, giving $X(x) \equiv 0$.

If, on the other hand, we assume that $\lambda < 0$, and write $\lambda = -\gamma^2$ for some $\gamma > 0$, then the equation for X becomes $X'' = \gamma^2 X$, which has the solution

$$X(x) = Ce^{\gamma x} + De^{-\gamma x}.$$

The boundary conditions then give

$$\begin{cases} C + D = 0 \\ Ce^{\gamma l} + De^{-\gamma l} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} C = -D \\ Ce^{2\gamma l} = C \end{cases} \quad \Rightarrow \quad C = D = 0,$$

which again results in the identically zero solution $X(x) \equiv 0$. So there are no nonpositive eigenvalues.

PART 2

Diffusion-Type Problems

LESSON 2

Diffusion-Type Problems (Parabolic Equations)

PURPOSE OF LESSON: To show how parabolic PDEs are used to model heat-flow and diffusion-type problems. The physical meaning of different terms (such as u_t, u_x, u_{xx}, and u) are explained and a few examples of parabolic equations presented.

The idea of an *initial-boundary-value problem* is introduced along with an example. One of the major goals of this lesson is to give the reader an intuitive feeling for parabolic-type problems.

We begin this lesson by introducing a simple physical problem and showing how it can be described by means of a mathematical model (which will involve a PDE). We then complicate the problem and show how new partial differential equations can describe the new situations. The partial differential equations in this lesson are not derived or solved now, but will be in later lessons.

A Simple Heat-Flow Experiment

Suppose we have the following simple experiment that we break into steps:

STEP 1 We start with a reasonably long (say $L = 2$ m) rod (say copper) 2 cm in diameter whose lateral sides (but not the ends) we wrap with insulation. We could even use copper tubing provided we pour some sort of insulation down the inside. In other words, heat can flow in and out of the rod *at the ends*, but not across the lateral boundary.

STEP 2 Next, we place this rod in an environment whose temperature is fixed at some temperature T_0 (degrees °C) for a sufficiently long time, so that the temperature of the entire rod comes to a steady-state temperature similar to the environment. For simplicity, we let the temperature of the environment $T_0 = 10$°C.

STEP 3 We take the rod out of the environment at a time that we call $t = 0$ and attach two *temperature elements* to the ends of the rod. The purpose of these elements is to keep the ends at specific temperatures T_1 and T_2 (say $T_1 = 0°C$ and $T_2 = 50°C$). In other words, two thermostats constantly monitor the temperature at the ends of the rod, and if the temperatures differ from their prescribed values T_1 and T_2, strong heating (or cooling) elements come into operation to adjust the temperature accordingly. Our experiment is illustrated in Figure 2.1.

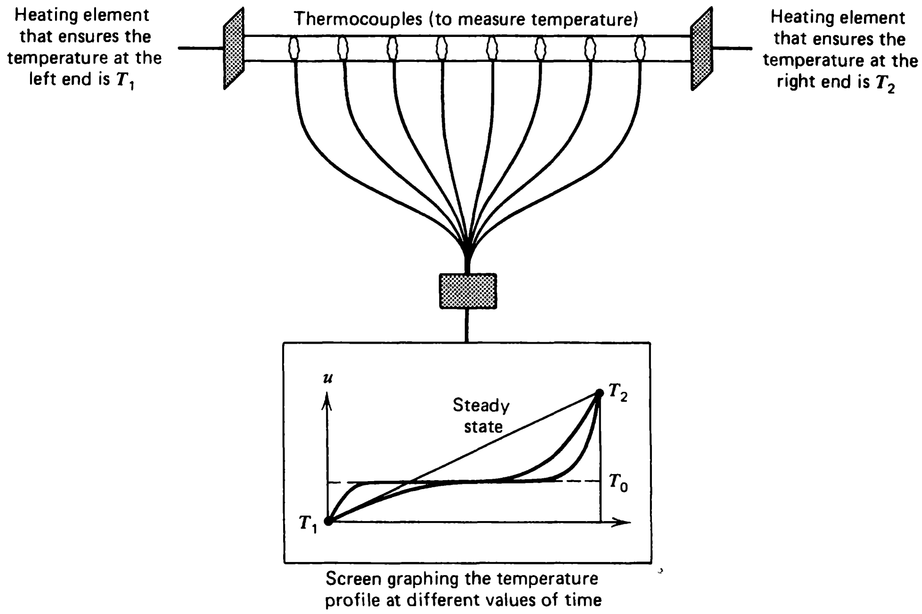


FIGURE 2.1 Schematic diagram of the experiment.

STEP 4 We now monitor the temperature profile of the rod on some type of display. (Why we want to perform an experiment of this kind is another question; we will talk about that later.) This completes our discussion of the experiment. The main purpose of this lesson is to show how this physical problem (and variations of it) can be explained (modeled) by parabolic PDEs.

The Mathematical Model of the Heat-Flow Experiment

The description of our physical problem requires three types of equations

1. The *PDE* describing the physical phenomenon of heat flow.
2. The *boundary conditions* describing the physical nature of our problem on the boundaries.
3. The *initial conditions* describing the physical phenomenon at the start of the experiment.

The Heat Equation

The basic equation of *one-dimensional* heat flow is the relationship

(2.1) $\boxed{\text{PDE} \quad u_t = \alpha^2 u_{xx} \quad 0 < x < L \quad 0 < t < \infty}$

which relates the quantities

u_t = the *rate of change* in temperature with respect to time (measured in deg/sec)

and

u_{xx} = the *concavity* of the temperature profile $u(x,t)$ (which essentially compares the temperature at one point to the temperature at neighboring points).

This equation will be derived from the basic *conservation of heat equation* in later lessons, but for the time being, we examine it by itself. This equation simply says that the temperature $u(x,t)$ (at some point along the rod x and at some point in time t) is increasing ($u_t > 0$) or decreasing ($u_t < 0$) according to whether u_{xx} is positive or negative. Figure 2.2 illustrates the change in temperature at different points along the rod.

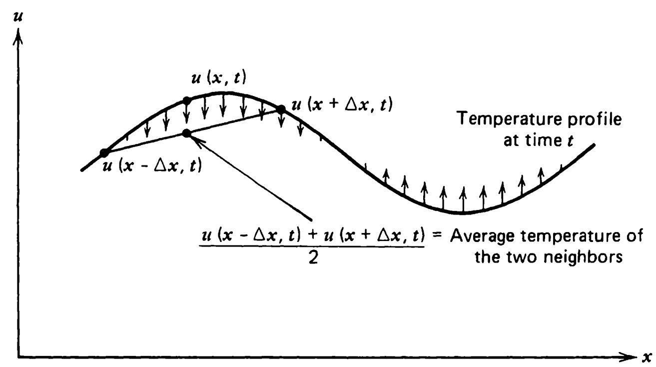


FIGURE 2.2 Arrows indicating change in temperature according to $u_t = \alpha^2 u_{xx}$.

To see how u_{xx} can be interpreted to measure heat flow, suppose we approximate u_{xx} by the difference quotient

$$u_{xx}(x,t) \cong \frac{1}{\Delta x^2}\left[u(x + \Delta x,t) - 2u(x,t) + u(x - \Delta x,t)\right]$$

Since this can be rewritten

$$u_{xx}(x,t) \cong \frac{2}{\Delta x^2}\left[\frac{u(x+\Delta x,t)+u(x-\Delta x,t)}{2} - u(x,t)\right]$$

we have the following interpretation of u_{xx}:

1. If the temperature $u(x,t) <$ average of the two neighboring temperatures, then $u_{xx} > 0$ (here, the net flow of heat into x is positive).
2. If the temperature $u(x,t) =$ average of the two neighboring temperatures, then $u_{xx} = 0$ (here the net flow of heat into x is zero).
3. If the temperature $u(x,t) >$ average of the two neighboring temperatures, then $u_{xx} < 0$ (here the net flow of heat into x is negative).

This is illustrated in Figure 2.2. In other words, if the temperature at a point x is greater than the average of the temperature at two nearby points $x - \Delta x$ and $x + \Delta x$, then the temperature at x will be decreasing. Furthermore, the exact rate of decrease u_t is proportional to this difference. The proportionality constant α^2 is a property of the material, and we will discuss this constant more in the next few lessons.

Boundary Conditions

All physical problems have boundaries of some kind, so we must describe mathematically what goes on there in order to adequately describe the problem. In our experiment, the boundary conditions (BCs) are quite easy. Since the temperature u was fixed for all time $t > 0$ at T_1 and T_2 at the two ends $x = 0$ and $x = L$, we would simply say

$$\text{(2.2)} \qquad \text{BCs}\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \qquad 0 < t < \infty$$

Initial Conditions

All physical problems must start from some value of time (generally called $t = 0$), so we must specify the physical apparatus at this time. Since we started monitoring the rod temperature in our example from the time the rod had achieved a constant temperature of T_0, we have

$$\text{(2.3)} \qquad \boxed{\text{IC} \quad u(x,0) = T_0 \quad 0 \leq x \leq L}$$

We have now mathematically described the experiment. By writing equations (2.1), (2.2), and (2.3) together, we have what is called an *initial-boundary-value problem* (IBVP)

$$
(2.4) \qquad
\begin{array}{lll}
\text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < L \qquad 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = T_0 & 0 \leqslant x \leqslant L
\end{array}
$$

The interesting thing here, which is not at all obvious, is that there is *only one function* $u(x,t)$ that satisfies the problem (2.4), and that function will describe the temperature of the rod. Hence, our goal in the near future will be to find that unique solution $u(x,t)$ to (2.4).

Before finishing this lesson, we will discuss some variations of this basic problem. We start with a few modifications of the heat equation $u_t = \alpha^2 u_{xx}$.

More Diffusion-Type Equations

Lateral Heat Loss Proportional to the Temperature Difference

The equation

$$u_t = \alpha^2 u_{xx} - \beta\,(u - u_0) \qquad \beta > 0$$

describes heat flow in the rod with both diffusion $\alpha^2 u_{xx}$ along the rod and heat loss (or gain) across the *lateral* sides of the rod. Heat loss ($u > u_0$) or gain ($u < u_0$) is proportional to the difference between the temperature $u(x,t)$ of the rod and the surrounding medium u_0 (with β the proportionality constant). If β is very large in contrast to α^2, then the flow of heat *back and forth* along the rod will be small in contrast to the flow *in and out the sides,* and, hence, the heat will drain out the sides (at each point) according to the approximate equation $u_t = -\beta\,(u - u_0)$.

In chemistry where u may stand for concentration, the equation

$$u_t = \alpha^2 u_{xx} - \beta(u - u_0)$$

says that the rate of change (u_t) of the substance is due both to the diffusion $\alpha^2 u_{xx}$ (in the x-direction) and to the fact that the substance is being created ($u < u_0$) or destroyed ($u > u_0$) by a chemical reaction proportional to the difference between two concentrations u and u_0.

Internal Heat Source

The *nonhomogeneous* equation

$$u_t = \alpha^2 u_{xx} + f(x,\,t)$$

corresponds to the situation where the rod is being supplied with an internal heat source (everywhere along the rod and for all time t). It may be that a wire carrying electrical current passes through the rod and the resistance generates a constant heat source $f(x, t) = K$.

Diffusion-convection Equation

Suppose a pollutant is being carried along in a stream moving with velocity v. It is obvious that the concentration $u(x,t)$ of the substance changes as a function of both x (positive x measures the distance downstream) and time t. The rate of change u_t is measured by the *diffusion-convection equation*

$$u_t = \alpha^2 u_{xx} - v u_x$$

The term $\alpha^2 u_{xx}$ is the diffusion contribution and $-vu_x$ is the convection component. Whether the pollutant primarily diffuses or convects depends on the relative size of the two coefficients α^2 and v. You have probably seen smoke rising from a smoke stack. Here, the smoke particles are *convected* upward with the hot air and, at the same time, *diffuse* within the air currents.
In addition to these modifications in the heat equation, the *boundary conditions* of the rod can also be changed to correspond to other physical situations. We will discuss some of these modifications in Lesson 3.

NOTES

The heat equation $u_t = \alpha^2(x)u_{xx}$ with a variable coefficient $\alpha(x)$ would correspond to a problem where the diffusion within the rod depends on x (the material is *nonhomogeneous*). For example, if copper and steel slabs were placed next to each other (see Figure 2.3) and if the left side of the copper slabs were fixed at

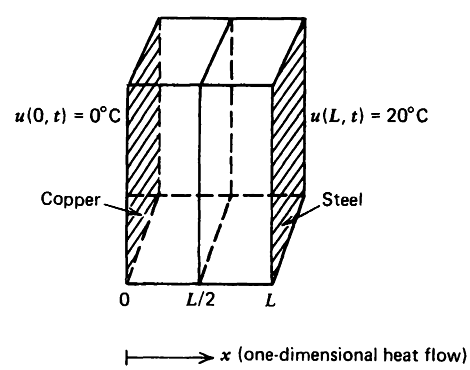


FIGURE 2.3

$u(0,t) = 0°C$ and the right side of the steel sheet were fixed at $u(L,t) = 20°C$, then the PDE that describes the heat flow would be

$$u_t = \alpha^2(x)u_{xx} \qquad 0 < x < L$$

$$\text{where } \alpha(x) = \begin{cases} \alpha_1 \text{ (diffusion coefficient of copper)} & 0 < x < L/2 \\ \alpha_2 \text{ (diffusion coefficient of steel)} & L/2 < x < L \end{cases}$$

PROBLEMS

1. If the initial temperature of the rod were

$$u(x,0) = \sin \pi x \qquad 0 \leq x \leq 1$$

and if the BCs were

$$u(0,t) = 0$$
$$u(1,t) = 0$$

what would be the behavior of the rod temperature $u(x,t)$ for later values of time?

HINT Use the physical interpretation of the heat equation $u_t = \alpha^2 u_{xx}$.

2. Suppose the rod has a constant internal heat source, so that the basic equation describing the heat flow within the rod is

$$u_t = \alpha^2 u_{xx} + 1 \qquad 0 < x < 1$$

Suppose we fix the boundaries' temperatures by $u(0,t) = 0$ and $u(1,t) = 1$. What is the steady-state temperature of the rod? In other words, does the temperature $u(x,t)$ converge to a constant temperature $U(x)$ independent of time?

HINT Set $u_t = 0$. It would be useful to graph this temperature. Also start with an initial temperature of zero and draw some temperature profiles.

3. Suppose a metal rod loses heat across the lateral boundary according to the equation

$$u_t = \alpha^2 u_{xx} - \beta u \qquad 0 < x < 1$$

and suppose we keep the ends of the rod at $u(0,t) = 1$ and $u(1,t) = 1$. Find the steady-state temperature of the rod (graph it). Where is heat flowing in this problem?

4. Suppose a laterally insulated metal rod of length $L = 1$ has an initial temperature of $\sin(3\pi x)$ and has its left and right ends fixed at temperatures zero and 10°C. What would be the IBVP that describes this problem?

*Note that the boundary and initial data do not match up in this problem.

OTHER READING

1. *Equations of Mathematical Physics* by A. N. Tikhonov and A. A. Samarskii. Macmillan, 1963; Dover, 1990. An encyclopedia of information; contains many good examples and problems.

LESSON 3

Boundary Conditions for Diffusion-Type Problems

PURPOSE OF LESSON: To show how heat-flow and diffusion-type problems can give rise to a variety of boundary conditions and to introduce the important concept of *flux*.

Three important types of BCs discussed are

1. $u = g(t)$ (temperature specified on the boundary).
2. $\dfrac{\partial u}{\partial n} + \lambda u = g(t)$ (temperature of the *surrounding medium* is specified; n is the *outward normal* direction to the boundary).
3. $\dfrac{\partial u}{\partial n} = g(t)$ (heat flow across the boundary specified).

When describing the various types of boundary conditions that can occur for heat-flow problems, three basic types generally come to mind. Lesson 3 discusses these three kinds of BCs and gives an example of how they occur in experiments.

Type 1 BC (Temperature specified on the boundary)

Consider the heat flow in the one-dimensional rod illustrated in Figure 3.1 and suppose we make the ends of the rod follow the temperature curves $g_1(t)$ and $g_2(t)$.

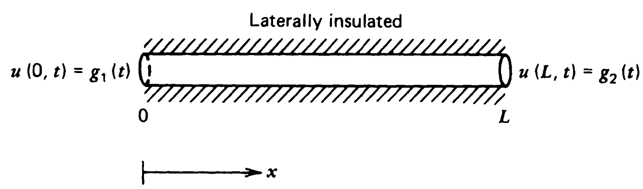


FIGURE 3.1 Temperature specified on the boundary.

As we mentioned in the previous lesson, an apparatus that keeps the ends at specified temperatures requires a thermostat at each end and heating elements to adjust the temperature accordingly. Problems with BCs of this kind are fairly common. It may even be that the goal of the problem is to find the *boundary temperatures* (boundary control) $g_1(t)$ and $g_2(t)$ that will force the temperature to behave in a suitable manner. In the steel industry, it is often necessary to determine the *boundary controls* so that the temperature of the metal inside the furnace changes over time but the temperature *gradient* from one point to another is small.

Similar types of BCs also apply to higher dimensional domains, for example, in two dimensions, we could imagine the interesting problem of finding the temperature inside the circular disc (of radius R) when the boundary temperature is specified in polar coordinates to be

$$u(R,\theta,t) = \cos t \sin \theta$$

See Figure 3.2.

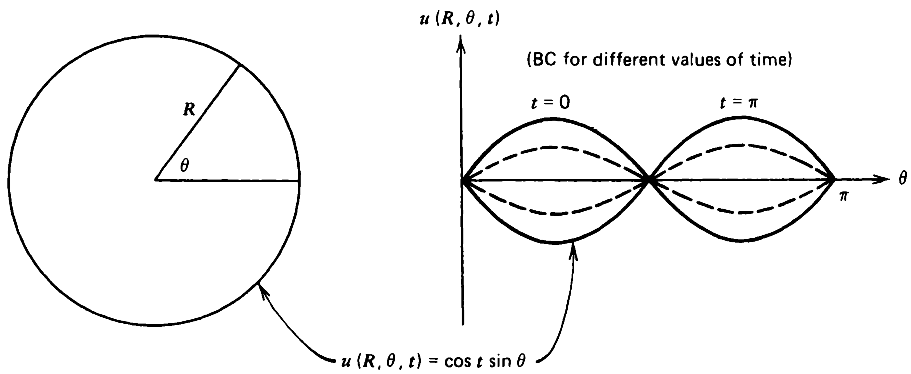


FIGURE 3.2 Oscillating boundary temperature.

Of course, we'd have to have an initial temperature to get this experiment started, but in this case, the effects of our IC would vanish after a short period of time, and the resulting temperature inside the circle would depend on the boundary temperature.

Type 2 BC (Temperature of the surrounding medium specified)

Suppose we consider again our laterally insulated copper rod, but now instead of requiring the two boundaries to be specified at temperatures $g_1(t)$ and $g_2(t)$, we only bring them in contact with surrounding mediums that have those temperatures. In other words, suppose the left side of the rod is enclosed in a container of liquid that has a changing temperature $g_1(t)$, while the right end is enclosed in another liquid with temperature $g_2(t)$ (Figure 3.3).

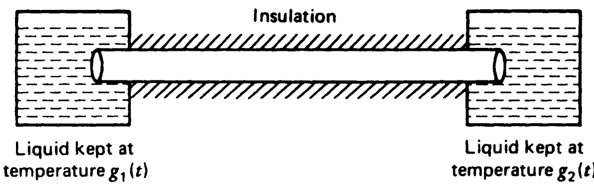


FIGURE 3.3 Convection cooling at the boundaries.

By specifying these types of BCs, we cannot say the boundary temperatures of the rod will be the same as the liquid temperatures $g_1(t)$ and $g_2(t)$, but we do know (Newton's law of cooling) that whenever the rod temperature at one of the boundaries is *less* than the respective liquid temperatures, then heat will flow into the rod at a rate proportional to this difference. In other words, for the one-dimensional rod with boundaries at $x = 0$ and L, Newton's law of cooling states

$$(3.1)\qquad \begin{cases} \text{Outward flux of heat (at } x = 0) = h[u(0,t) - g_1(t)] \\ \text{Outward flux of heat (at } x = L) = h[u(L,t) - g_2(t)] \end{cases}$$

where h is a **heat-exchange coefficient,** which is a measure of how many calories flow across the boundary per unit of temperature difference per second per cm and the **outward flux of heat** is the number of calories crossing the ends of the rod per second. Note that the outward flux of heat will be positive at either end provided the temperature of the rod is greater than the surrounding medium. Equations (3.1) can now be used in conjunction with what is known as Fourier's Law of Cooling to arrive at our BCs. Fourier's law gives us another representation (the first one is 3.1) for the outward flux of heat and by setting these two representations equal to each other, we get our BCs. First, we state Fourier's law (proven experimentally):

(3.2) | Outward flux of heat across a boundary is proportional to the inward normal derivative across the boundary. |

This law says that if the temperature is increasing rapidly in the direction *outward* from the boundary of D (Figure 3.4), then heat will flow *from* the surrounding medium *into* the domain D.

In our one-dimensional problem, Fourier's law takes the form:

$$(3.3)\qquad \begin{cases} \text{Outward flux of heat (at } x = 0) = k\,\dfrac{\partial u(0,t)}{\partial x} \\ \text{Outward flux of heat (at } x = L) = -k\,\dfrac{\partial u(L,t)}{\partial x} \end{cases}$$

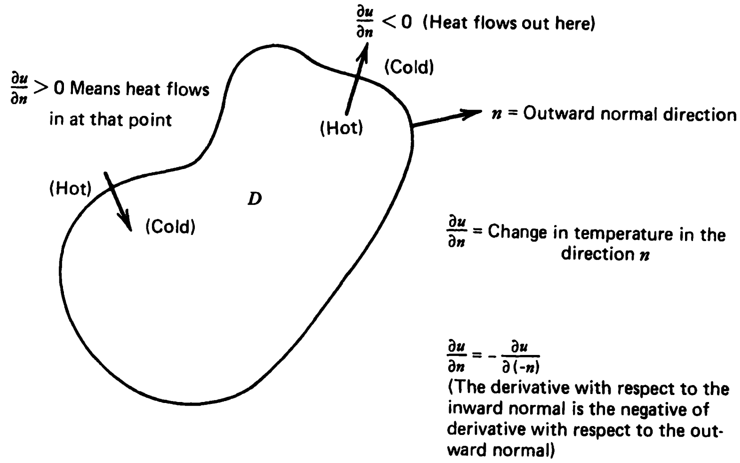


FIGURE 3.4 Illustration of Fourier's law.

where k is the **thermal conductivity** of the metal, which is a measure of how well the material conducts heat. (Poorly conducting materials have values near zero in cgs. units, while copper and aluminum have values close to one.)

Fourier's law (3.3) actually holds anywhere inside the rod and not just at the boundary; for example,

$$\boxed{\text{Flux of heat crossing } x_0 \text{ (from left to right)} = -kA\frac{\partial u}{\partial x}(x_0,t)} \tag{3.4}$$

See Figure 3.5.

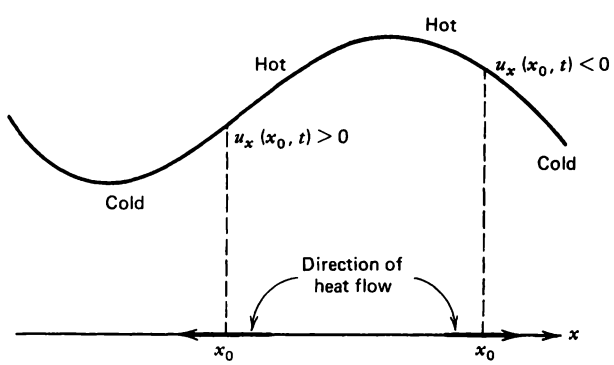


FIGURE 3.5 Another illustration of Fourier's law.

Fourier's law (3.4) says that if $u_x(x_0,t) < 0$, then heat will flow from *left to right*; if $u_x(x_0,t) > 0$, then the flow of heat through point x_0 will be from *right to left* (heat always flows from high to low temperatures).

Finally, if we use the two expressions (3.1) and (3.3) for heat flux, we have our desired BCs for Figure 3.3 in purely mathematical terms; namely,

$$\text{BCs}\quad \begin{cases} \dfrac{\partial u(0,t)}{\partial x} = \dfrac{h}{k}\,[u(0,t) - g_1(t)] \\[2ex] \dfrac{\partial u(L,t)}{\partial x} = -\dfrac{h}{k}\,[u(L,t) - g_2(t)] \end{cases} \qquad 0 < t < \infty$$

Quite often, the constant h/k is simply written as λ, and so we have the BCs for heat flow across the boundary

$$\begin{aligned} u_x(0,t) &= \lambda\,[u(0,t) - g_1(t)] \\ u_x(L,t) &= -\lambda\,[u(L,t) - g_2(t)] \end{aligned} \tag{3.5}$$

In higher dimensions, we have similar BCs; for example, if the boundary of a circular disc is interfaced with a moving liquid that has a temperature $g(\theta,t)$, our BC would be

$$\frac{\partial u}{\partial r}(R,\theta,t) = -\frac{h}{k}\,[u(R,\theta,t) - g(\theta,t)]$$

Here, $\dfrac{\partial u}{\partial r}(R,\theta,t)$ represents the outward normal derivative (in the positive r-direction) of u evaluated at a point (R,θ) on the boundary. This type of BC would be called a *linear* BC (since it is linear in u and u_r) but nonhomogeneous due to the right-hand side $g(\theta,t)$.

Type 3 BC (Flux specified—including the special case of insulated boundaries)

Insulated boundaries are those that do not allow any flow of heat to pass, and, hence, the normal derivative (inward or outward) must be zero on the boundary (since the normal derivative is proportional to the flux). In the case of the one-dimensional rod with insulated ends at $x = 0$ and $x = L$, the BCs are

$$\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0 \end{aligned} \qquad 0 < t < \infty$$

In two-dimensional domains, an insulated boundary would mean that the *normal derivative* of the temperature across the boundary is zero. For example, if the circular disc were insulated on the boundary, then the BC would be $u_r(R,\theta,t) = 0$ for all $0 \leq \theta < 2\pi$ and all $0 < t < \infty$.

On the other hand, if we specify the amount of heat entering across the boundary of our disc, the BC is

$$u_r(R,\theta,t) = f(\theta,t)$$

where $f(\theta,t)$ would represent the amount of heat crossing *into* the circular disc from an outside heating source.

We now illustrate different types of BCs.

Typical BCs for One-Dimensional Heat Flow

Suppose we have a copper rod 200 cm long that is laterally insulated and has an initial temperature of 0°C. Suppose the top of the rod ($x = 0$) is insulated, while the bottom ($x = 200$) is immersed in moving water that has a constant temperature of $g_2(t) = 20°C$ (Figure 3.6).

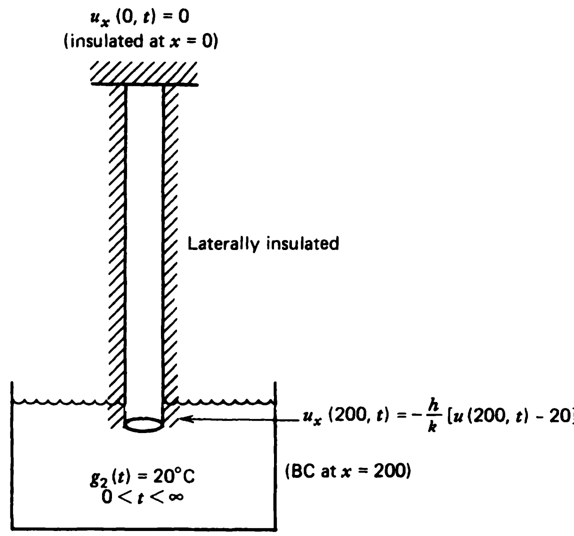


FIGURE 3.6 Initial-boundary-value problem.

The mathematical model for this problem would be the following four equations:

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 200 \qquad 0 < t < \infty$$

$$(3.6) \qquad \text{BCs} \qquad \begin{cases} u_x(0,t) = 0 \\ u_x(200,t) = -\dfrac{h}{k}[u(200,t) - 20] \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = 0°C \qquad 0 \le x \le 200$$

where

α^2 = 1.16 cm²/sec (diffusivity constant for copper)

k = 0.93 cal/cm-sec°C (thermal conductivity of copper)

h = heat exchange coefficient. To find h is a hard problem in itself. It measures the rate that heat is being exchanged between the bottom of the rod and the surrounding water. It is a function of how fast the water is being circulated, the nature of the interface, and so forth. The reader would have to carry out an experiment to determine its value.

NOTES

1. A typical heat-flow problem inside a square is shown in Figure 3.7.

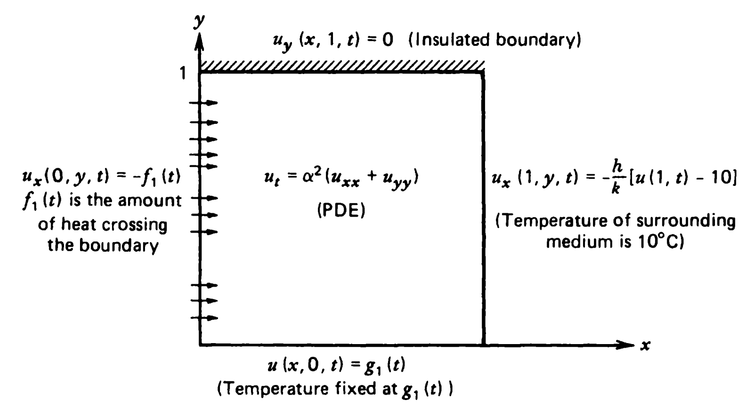


FIGURE 3.7 Typical BCs for diffusion problems inside a square.

In this problem, after we specify the initial temperature $u(x,y,0)$ at $t = 0$ inside the square, the PDE and BCs in the diagram will take over for $0 < t < \infty$ and determine the subsequent temperature values $u(x,y,t)$. Whatever the temperature is, however, it must satisfy the BCs in Figure 3.7.

2. Note that the BC

$$u_r(R,\theta,t) = -\frac{h}{k}[u(R,\theta,t) - g(\theta,t)]$$

on the circle will not require the boundary temperature to be $g(\theta,t)$, but when the heat-exchange coefficient h is large, then the BC essentially says that the boundary temperature $u(R,\theta,t)$ is almost equal to $g(\theta,t)$.

PROBLEMS

1. Draw rough sketches of the solution to the IBVP (3.6) for different values of time. Do your sketches satisfy the BCs? What is the steady-state temperature of the rod? Is this obvious based on your intuition?
2. What is your interpretation of the initial-boundary-value problem?

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u_x\,(1,t) = 1 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \sin\,(\pi x) \qquad 0 \leq x \leq 1$$

   Can you draw rough sketches of the solution for different values of time? Will the solution come to a steady state; is this obvious?
3. What is your physical interpretation of the problem?

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \sin\,(\pi x) \qquad 0 \leq x \leq 1$$

   Can you draw rough sketches of this solution for various values of time? What about the steady-state temperature?
4. Suppose a metal rod laterally insulated has an initial temperature of 20°C but immediately thereafter has one end fixed at 50°C. The rest of the rod is immersed in a liquid solution of temperature 30°C. What would be the IBVP that describes this problem?

OTHER READING

1. *Conduction of Heat in Solids* by H. S. Carslaw and J. C. Jaeger. Oxford University Press, 1959. An excellent reference that discusses BCs of many physical problems.

2. *Partial Differential Equations in Biology* by C. S. Peskin. Courant Institute of Mathematical Sciences, 1976. Several biological phenomena such as nerve cells, the inner ear, and the cardiovascular system are modeled by PDEs.

LESSON 4

Derivation of the Heat Equation

PURPOSE OF LESSON: To show how the one-dimensional heat equation

$$u_t = \alpha^2 u_{xx} + f(x,t)$$

is derived from the basic principle of *conservation of heat.* Physical concepts such as *thermal conductivity, thermal capacity,* and *density* are discussed, and it is shown how the rate of heat transfer depends on these three basic physical parameters. A few variations of the basic heat equation are also discussed.

In all areas of science, we begin with a given set of assumptions that are taken to be self-evident and from which all other ideas are derived. Of course, what is self-evident to one person may hold doubts for others. The history of science consists of pushing back the basic axioms further and further so that there is a universally agreed upon starting point.

For example, one person may think that all relevant facts will spring from a basic assumption, say assumption B. From assumption B, he or she may prove theorem C, which in turn proves theorem D, which in turn proves others (Figure 4.1).

? $\longrightarrow$ Assumption A $\longrightarrow$ Assumption B $\longrightarrow$ Theorem C $\longrightarrow$ Theorem D $\longrightarrow$?

FIGURE 4.1 The axiomatic method.

This of course is progress—the more new results a person can prove, the better. Physicists, chemists, and biologists all proceed in this basic manner.

On the other hand, instead of proving new theorems we may ask if it is possible to find a new assumption, say assumption A, more basic than assumption B, so that assumption B can be proven from A. In this way, we are pushing back the frontiers of knowledge. In the general area of heat flow, the concept of *conservation of energy* (heat energy) is the basis from which other principles are derived (Figure 4.2).

Conservation of energy (Assumption) $\longrightarrow$ $u_t = \alpha^2 u_{xx} + f(x,t)$ $\longrightarrow$ Other properties of heat flow

FIGURE 4.2 Conservation of energy: the cornerstone of heat-flow problems.

We could, of course, forget this lesson and use the heat equation as the starting point (some people may think it is self-evident in itself), but this would be shortchanging serious students, since conservation of energy assumptions are basic to science. Scientists often begin modeling specific problems by writing conservation of energy relationships and then rewriting them as partial differential equations.

We now turn to the goal of the lesson—to derive the heat equation from the conservation of heat equation.

Derivation of the Heat Equation

Suppose we have a one-dimensional rod of length L for which we make the following assumptions:

1. The rod is made of a single homogeneous conducting material.
2. The rod is laterally insulated (heat flows only in the x-direction).
3. The rod is thin (the temperature at all points of a cross section is constant).

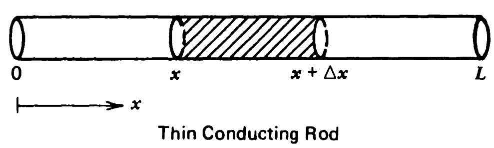


FIGURE 4.3 Thin conducting rod.

If we apply the principle of conservation of heat to the segment $[x,x + \Delta x]$, we can claim

(4.1)
$$\begin{aligned}&\text{Net change of heat inside } [x,x+\Delta x]\\ &= \text{Net flux of heat across the boundaries}\\ &+ \text{Total heat generated inside } [x,x+\Delta x]\end{aligned}$$

Now, inasmuch as the total amount of heat (in calories) inside $[x,x + \Delta x]$ at any time t is measured by (see reference 1 in this lesson)

$$\text{Total heat inside } [x,x+\Delta x] = \int_x^{x+\Delta x} c\rho A u(s,t)\, ds$$

where

c = thermal capacity of the rod (measures the ability of the rod to store heat).
ρ = density of the rod.
A = cross-section area of the rod

we can write the conservation of energy equation (4.1) via calculus as

$$\frac{d}{dt}\int_x^{x+\Delta x} c\rho A u(s,t)\,ds = c\rho A \int_x^{x+\Delta x} u_t(s,t)\,ds \tag{4.2}$$
$$= kA\,[u_x(x+\Delta x,t) - u_x(x,t)] + A\int_x^{x+\Delta x} f(s,t)\,ds$$

where

k = thermal conductivity of the rod (measures the ability to conduct heat).
$f(x,t)$ = external heat source (calories per cm per sec).

The problem now is to replace equation (4.2) by one that does not contain integrals. The reader may recall the Mean Value Theorem from calculus.

Mean Value Theorem

If $f(x)$ is a continuous function on $[a,b]$, then there exists at least one number ξ, $a < \xi < b$ that satisfies

$$\int_a^b f(x)dx = f(\xi)\,(b-a)$$

Applying this result to equation (4.2), we arrive at the following equation:

$$c\rho A u_t(\xi_1,t)\Delta x = kA[u_x(x+\Delta x,t) - u_x(x,t)] + Af(\xi_2,t)\,\Delta x \quad x < \xi < x + \Delta x$$

or

$$u_t(\xi,t) = \frac{k}{c\rho}\left\{\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right\} + \frac{1}{c\rho}f(\xi,t)$$

Finally letting $\Delta x \to 0$, we have the desired result

$$\boxed{u_t(x,t) = \alpha^2 u_{xx}(x,t) + F(x,t)} \tag{4.3}$$

where

$$\alpha^2 = \frac{k}{c\rho} \qquad \text{(called the diffusivity of the rod)}$$

$$F(x,t) = \frac{1}{c\rho}f(x,t) \qquad \text{(heat source density)}$$

This completes our discussion. Before we close, however, suppose the rod were not laterally insulated and that heat can flow in and out across the lateral boundary at a rate proportional to the difference between the temperature $u(x,t)$ and the surrounding medium that we keep at zero. In this case, the conservation of heat principle will give.

$$u_t = \alpha^2 u_{xx} - \beta u + F(x;t) \tag{4.4}$$

where β = rate constant for the lateral heat flow ($\beta > 0$).

NOTES

1. The constant k is the **thermal conductivity** of the rod and a measure of the heat flow (in calories) that is transmitted per second through a plate 1 cm thick across an area of 1 cm^2 when the temperature difference is 1°C; values for k can be found in *The Handbook of Chemistry and Physics.* Typical values of k are close to 1 for copper and near zero for insulating-type materials.

   If the material of the rod is uniform, then k will not depend on x. For some materials, the value of k depends on the temperature u and hence the heat equation

   $$u_t = \frac{1}{c\rho}\frac{\partial}{\partial x}\{k(u)u_x\}$$

   is *nonlinear*. Most of the time, however, k changes very slowly with u and this nonlinearity is neglected.
2. The constant c is known as the **thermal capacity** (or specific heat) of the substance and measures the amount of energy the substance can store. For example, a baked potato would have a large thermal capacity, since it can store a large amount of heat per unit mass of potato (that's why it takes a long time to heat). Technically, the thermal capacity is the amount of heat (in calories) necessary to produce a 1°C change in temperature of 1 g of the substance. For most of our problems, c is taken as a constant independent of x and u; typical values can be found in *The Handbook of Chemistry and Physics*.
3. The units of some of the basic quantities of heat flow (in the cgs. measurement system) are

   u = temperature (degrees centigrade).
   u_t = rate of change in temperature (°C/sec).

u_x = slope of temperature curve (°C/cm).
u_{xx} = concavity of temperature curve (°C/cm^2).
c = thermal capacity (cal/g-°C).
k = thermal conductivity (cal/cm-sec-°C).
ρ = density (g/cm^3).
α^2 = diffusivity (cm^2/sec).

4. Note that the diffusivity $\alpha^2 = \dfrac{k}{c\rho}$ of a material is proportional to the conductivity k of the material and inversely proportional to the density ρ and thermal capacity c; this should have some intuitive appeal to the reader.

PROBLEMS

1. Substitute the units of each quantity $u, u_t, \ldots$ into the equation

$$u_t = \alpha^2 u_{xx} - \beta u$$

to see that every term has the same units of °C/sec.

2. Substitute the units of each quantity into the equation

$$u_t = \alpha^2 u_{xx} - v u_x$$

where v has units of velocity to see that every term has the same units.

3. Derive the heat equation

$$u_t = \frac{1}{c\rho}\frac{\partial}{\partial x}[k(x)u_x] + f(x,t)$$

for the situation where the thermal conductivity $k(x)$ depends on x.

4. Suppose $u(x,t)$ measures the concentration of a substance in a moving stream (moving with velocity v). Suppose the concentration $u(x,t)$ changes both by diffusion and convection; derive the equation

$$u_t = \alpha^2 u_{xx} - v u_x$$

from the fact that at any instant of time, the total mass of the material is not created or destroyed in the region $[x, x + \Delta x]$.

HINT Write the conservation equation

Change of mass inside $[x, x + \Delta x]$
= Change due to *diffusion* across the boundaries
+ Change due to the material being *carried* across the boundaries

OTHER READING

1. *Applied Mathematics for the Engineer and Physicist* by L. A. Pipes. McGraw-Hill, 1958. An older reference, but still a good one for the practicing scientist.

2. *Equations of Mathematical Physics* by A. N. Tikhonov and A. A. Samarskii. Macmillan, 1963; Dover, 1990. A good text for derivations of equations. A companion volume, *A Collection of Problems in* [*on*] *Mathematical Physics*, by B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (Pergamon, 1964; Dover, 1988) is available, which will give the student additional experience with solving problems in partial differential equations.

LESSON 5

Separation of Variables

PURPOSE OF LESSON: To introduce the powerful method of separation of variables and to show how this method can be used to solve a well-known diffusion problem. Inasmuch as the method is not well understood by some students due to its complicated algebraic nature, several intuitive explanations are given along the way.

The basic idea is to break down the *initial conditions* of the problem into simple components, find the response to each component, and then add up these individual responses. This gives the response to the *arbitrary initial condition.*

The actual step-by-step methodology of separation of variables somewhat hides this basic interpretation, but that's what's going on nevertheless.

Separation of variables is one of the oldest techniques for solving initial-boundary-value problems (IBVPs) and applies to problems where

1. The PDE is linear and homogeneous (not necessarily constant coefficients).
2. The boundary conditions are of the form

$$\alpha u_x(0,t) + \beta u(0,t) = 0$$
$$\gamma u_x(1,t) + \delta u(1,t) = 0$$

where α, β, γ, and δ are constants (boundary conditions of this form are called **linear homogeneous BCs**).

It dates back to the time of Joseph Fourier (in fact, it's occasionally called *Fourier's method*) and is probably the most widely used method of solution (when applicable).

Instead of showing how the method works in general, let's apply it to a specific problem (later we will discuss it in more generality). Consider the IBVP (diffusion problem)

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leq x \leq 1$$

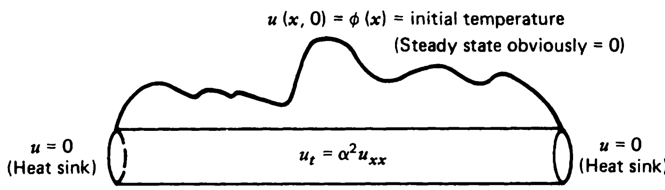


FIGURE 5.1 Diagram of the diffusion problem.

Before getting to separation of variables, let's first think about our problem. Here we have a finite rod where temperature at the ends is fixed at zero (suppose it's a temperature problem where zero means so many degrees). We are also given data for the problem in the form of an initial condition; our goal is to find the temperature $u(x,t)$ at later points in time.

Now for the method itself—but first an overview.

Overview of Separation of Variables

Separation of variables looks for simple-type solutions to the PDE of the form

$$u(x,t) = X(x)T(t)$$

where $X(x)$ is some function of x and $T(t)$ is some function of t. The solutions are simple because any temperature $u(x,t)$ of this form will retain its basic "shape" for different values of time t (Figure 5.2).

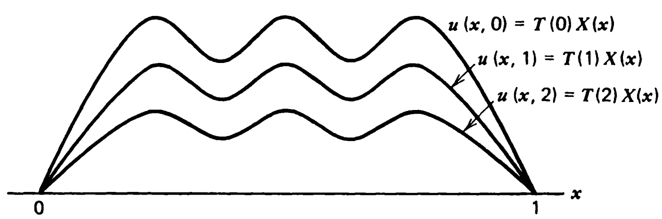


FIGURE 5.2 Graph of $X(x)T(t)$ for different values of t.

The general idea is that it is possible to find an infinite number of these solutions to the PDE (which, at the same time, also satisfy the BCs). These simple functions $u_n(x,t) = X_n(x)T_n(t)$ (called **fundamental solutions**) are the building blocks of our problem, and the solution $u(x,t)$ we are looking for is found by adding the simple fundamental solutions $X_n(x)T_n(t)$ in such a way that the resulting sum

$$\sum_{n=1}^{\infty} A_n X_n(x) T_n(t)$$

satisfies the initial conditions. Inasmuch as this sum still satisfies the PDE and the BCs, we now have the solution to our problem. Let's now carry this out in detail.

Separation of Variables

STEP 1 (Finding elementary solutions to the PDE)

We wish to find the function $u(x,t)$ that satisfies the following four conditions:

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leqslant x \leqslant 1$$

To begin, we look for solutions of the form $u(x,t) = X(x)T(t)$ by substituting $X(x)T(t)$ into the PDE and solving for $X(x)T(t)$. Making this substitution gives

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Now, here is the part that makes all this work: If we *divide* each side of this equation by $\alpha^2 X(x)T(t)$, we have

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

and obtain what is called **separated variables**, that is, the left side of the equation depends only on t and the right side, only on x. Inasmuch as x and t are *independent of each other*, each side must be a fixed constant (say k); hence, we can write

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = k$$

or

$$\begin{aligned} T' - k\alpha^2 T &= 0 \\ X'' - kX &= 0 \end{aligned}$$

So now we can solve each of these two ODEs, multiply them together to get a solution to the PDE (note that we have essentially changed a second-order PDE to two ODEs). However, we now make an important observation, namely, that we want the separation constant k to be *negative* (or else the $T(t)$ factor doesn't go to zero as $t \to \infty$). With this in mind, it is general practice to rename $k = -\lambda^2$, where λ is nonzero ($-\lambda^2$ is guaranteed to be negative). Calling our separation constant by its new name, we can now write the two ODEs as

$$T' + \lambda^2\alpha^2 T = 0$$
$$X'' + \lambda^2 X = 0$$

We will now solve these equations. Both equations are standard-type ODEs and have solutions

$$T(t) = Ae^{-\lambda^2\alpha^2 t} \qquad (A \text{ an arbitrary constant})$$
$$X(x) = A\sin(\lambda x) + B\cos(\lambda x) \qquad (A,\ B \text{ arbitrary})$$

and hence all functions

$$u(x,t) = e^{-\lambda^2\alpha^2 t}[A\sin(\lambda x) + B\cos(\lambda x)]$$

(with A, B, and λ arbitrary) will satisfy the PDE $u_t = \alpha^2 u_{xx}$; this verification is problem 1 in the problem set. At this point, we have an infinite number of functions that satisfy the PDE.

STEP 2 (Finding solutions to the PDE and the BCs)

We are now to the point where we have many solutions to the PDE but not all of them satisfy the BCs or the IC. The next step is to choose a certain *subset* of our current crop of solutions

$$e^{-\lambda^2\alpha^2 t}[A\sin(\lambda x) + B\cos(\lambda x)] \tag{5.1}$$

that satisfy the boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0$$

To do this, we substitute our solutions (5.1) into these BCs, getting

$$u(0,t) = Be^{-\lambda^2\alpha^2 t} = 0 \Rightarrow B = 0$$
$$u(1,t) = Ae^{-\lambda^2\alpha^2 t}\sin\lambda = 0 \Rightarrow \sin\lambda = 0$$

This last BC restricts the separation constant λ from being any nonzero number, it must be a root of the equation $\sin\lambda = 0$. In other words, in order that $u(1,t) = 0$, it is necessary to *pick*

$$\lambda = \pm\pi,\ \pm 2\pi,\ \pm 3\pi,\ \ldots$$

or

$$\lambda_n = \pm n\pi \qquad n = 1, 2, 3, \ldots$$

Note that the last BC could also imply $A = 0$, but if we choose this, we would get the zero solution in (5.1).

We have now finished the second step; we have found an infinite number of functions

$$\boxed{u_n(x,t) = A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \qquad n = 1, 2, \ldots} \tag{5.2}$$

each one satisfying the PDE and the BCs.* These are the building blocks of the problem, and our desired solution will be a certain sum of these simple functions; the specific sum will depend on the initial conditions. See Figure 5.3 for the graphs of these fundamental solutions $u_n(x,t)$:

STEP 3 (Finding the solution to the PDE, BCs, and the IC)

The last step (and probably the most interesting from a mathematical point of view) is to add the fundamental solutions

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)$$

in such a way (pick the coefficients A_n) that the initial condition

$$u(x,0) = \phi(x)$$

is satisfied. Substituting the sum into the IC gives

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \tag{5.3}$$

This equation leads us to the interesting question asked by the French mathematician Joseph Fourier, is it possible to expand the initial temperature $\phi(x)$ as the sum of the elementary functions as follows:

$$A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + \ldots$$

The answer to this question is yes provided $\phi(x)$ is a reasonably nice function—continuous. Hence, the question now becomes how to find the coefficients A_n.

* Notice that the functions u_n and u_{-n} are essentially the same except for a minus sign.

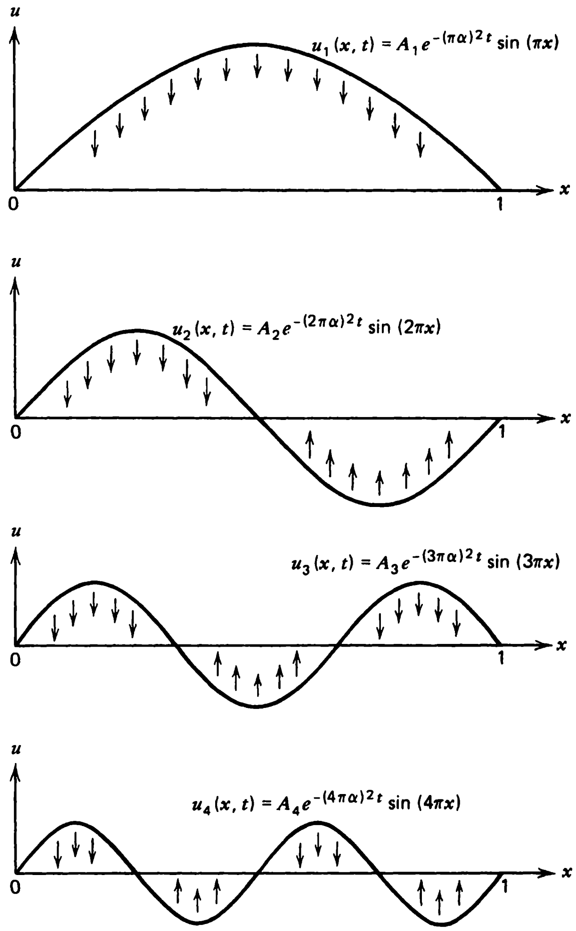


FIGURE 5.3 Fundamental solutions $u_n(x,t) = A_ne^{-(n\pi\alpha)^2t} \sin(n\pi x)$.

This is actually very easy: One uses a property of the functions

$$\{\sin(n\pi x); \quad n = 1, 2, \ldots\}$$

known as **orthogonality**. It turns out (see problem 2) that these functions are orthogonal to each other in the sense

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$$

This property can be illustrated by looking at the graphs of these functions (Figure 5.4).

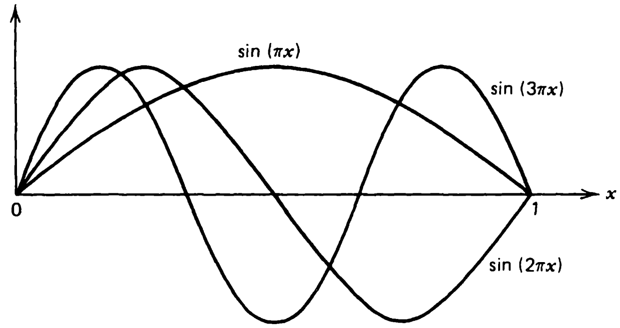


FIGURE 5.4 Orthogonal sequence of functions.

So, we are now in position to solve for the coefficients in the expression

$$\phi(x) = A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + A_4 \sin(4\pi x) + \ldots$$

We *multiply* each side of this equation by $\sin(m\pi x)$ (m, an arbitrary integer) and *integrate* from zero to one; doing this, we get

$$\int_0^1 \phi(x) \sin(m\pi x)\, dx = A_m \int_0^1 \sin^2(m\pi x)\, dx = \frac{1}{2} A_m$$

(all other terms drop out due to orthogonality). Solving for A_m gives

$$A_m = 2 \int_0^1 \phi(x) \sin(m\pi x)\, dx$$

We're done; the solution is

(5.4) $$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)}$$

where the coefficients A_n are given by

(5.5) $$\boxed{A_n = 2\int_0^1 \phi(x) \sin(n\pi x)\, dx}$$

We can check this answer to see that it satisfies all four of our original conditions in the problem. This ends step 3.

Many students are disappointed when they finally discover that the solution is this complicated, and many hardly give the solution a second look (that's too bad). The solution is not all that difficult if one takes the time to analyze it; in fact, the more complicated it is, the more information it contains. Here are a few notes that will help you interpret this solution.

NOTES

1. Observe that the only difference between the *Fourier sine expansion* of $\phi(x)$ in (5.3) and the solution (5.4) is the insertion of the time factor

$$e^{-(n\pi\alpha)^2 t}$$

in each term. Hence, if our IC were a very simple expression like

$$\phi(x) = \sin(\pi x) + \frac{1}{2}\sin(3\pi x)$$

then the solution would simply be

$$u(x,t) = e^{-(\pi\alpha)^2 t}\sin(\pi x) + \frac{1}{2}e^{-(3\pi\alpha)^2 t}\sin(3\pi x)$$

In this case, it's obvious that if we expanded $\phi(x)$ as a Fourier sine series, we would get

$$\begin{aligned} A_1 &= 1 \\ A_2 &= 0 \\ A_3 &= 1/2 \\ A_4 &= A_5 = \ldots = 0 \end{aligned}$$

2. We can interpret the solution (5.4) in the following manner: We expand the initial temperature $\phi(x)$ as a sum of simple functions, $A_n \sin(n\pi x)$ and then find the response to each of these (which is $A_n e^{-(n\pi\alpha)^2 t}\sin(n\pi\alpha)$; and then add these individual responses to get the solution corresponding to the IC $u(x,0) = \phi(x)$.
3. The terms in the solution

$$u(x,t) = A_1 e^{-(\pi\alpha)^2 t}\sin(\pi x) + A_2 e^{-(2\pi\alpha)^2 t}\sin(2\pi x) + \ldots$$

are functions of x and t. Note that the terms further out in the series get small very fast due to the factor

$$e^{-(n\pi\alpha)^2 t}$$

Hence, for long time periods, the solution is approximately equal to the first term

$$u(x,t) \cong A_1 e^{-(\pi\alpha)^2 t} \sin(\pi x)$$

which is the shape of a damped sine curve (Figure 5.5).

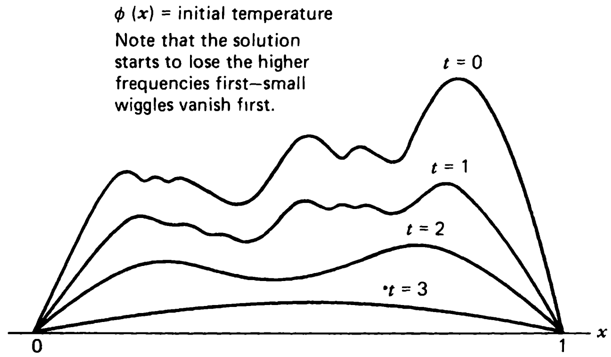


FIGURE 5.5 Higher-order terms damp faster in diffusion problems.

PROBLEMS

1. Show that $u(x,t) = e^{-\lambda^2\alpha^2 t}[A \sin(\lambda x) + B \cos(\lambda x)]$ satisfies the PDE $u_t = \alpha^2 u_{xx}$ for arbitrary A, B, and λ.

2. Show $\int_0^1 \sin(\pi m x)\sin(\pi n x)\,dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$

   HINT Use the identity

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

3. Find the Fourier sine expansion of $\phi(x) = 1 \quad 0 \leq x \leq 1$. Draw the first three or four terms.

4. Using the results of problem 3, what is the solution to the IBVP

$$\text{PDE} \quad u_t = u_{xx} \quad 0 < x < 1$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 1 \quad 0 \leq x \leq 1$$

(Note that this problem is physically impossible, since we are pulling the temperature from one to zero instantaneously. In most problems, if the BCs are zero, then the initial temperature $\phi(x)$ should also be zero at $x = 0$ and $x = 1$.)

5. What is the solution to problem 4 if the IC is changed to

$$u(x,0) = \sin(2\pi x) + \frac{1}{3}\sin(4\pi x) + \frac{1}{5}\sin(6\pi x)$$

6. What would be the solution to problem 4 if the IC were

$$u(x,0) = x - x^2 \qquad 0 < x < 1$$

OTHER READING

Partial Different Equations of Mathematical Physics by Tyn Myint-U. Elsevier, 1973. A well-written text slightly more advanced than the current one; Chapter 6. A large chapter on separation of variables with several good problems.

LESSON 6

Transforming Nonhomogeneous BCs into Homogeneous Ones

PURPOSE OF LESSON: To show how the initial-boundary-value problem

$$\text{PDE} \quad u_t - \alpha^2 u_{xx} = f(x,t)$$

$$\text{BCs} \quad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = g_1(t) \\ \alpha_2 u_x(L,t) + \beta_2 u(L,t) = g_2(t) \end{cases}$$

$$\text{IC} \quad u(x,0) = \phi(x)$$

can be transformed into a new one (with *zero* BCs) like

$$\begin{aligned} &U_t - \alpha^2 U_{xx} = F(x,t) \\ &\alpha_1 U_x(0,t) + \beta_1 U(0,t) = 0 \\ &\alpha_2 U_x(L,t) + \beta_2 U(L,t) = 0 \\ &U(x,0) = \phi(x) \end{aligned}$$

This new problem can then be solved by

1. Separation of variables if the new PDE just happens to be homogeneous [$F(x,t) = 0$].
2. Integral transforms and eigenfunction expansions if $F(x,t) \neq 0$.

Although the method of separation of variables that we discussed in the last lesson is very powerful and gives us a nice series solution, the reader should realize it doesn't apply to all problems. In order for separation of variables to apply, the BCs must be of the following form (*linear homogeneous* BCs):

$$\begin{aligned} &\alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ &\alpha_2 u_x(L,t) + \beta_2 u(L,t) = 0 \end{aligned} \tag{6.1}$$

The purpose of this lesson is to show how problems with *nonhomogeneous* BC like

$$
\begin{array}{llll}
 & \text{PDE} & u_t = \alpha^2 u_{xx} & \\
(6.2) & \text{BCs} & \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = g_1(t) \\ \alpha_2 u_x(L,t) + \beta_2 u(L,t) = g_2(t) \end{cases} & \text{(nonhomogeneous BCs)} \\
 & \text{IC} & u(x,0) = \phi(x) &
\end{array}
$$

can be solved by transforming them into others with zero BCs. The new problem can then be solved by other methods (like eigenfunction expansions). We start our discussion by transforming an extremely simple problem with nonhomogeneous BCs into one with zero BCs.

Transforming Nonhomogeneous BCs to Homogeneous Ones

Consider heat flow in an insulated rod where the two ends are kept at constant temperatures k_1 and k_2; that is,

$$
\begin{array}{llll}
 & \text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < L \quad 0 < t < \infty \\
(6.3) & \text{BCs} & \begin{cases} u(0,t) = k_1 \\ u(L,t) = k_2 \end{cases} & 0 < t < \infty \\
 & \text{IC} & u(x,0) = \phi(x) & 0 \leq x \leq L
\end{array}
$$

The difficulty here is that since the BCs are not homogeneous, we cannot solve this problem by separation of variables. However, it is obvious that the solution will have a steady-state solution (solution when $t = \infty$) that varies *linearly* (in x) between the boundary temperatures k_1 and k_2 (Figure 6.1).

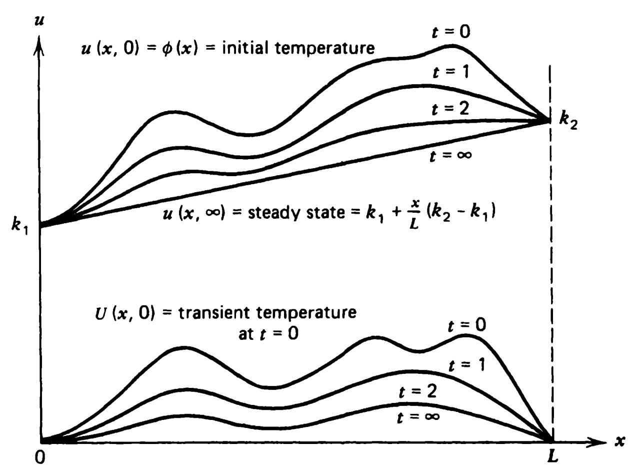


FIGURE 6.1 Solution of (6.3) for various values of time.

In other words, it seems reasonable to think of our temperature $u(x,t)$ as the sum of two parts

$$u(x,t) = \underbrace{\text{steady state}}_{\substack{\text{Eventual solution}\\ \text{for large time}}} + \underbrace{\text{transient}}_{\substack{\text{Part of the solution that}\\ \text{depends on the IC (and}\\ \text{will go to zero)}}}$$

$$= \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x,t)$$

This being the case, our goal is to find the *transient* $U(x,t)$. By substituting

$$u(x,t) = \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x,t)$$

in the original problem (6.3), we will arrive at a new problem in $U(x,t)$. We can then solve this new one for $U(x,t)$ and add it to the steady state to get $u(x,t)$. Carrying out this simple substitution in (6.3) gives us

$$\text{PDE} \qquad U_t = \alpha^2 U_{xx} \qquad 0 < x < L$$

$$\text{(6.4)} \qquad \text{BCs} \qquad \begin{cases} U(0,t) = 0 \\ U(L,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad U(x,0) = \underbrace{\phi(x) - \left[k_1 + \frac{x}{L}(k_2 - k_1)\right]}_{\bar{\phi}(x) \text{ = new IC—but known}}$$

This problem (fortunately) has a homogeneous PDE as well as homogeneous BCs, and so we can solve it by separation of variables; in fact, the reader probably remembers the solution:

$$\text{(6.5)} \qquad U(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x/L)$$

where

$$a_n = \frac{2}{L}\int_0^L \bar{\phi}(\xi) \sin(n\pi\xi/L)\, d\xi$$

So much for rods with *fixed* temperatures at the boundaries. What about more realistic-type derivative BCs with *time-varying* right-hand sides? The ideas are similar to the previous problem but a little more complicated.

Transforming Time Varying BCs to Zero BCs

Consider the typical problem

$$
\text{(6.6)} \qquad
\begin{array}{ll}
\text{PDE} & u_t = \alpha^2 u_{xx} \qquad 0 < x < L \qquad 0 < t < \infty \\[2ex]
\text{BCs} & \begin{cases} u(0,t) = g_1(t) \qquad 0 < t < \infty \\ u_x(L,t) + hu(L,t) = g_2(t) \end{cases} \\[2ex]
\text{IC} & u(x,0) = \phi(x) \qquad 0 \leqslant x \leqslant L
\end{array}
$$

To change these nonzero BCs to homogeneous ones, we (after some trial and error) seek a solution of the form

$$
u(x,t) = A(t)[1 - x/L] + B(t)[x/L] + U(x,t) \tag{6.7}
$$

where $A(t)$ and $B(t)$ are chosen so that the steady-state part

$$
S(x,t) = A(t)[1 - x/L] + B(t)[x/L] \tag{6.8}
$$

satisfies the BCs of the problem. In this way, the transformed problem in $U(x,t)$ will have homogeneous BCs. Substituting $S(x,t)$ into the BCs

$$
\begin{aligned}
&S(0,t) = g_1(t) \\
&S_x(L,t) + hS(L,t) = g_2(t)
\end{aligned}
$$

gives us two equations in which we can solve for $A(t)$ and $B(t)$. Doing this, we get

$$
\begin{aligned}
A(t) &= g_1(t) \\
B(t) &= \frac{g_1(t) + Lg_2(t)}{1 + Lh}
\end{aligned} \tag{6.9}
$$

Hence, we have

$$
u(x,t) = g_1(t)[1 - x/L] + \frac{g_1(t) + Lg_2(t)}{1 + Lh}[x/L] + U(x,t)
$$

and so if we substitute this into the original problem (6.6), we get our transformed problem in $U(x,t)$ (the reader should do this)

$$
\begin{array}{llll}
 & \text{PDE} & U_t = \alpha^2 U_{xx} - S_t & \text{(nonhomogeneous PDE)} \\
(6.10) & \text{BCs} & \begin{cases} U_x(L,t) + hU(L,t) = 0 \\ U(0,t) = 0 \end{cases} & \text{(homogeneous BCs)} \\
 & \text{IC} & U(x,0) = \phi(x) - S(x,0) & \text{(new IC—but known)}
\end{array}
$$

We now have our new problem with zero BCs (unfortunately, the PDE is nonhomogeneous). We can't solve this problem by separation of variables, but if the reader can wait for a few lessons, we will solve it by integral transforms and eigenfunction expansions.

NOTES

1. Our goal in this lesson was to transform problems with nonhomogeneous BCs into those with zero BCs. In so doing, if the new PDE just happens to be homogeneous, we are fortunate (like the first example) because we can then solve the problem by separation of variables.

   If, on the other hand, the new transformed PDE is nonhomogeneous, then we must solve the new problem by some other method.
2. The most general nonhomogeneous linear BCs

$$
\begin{aligned}
\alpha_1 u_x(0,t) + \beta_1 u(0,t) &= g_1(t) \\
\alpha_2 u_x(L,t) + \beta_2 u(L,t) &= g_2(t)
\end{aligned}
$$

   can also be transformed into zero BCs in a manner similar to the technique in the second example. Of course, the new PDE would most likely be nonhomogeneous.
3. Some methods of solution do not require the BCs to be homogeneous at all, and, hence, it isn't necessary to make any preliminary transformation. Later, when we study the *Laplace transform*, we will see it isn't necessary to have zero BCs (it's just that it's sometimes easier).
4. For BCs of the form

$$
\begin{aligned}
u(0,t) &= g_1(t) \\
u(L,t) &= g_2(t)
\end{aligned}
$$

   the method discussed in the second example will give us the transformation

$$
u(x,t) = \{g_1(t) + \frac{x}{L}[g_2(t) - g_1(t)]\} + U(x,t)
$$

PROBLEMS

1. Solve the initial-boundary-value problem

$$
\begin{array}{lll}
\text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < 1 \\
\text{BCs} & \begin{cases} u(0,t) = 1 \\ u_x(1,t) + hu(1,t) = 1 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = \sin(\pi x) + x & 0 \le x \le 1
\end{array}
$$

by transforming it into homogeneous BCs and then solving the transformed problem. Does the solution agree with your intuition of the problem?

2. Transform

$$
\begin{array}{lll}
\text{PDE} & u_t = u_{xx} & 0 < x < 1 \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 1 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = x^2 & 0 \le x \le 1
\end{array}
$$

to zero BCs and solve the new problem. What will the solution to this problem look like for different values of time? Does the solution agree with your intuition? What is the steady-state solution? What does the transient solution look like?

3. Transform

$$
\begin{array}{lll}
\text{PDE} & u_t = u_{xx} & 0 < x < 1 \\
\text{BCs} & \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) + hu(1,t) = 1 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = \sin(\pi x) & 0 \le x \le 1
\end{array}
$$

into a new problem with zero BCs; is the new PDE homogeneous?

OTHER READING

Analysis and Solution of Partial Differential Equations by R. L. Street. Brooks-Cole, 1973. This excellent text contains an extensive section on transforms of the type we discuss in this lesson and a good section on separation of variables.

LESSON 7

Solving More Complicated Problems by Separation of Variables

PURPOSE OF LESSON: To show how more complicated heat-flow problems can be solved by separation of variables. This lesson essentially consists of a worked problem that will give the reader more familiarity with the method. Hopefully, the reader will be able to extrapolate the ideas presented here to solve problems on his or her own.

Eigenvalue problems, known as *Sturm-Liouville problems*, are introduced, and some properties of these general problems are discussed.

The purpose of this lesson is to solve an *initial-boundary-value problem* by the separation of variables method that the reader might have trouble working on his or her own. Hopefully, the reader can extrapolate from this problem to other problems not specifically mentioned in this text.

We start with a one-dimensional heat-flow problem where one of the BCs contains derivatives.

Heat-Flow Problem with Derivative BC

Consider an apparatus

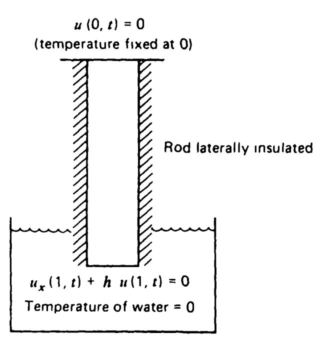


FIGURE 7.1 Diagram for the initial-boundary-value problem.

in which we fix the temperature at the top of the rod at $u(0,t) = 0$ and immerse the bottom of the rod in a solution of water fixed at the same temperature of zero (zero refers to some reference temperature). The natural flow of heat (Newton's law of cooling) says that the BC at $x = 1$ is

$$u_x(1,t) = -hu(1,t)$$

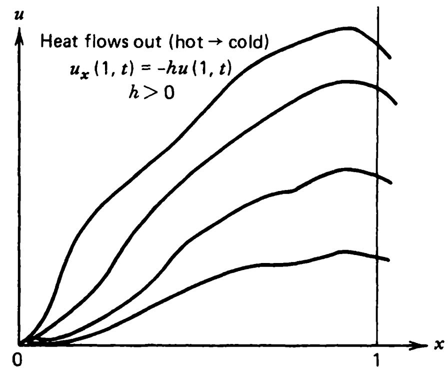


FIGURE 7.2 The nature of curves with BCs $\begin{cases} u(0,t) = 0 \\ u_x(1,t) = -hu(1,t) \end{cases}$

Suppose now the *initial temperature* of the rod is $u(x,0) = x$, but instantaneously thereafter ($t > 0$), we apply our BCs. To find the ensuing temperature, we must solve the IBVP

$$\begin{array}{llll} & \text{PDE} & u_t = \alpha^2 u_{xx} \quad 0 < x < 1 & 0 < t < \infty \\ (7.1) & \text{BCs} & \begin{cases} u(0,t) = 0 \\ u_x(1,t) + hu(1,t) = 0 \end{cases} & \text{(homogeneous BCs)} \\ & \text{IC} & u(x,0) = x \quad 0 \leq x \leq 1 & \end{array}$$

To apply the separation of variables method, we carry out the following steps:

STEP 1 (Separating the PDE into two ODEs)
Substituting $u(x,t) = X(x)T(t)$ into the PDE gives

$$XT' = \alpha^2 X''T$$

and dividing by $\alpha^2 XT$, we get

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X}$$

Since the left-hand side depends *only* on time and the right-hand side depends only on x (and since x and t are independent), both sides of this equation must be constants. Setting them both equal to μ gives the two ODEs

$$\begin{aligned} T' - \mu\alpha^2 T &= 0 \\ X'' - \mu X &= 0 \end{aligned} \tag{7.2}$$

We have now completed the *separation process*.

STEP 2 (Finding the separation constant)
First of all, μ must not be positive or else $T(t)$ will grow exponentially to infinity (which would make $u = XT$ go to infinity—which we can reject on physical grounds).

Secondly, suppose $\mu = 0$. This being the case, we have

$$X'' = 0$$

and thus

$$X(x) = A + Bx$$

But since the BCs of the problem are

$$\begin{aligned} u(0, t) &= X(0)T(t) = 0 \\ u_x(1,t) + h\,u(1,t) &= X'(1)T(t) + h\,X(1)\,T(t) = 0 \end{aligned}$$

we could conclude that

$$\begin{aligned} X(0) &= 0 \\ X'(1) + hX(1) &= 0 \end{aligned} \Rightarrow \begin{aligned} A &= 0 \\ B &= 0 \end{aligned}$$

which would mean $u(x,t) = 0$. In other words, $\mu = 0$ gives only $u = 0$; hence we throw it out (we are looking for nonzero solutions).

Finally, if $\mu < 0$, we call $\mu = -\lambda^2$ and write the two ODEs (7.2) as

$$\begin{aligned} T' + \lambda^2\alpha^2 T &= 0 \\ X'' + \lambda^2 X &= 0 \end{aligned}$$

which gives us solutions

$$\begin{aligned} T(t) &= Ae^{-(\lambda\alpha)^2 t} \\ X(x) &= B\sin(\lambda x) + C\cos(\lambda x) \end{aligned}$$

Hence, what we have is that *any function*

$$(7.3) \qquad u(x, t) = e^{-(\lambda\alpha)^2 t}\,[A \sin(\lambda x) + B \cos(\lambda x)]$$

for *any* λ and *any* A and B will satisfy the PDE (the reader can verify this calculation on his or her own). What we'd like to do now is find out how many of these functions will satisfy the BCs

$$(7.4) \qquad \begin{aligned} &u(0,t) = 0 \\ &u_x(1,t) + hu(1, t) = 0 \end{aligned}$$

Substituting the solution (7.3) into the BCs (7.4) gives us conditions on λ, A, and B that must be satisfied; namely,

$$\begin{aligned} &Be^{-(\lambda\alpha)^2 t} = 0 \Rightarrow B = 0 \\ &A\lambda e^{-(\lambda\alpha)^2 t} \cos \lambda + hAe^{-(\lambda\alpha)^2 t} \sin \lambda = 0 \end{aligned}$$

Performing a little algebra on this last equation gives us our desired condition on λ

$$\tan \lambda = -\lambda/h$$

In other words, to find λ, we must find the intersections of the curves $\tan \lambda$ and $-\lambda/h$ (Figure 7.3).

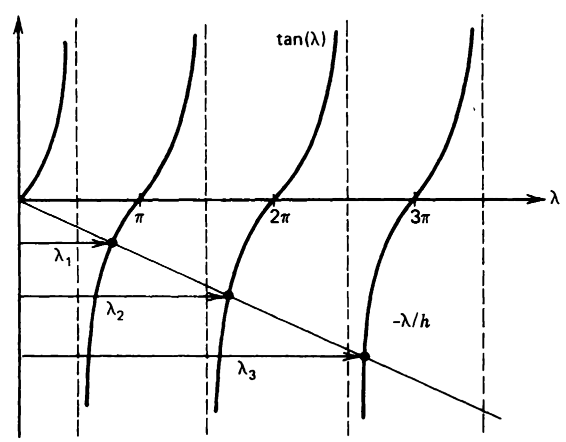


FIGURE 7.3 Graph showing intersections of tan (λ) and $-\lambda/h$.

These values $\lambda_1, \lambda_2, \ldots$ can be computed numerically for a given h on a computer and are called the **eigenvalues** of the boundary-value problem

$$(7.5) \qquad \begin{aligned} &X'' + \lambda^2 X = 0 \\ &X(0) = 0 \\ &X'(1) + hX(1) = 0 \end{aligned}$$

In other words, they are the values of λ for which there exists a *nonzero solution*. The eigenvalues λ_n of (7.5), which, in this case, are the roots of $\tan \lambda = -\lambda/h$, have been computed (for $h = 1$) numerically, and the first five values are listed in Table 7.1.

TABLE 7.1 Roots of $\tan \lambda = -\lambda$

n	λ_n
1	2.02
2	4.91
3	7.98
4	11.08
5	14.20

The solutions of (7.5) corresponding to the eigenvalues λ_n are called the **eigenfunctions** $X_n(x)$, and for this problem, we have

$$X_n(X) = \sin(\lambda_n x)$$

See Figure 7.4.

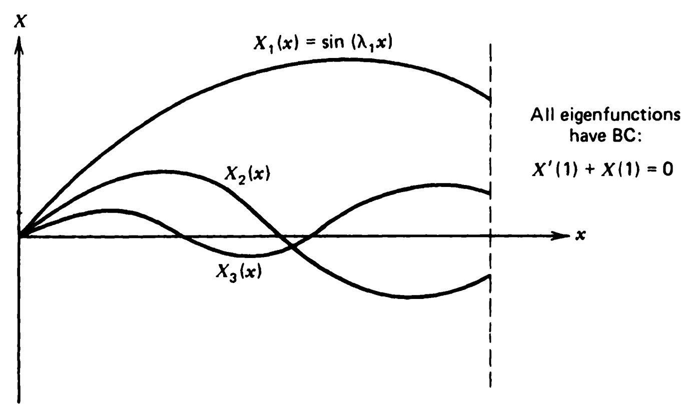


FIGURE 7.4 Eigenfunctions $X_n(x)$ of (7.5) for $h = 1$.

STEP 3 (Finding the fundamental solutions)
We now have an infinite number of functions (fundamental solutions),

$$u_n(x, t) = X_n(x)\, T_n(t) = e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x)$$

each one satisfying the PDE and the BCs. The final step is to add these functions together (the sum will still satisfy the PDE and BCs, since both the PDE and BCs are *linear* and *homogeneous*) in such a way that they agree with the IC when $t = 0$; that is, we sum

$$u(x, t) = \sum_{n=1}^{\infty} a_n X_n(x)\, T_n(t)$$

$$= \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x)$$

so that the IC $u(x,0) = x$ is satisfied. In other words,

$$u(x, 0) = x = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) \tag{7.6}$$

This brings us to our final step.

STEP 4 (Expansion of the IC as a sum of eigenfunctions)
To find the constants a_n in the eigenfunction expansion (7.6), we must multiply each side of the equation by $\sin(\lambda_m x)$ and integrate x from 0 to 1; that is;

$$\int_0^1 \xi \, sin\, (\lambda_m \xi)\, d\xi = \sum_{n=1}^{\infty} a_n \int_0^1 \sin(\lambda_n \xi) \sin(\lambda_m \xi)\, d\xi$$

$$= a_m \int_0^1 \sin^2(\lambda_m \xi)\, d\xi$$

$$= a_m \left(\frac{\lambda_m - \sin \lambda_m \cos \lambda_m}{2\lambda_m} \right)$$

Solving for a_m (we'll change the notation to a_n), we get our desired result

$$a_n = \frac{2\lambda_n}{(\lambda_n - \sin \lambda_n \cos \lambda_n)} \int_0^1 \xi \sin(\lambda_n \xi)\, d\xi \tag{7.7}$$

In other words, our solution to (7.1) is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x) \tag{7.8}$$

where the constants a_n are given by (7.7). In this problem, the first five constants a_n have been computed and are listed in Table 7.2.

TABLE 7.2 Coefficients a_n in (7.8)

n	a_n
1	0.24
2	0.22
3	−0.03
4	−0.11
5	−0.09

Hence, the first three terms of the IBVP

$$
\begin{array}{llll}
 & \text{PDE} & u_t = u_{xx} \quad 0 < x < 1 & 0 < t < \infty \\
(7.9) & \text{BCs} & \begin{cases} u(0,t) = 0 \\ u_x(1,t) + u(1,t) = 0 \end{cases} & 0 < t < \infty \\
 & \text{IC} & u(x,0) = x \quad 0 \le x \le 1 &
\end{array}
$$

are

$$u(x, t) = 0.24\, e^{-4t} \sin(2x) + 0.22\, e^{-24t} \sin(4.9x) + 0.03 e^{-63.3t} \sin(7.98x) + \dots$$

The graph of this solution is drawn for various values of time in Figure 7.5. The reader can ask himself or herself if this solution agrees with his or her intuition and whether or not it satisfies the BCs of the problem.

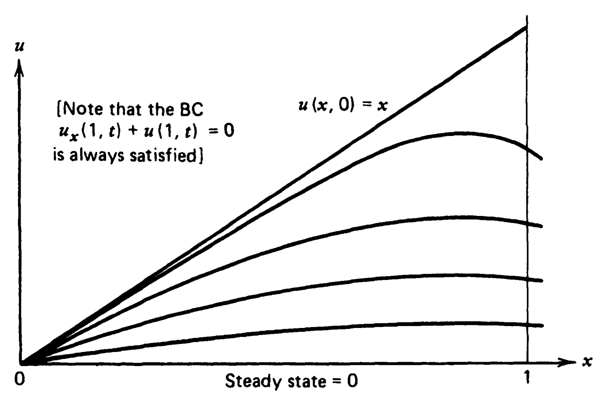


FIGURE 7.5 Solution to (7.8).

NOTES

The eigenvalue problem (7.5) is a special case of the general problem

$$
(7.10) \qquad \begin{array}{ll} \text{ODE} & [p(x)\,y']' - q(x)y + \lambda\, r(x)y = 0 \qquad 0 < x < 1 \\ \text{BCs} & \begin{cases} \alpha_1 y\,(0) + \beta_1 y'(0) = 0 \\ \alpha_2 y\,(1) + \beta_2 y'(1) = 0 \end{cases} \end{array}
$$

known as the **Sturm-Liouville problem**. When we solve PDEs by separation of variables with linear homogeneous BCs, the ODE in $X(x)$ along with its BCs will always be some particular Sturm-Liouville problem. We observe that the eigenvalue problem (7.5) is a special case of (7.10).

What Sturm and Liouville proved is that under suitable conditions on the functions $p(x)$, $q(x)$, and $r(x)$, the problem (7.10) has

1. An infinite sequence of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots \to \infty$$

2. Corresponding to *each* eigenvalue λ_n, there is *one* nonzero solution $y_n\,(x)$ [not including other constant multiples of $y_n(x)$].

3. If $y_n(x)$ and $y_m(x)$ are two *different* eigenfunctions (corresponding to $\lambda_n \neq \lambda_m$), then they are *orthogonal* with respect to the *weight function* $r(x)$ on the interval of [0,1]; that is, they satisfy

$$\int_0^1 r(x)\, y_n(x)\, y_m(x)\, dx = 0$$

More details of Sturm-Liouville-type problems can be found in references 1 and 2.

PROBLEMS

1. Solve the following heat-flow problem:

$$
\begin{array}{lll} \text{PDE} & u_t = u_{xx} & 0 < x < 1 \qquad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = x & 0 \leqslant x \leqslant 1 \end{array}
$$

by separation of variables. Does your solution agree with your intuition? What is the steady-state solution?

2. What are the eigenvalues and eigenfunctions of the Sturm-Louiville problem

$$\text{ODE} \quad X'' + \lambda X = 0 \quad 0 < x < 1$$

$$\text{BCs} \quad \begin{cases} X(0) = 0 \\ X'(1) = 0 \end{cases}$$

What are the functions $p(x)$, $q(x)$, and $r(x)$ in the general Sturm-Liouville problem for this equation?

3. Solve the following problem with insulated boundaries:

$$\text{PDE} \quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = x \quad 0 \leqslant x \leqslant 1$$

Does your solution agree with your interpretation of the problem? What is the steady-state solution?; does this make sense?

4. What are the eigenvalues and eigenfunctions of

$$\text{ODE} \quad X'' + \lambda X = 0 \quad 0 < x < 1$$

$$\text{BCs} \quad \begin{cases} X'(0) = 0 \\ X'(1) = 0 \end{cases}$$

OTHER READING

1. *Elementary Differential Equations and Boundary-Value Problems* by W. E. Boyce and R. C. DiPrima. John Wiley & Sons, 1965. Chapter 11. This is an ordinary-differential-equations text that contains an excellent section on the Sturm-Liouville problem, one of the better undergraduate texts in ODEs.

2. *Advanced Engineering Mathematics* by E. Kreyszig. John Wiley & Sons, 1967. This text contains many worked examples of typical problems; very readable.

LESSON 8

Transforming Hard Equations into Easier Ones

PURPOSE OF LESSON: To show how one can transform a PDE in $u(x,t)$ into a new (easier) one in a new variable $w(x,t)$. The transformation is generally based on intuition, and in this lesson, the PDEs

$$u_t = \alpha^2 u_{xx} - \beta u$$
$$u_t = \alpha^2 u_{xx} - v u_x$$

are transformed into the simple heat equation

$$w_t = \alpha^2 w_{xx}$$

by means of the transformations

$$u(x,t) = e^{-\beta t} w(x,t)$$
$$u(x,t) = e^{v[x - vt/2]/2\alpha^2} w(x,t)$$

After the transformations are made, the heat equation (the easy one) can be solved for $w(x,t)$, hence,

$$u = e^{-\beta t} w(x,t)$$
$$u = e^{v[x - vt/2]/2\alpha^2} w(x,t)$$

are the solutions of the original equations (of course, the BCs and the IC must be transformed too).

The reader may get the impression from the last two lessons that the only type of PDE that can be solved by separation of variables is

$$u_t = \alpha^2 u_{xx}$$

It is true the heat equation is the easiest parabolic PDE to solve by separation of variables, but it is in no way the *only* equation we can solve by this technique.

As mentioned earlier, as long as the equation is *linear* and *homogeneous*, we can separate variables. For example, two-dimensional heat flow inside a circle would be described by the equation

$$u_t = \alpha^2\left[u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right]$$

and although it has *variable coefficients,* it can still be separated into three ODEs.

This lesson will show the reader that sometimes a PDE doesn't have to be attacked directly but that the original PDE can be *transformed* into an easier one. In this way, the easier problem can be solved (by separation of variables or some other technique). We now present an example that illustrates this technique.

Transforming a Heat-Flow Problem with Lateral Heat Loss into an Insulated Problem

Consider the following problem:

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} - \beta u \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$(8.1) \qquad \text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leq x \leq 1$$

where the term $-\beta u$ represents heat flow across the lateral boundary (Figure 8.1).

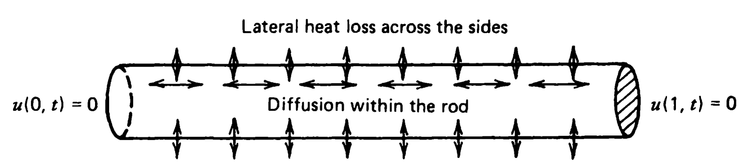


FIGURE 8.1 Heat flow described by $u_t = \alpha^2 u_{xx} - \beta u$.

The goal of this lesson is to introduce a *new temperature* $w(x,t)$ in place of $u(x,t)$, so that the PDE in w is simpler than the original one

$$u_t = \alpha^2 u_{xx} - \beta u$$

This is a common technique in PDEs, and the transformation is generally based on an intuitive feeling of how the solution of the original PDE behaves. For example, in our problem (8.1), the temperature $u(x,t)$ at any point x_0 is changing as a result of two phenomena

1. *diffusion* of heat within the rod (due to $\alpha^2 u_{xx}$).
2. *heat flow* across the lateral boundary (due to $-\beta u$).

The important point is that if there were *no* diffusion *within* the rod ($\alpha = 0$), then the temperature at each point x_0 would "damp" exponentially to zero according to

$$u(x_0,t) = u(x_0,0)e^{-\beta t}$$

By means of this observation, we wonder if we can essentially decompose the temperature $u(x,t)$ of problem (8.1) into two factors

$$u(x,t) = e^{-\beta t}w(x,t) \tag{8.2}$$

or

$$\text{Noninsulated temperature} = e^{-\beta t} \quad \text{(insulated temperature)}$$

where $w(x,t)$ would represent the temperature due to diffusion only. Let's see what happens if we substitute this expression into problem (8.1); this is a routine calculation (the reader can do it on his or her own), and we arrive at

$$\begin{array}{lll} \text{PDE} & w_t = \alpha^2 w_{xx} & 0 < x < 1 \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} w(0,t) = 0 \\ w(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & w(x,0) = \phi(x) & 0 \leq x \leq 1 \end{array} \tag{8.3}$$

This is exactly the same problem we started with except that now the PDE doesn't contain $-\beta u$; so all we have to do to solve (8.1) is solve the transformed problem (8.3) and then multiply the solution $w(x,t)$ by $e^{-\beta t}$. In this case, we have already solved (8.3) previously by the separation of variables method and found

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \\ a_n &= 2\int_0^1 \phi(\xi)\sin(n\pi\xi)\,d\xi \end{aligned} \tag{8.4}$$

and, hence, the solution of the original problem (8.1) is

$$u(x,t) = e^{-\beta t}w(x,t)$$

The following example in the notes is solved by this technique.

NOTES

1. To solve the problem

$$\begin{aligned} u_t &= u_{xx} - u \qquad 0 < x < 1 \qquad 0 < t < \infty \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= \sin(\pi x) + 0.5 \sin(3\pi x) \end{aligned}$$

   by the preceding strategy, we
   (a) neglect the convection term $-u$ for the time being.
   (b) solve the initial-boundary-value problem without the term $-u$ to get

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x) + 0.5\, e^{-(3\pi)^2 t} \sin(3\pi x)$$

   (c) multiply this solution by the convection factor $e^{-\beta t} = e^{-t}$ to get the solution

$$u(x,t) = e^{-t}\,[e^{-\pi^2 t} \sin(\pi x) + 0.5\, e^{-(3\pi)^2 t} \sin(3\pi x)]$$

2. The *diffusion-convection* equation

$$u_t = \alpha^2 u_{xx} - v u_x$$

   (v is a constant) can also be transformed to

$$w_t = \alpha^2 w_{xx}$$

   In this case, the transformation is

$$u(x,t) = e^{v(x - vt/2)/2\alpha^2}\, w(x,t)$$

   This transformation essentially factors out the part of the solution (exponential factor) that is due to the moving medium. Note that the exponential factor consists of a moving exponential (moving to the right with velocity $v/2$). The reader will get a chance to use this transformation in the problem set.

PROBLEMS

1. Solve the diffusion problem

$$
\begin{array}{llll}
\text{PDE} & u_t = u_{xx} - u_x & 0 < x < 1 & 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty & \\
\text{IC} & u(x,0) = e^{x/2} & 0 \leq x \leq 1 &
\end{array}
$$

by transforming it into an easier problem. What does the solution look like? We could interpret this problem as describing the concentration $u(x,t)$ in a moving medium (moving from left to right with velocity $v = 1$) where the concentration at the *ends* of the medium are kept at zero (by some filtering device) and the *initial concentration* is $e^{x/2}$. Does your solution agree with this interpretation?

2. Solve the problem

$$
\begin{array}{llll}
\text{PDE} & u_t = u_{xx} - u + x & 0 < x < 1 & 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 1 \end{cases} & 0 < t < \infty & \\
\text{IC} & u(x,0) = 0 & 0 \leq x \leq 1 &
\end{array}
$$

by
   (a) changing the nonhomogeneous BCs to homogeneous ones.
   (b) transforming into a new equation without the term $-u$.
   (c) solving the resulting problem.

3. Solve

$$
\begin{array}{llll}
\text{PDE} & u_t = u_{xx} - u & 0 < x < 1 & 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty & \\
\text{IC} & u(x,0) = \sin(\pi x) & 0 \leq x \leq 1 &
\end{array}
$$

directly by separation of variables without making any preliminary transformation. Does your solution agree with the solution you would obtain if the transformation

$$u(x,t) = e^{-t}w(x,t)$$

were made in advance?

OTHER READING

Nonlinear Partial Differential Equations in Engineering by W. F. Ames. Academic Press, 1965. This text discusses many types of transformations for changing old problems into new ones, so that sometimes even nonlinear problems can be transformed into linear ones.

LESSON 9

Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

PURPOSE OF LESSON: To show how to solve the initial-boundary-value problem

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} + f(x,t) \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leq x \leq 1$$

A nonhomogeneous PDE can be solved by finding a series solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

where the $X_n(x)$ are the eigenfunctions we find when solving the associated homogeneous problem

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leq x \leq 1$$

and $T_n(t)$ are functions that can be found by solving a sequence of ODEs.

In Lesson 6, we discussed how transformations could be made to transform nonhomogeneous BCs into *homogeneous* ones. Unfortunately, the PDE was left nonhomogeneous by this process, and we were left with the problem

$$\text{(9.1)}\qquad \begin{array}{ll} \text{PDE} & u_t = \alpha^2 u_{xx} + f(x,t) \qquad 0 < x < 1 \qquad 0 < t < \infty \\ \text{BCs} & \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{cases} \qquad 0 < t < \infty \\ \text{IC} & u(x,0) = \phi(x) \qquad 0 \leq x \leq 1 \end{array}$$

The purpose of this lesson is to solve this problem by a method that is analogous to the method of *variation of parameters* in ODEs and is known as the **eigenfunction expansion method.**

The idea is quite simple. Inasmuch as the solution of (9.1) with $f(x,t) = 0$ (so-called corresponding homogeneous problem) is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} X_n(x)$$

where λ_n and $X_n(x)$ are the eigenvalues and eigenfunctions of the Sturm-Liouville problem,

$$\text{(9.2)}\qquad \begin{aligned} &X'' + \lambda^2 X = 0 \\ &\alpha_1 X'(0) + \beta_1 X(0) = 0 \\ &\alpha_2 X'(1) + \beta_2 X(1) = 0 \end{aligned}$$

we ask whether the solution of the nonhomogeneous problem (9.1) can be written in the slightly more general form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

The reason for this speculation is physically appealing, inasmuch as a source of heat $f(x,t)$ *within* the rod will give rise to a new time component and not the damping factor

$$e^{-(\lambda_n \alpha)^2 t}$$

as was the case when the only input into the problem was the IC.

To show how this method works, we apply it to a simple problem so the details aren't as complicated.

Solution by the Eigenfunction Expansion Method

Consider the nonhomogeneous problem

$$\text{PDE} \quad u_t = \alpha^2 u_{xx} + f(x,t) \quad 0 < x < 1 \quad 0 < t < \infty$$

$$(9.3) \quad \text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = \phi(x) \quad 0 \leq x \leq 1$$

To solve this problem, we divide the procedure into the following steps:

STEP 1 The basic idea in this method is to decompose the heat source $f(x,t)$ into simple components

$$f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + \ldots + f_n(t)X_n(x) + \ldots$$

and find the response $u_n(x,t)$ to *each* of these individual components $f_n(t)X_n(x)$. The solution to our problem is then

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

To determine how to decompose $f(x,t)$ into its component parts $f_n(t)X_n(x)$ is one of the major problems. It turns out that the $X_n(x)$ factors in this problem are the *eigenvectors* of the Sturm-Liouville system we get when solving the *associated homogeneous problem* to (9.3) by separation of variables; that is,

$$(9.4) \quad \begin{aligned} &u_t = \alpha^2 u_{xx} \quad \text{(note that } f(x,t) = 0\text{)} \\ &u(0,t) = 0 \\ &u(1,t) = 0 \\ &u(x,0) = \phi(x) \end{aligned}$$

in this case, the Sturm-Liouville problem we find when separating variables is

$$\begin{aligned} &X'' + \lambda^2 X = 0 \\ &X(0) = 0 \\ &X(1) = 0 \end{aligned}$$

and, hence, the $X_n(x)$ are

$$X_n(x) = \sin(n\pi x) \quad n = 1, 2, \ldots$$

Hence, our decomposition of the heat source has the form

$$(9.5) \quad f(x,t) = f_1(t)\sin(\pi x) + f_2(t)\sin(2\pi x) + \ldots + f_n(t)\sin(n\pi x) + \ldots$$

Finally, to find the functions $f_n(t)$, we merely multiply each side of this equation by $\sin(m\pi x)$ and integrate from zero to one (with respect to x); hence, we have

$$\int_0^1 f(x,t) \sin(m\pi x)\, dx = \sum_{n=1}^{\infty} f_n(t) \int_0^1 \sin(m\pi x) \sin(n\pi x)\, dx$$
$$= \frac{1}{2} f_m(t)$$

or (changing m to n)

$$f_n(t) = 2 \int_0^1 f(x,t) \sin(n\pi x)\, dx \tag{9.6}$$

This will give us an equation for the coefficients $f_n(t)$ in terms of the heat source $f(x,t)$.

STEP 2 (Find the response $u_n(x,t) = T_n(t)X_n(x)$ to input $f_n(t)X_n(x)$)
We now replace the heat source $f(x,t)$ by its decomposition

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x)$$

and try to find the individual responses

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$$

in other words, we seek the functions $T_n(t)$. Knowing these, the answer to our problem is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$$

Substituting $u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$ into the system

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x) \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= \phi(x) \end{aligned}$$

gives us

$$\sum_{n=1}^{\infty} T_n'(t) \sin(n\pi x) = -\alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 T_n(t) \sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x)$$

(9.7)
$$\sum_{n=1}^{\infty} T_n'(t) \sin 0 = 0 \qquad \text{(says nothing; zero = zero)}$$

$$\sum_{n=1}^{\infty} T_n(t) \sin(n\pi) = 0 \qquad \text{(says nothing; zero = zero)}$$

$$\sum_{n=1}^{\infty} T_n(0) \sin(n\pi x) = \phi(x)$$

Rewriting the PDE and the IC as

$$\text{PDE} \qquad \sum_{n=1}^{\infty} [T_n' + (n\pi\alpha)^2 T_n - f_n(t)] \sin(n\pi x) = 0$$

$$\text{IC} \qquad \sum_{n=1}^{\infty} T_n(0) \sin(n\pi x) = \phi(x)$$

we can see fairly easily that $T_n(t)$ will satisfy the simple initial value problem

(9.8)
$$T_n' + (n\pi\alpha)^2 T_n = f_n(t)$$

$$T_n(0) = 2\int_0^1 \phi(\xi) \sin(n\pi\xi)\, d\xi = a_n$$

This ODE is one of the easier ones to solve (use an integrating factor) and has the solution

(9.9)
$$T_n(t) = a_n e^{-(n\pi\alpha)^2 t} + \int_0^t e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau)\, d\tau$$

Hence, the solution of problem (9.3) is

(9.10)
$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} [a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)] + \sum_{n=1}^{\infty} \left[\sin(n\pi x) \int_0^t e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau)\, d\tau\right]$$

Transient part (because of the initial condition) — Steady state (because of the right-hand side $f(x,t)$)

We can see from this solution that the *temperature response* is due to *two parts*: the first part that is due to the IC and the second part that is due to the heat source $f(x,t)$. The phrase steady state is not the best phrase to describe the second part, since it doesn't necessarily come to rest (it may approach a periodic steady state, *if* $f(x,t)$ is periodic in t).

This completes the problem. Before stopping, however, we will show how this method can be applied to a specific example.

Solution of a Problem by the Eigenfunction-Expansion Method

Consider the simple problem

$$\text{(9.11)}\qquad \begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} + \sin(3\pi x) & 0 < x < 1 \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = \sin(\pi x) & 0 \leqslant x \leqslant 1 \end{array}$$

Our goal is to compute the coefficients $T_n(t)$ in the expansion

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$$

(the eigenfunctions $X_n(x)$ are still the same for this problem). If we substitute this expansion in the problem, we will get an ODE for the functions $T_n(t)$ In fact, we will get

$$T_n' + (n\pi\alpha)^2 T_n = f_n(t) = 2\int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases}$$

$$T_n(0) = 2\int_0^1 \sin(\pi\xi)\sin(n\pi\xi)\,d\xi = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Writing out these equations for $n = 1, 2, \ldots$, we see

$$(n = 1)\qquad \left.\begin{array}{l} T_1' + (\pi\alpha)^2 T_1 = 0 \\ T_1(0) = 1 \end{array}\right\} \Rightarrow T_1(t) = e^{-(\pi\alpha)^2 t}$$

$$(n = 2)\qquad \left.\begin{array}{l} T_2' + (2\pi\alpha)^2 T_2 = 0 \\ T_2(0) = 0 \end{array}\right\} \Rightarrow T_2(t) = 0$$

$$(n = 3)\qquad \left.\begin{array}{l} T_3' + (3\pi\alpha)^2 T_3 = 1 \\ T_3(0) = 0 \end{array}\right\} \Rightarrow T_3(t) = \frac{1}{(3\pi\alpha)^2}\left[1 - e^{-(3\pi\alpha)^2 t}\right]$$

$$(n \geqslant 4)\qquad \left.\begin{array}{l} T_n' + (n\pi\alpha)^2 T_n = 0 \\ T_n(0) = 0 \end{array}\right\} \Rightarrow T_n(t) = 0$$

Hence the solution of our problem is

$$u(x,t) = \underbrace{e^{-(\pi\alpha)^2 t} \sin(\pi x)}_{\substack{\text{Transient} \\ \text{(because of initial conditions)}}} + \underbrace{\frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi\alpha)^2 t}] \sin(3\pi x)}_{\substack{\text{Steady state} \\ \text{(because of the right-hand side of the PDE)}}} \tag{9.12}$$

NOTES

1. The method of eigenfunction expansion is one of the most powerful for solving nonhomogeneous PDEs. Later, when we study integral transforms, we will see that there are other methods for solving these types of problems.
2. The eigenfunctions $X_n(x)$ in the expansion *change* from problem to problem and depend on the PDE and BCs. The reader should look at problem 4 in the problem set to make sure he or she knows how to find the eigenfunctions $X_n(x)$.
3. If the reader remembers ODE theory, he or she will remember that solutions of equations corresponding to nonhomogeneous terms like

$$P_n(x)\, e^{\alpha x} \begin{cases} \sin(\beta x) \\ \cos(\beta x) \end{cases}$$

could be found by the method of *undetermined coefficients*. The same is true here. Problem (9.11) could be solved by this method. Any reader interested in this method should consult the reference.

PROBLEMS

1. The solution of the problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx} + \sin(3\pi x) & 0 < x < 1 \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = \sin(\pi x) & 0 \leq x \leq 1 \end{array}$$

is given by (9.12). Does this solution agree with your intuition? What does the solution look like?

2. Solve the problem

$$\text{PDE} \quad u_t = u_{xx} + \sin(\pi x) + \sin(2\pi x) \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 0 \quad 0 \leq x \leq 1$$

3. Solve the problem

$$\text{PDE} \quad u_t = u_{xx} + \sin(\pi x) \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 1 \quad 0 \leq x \leq 1$$

by the method of eigenfunction expansion.

4. Find the solution of

$$\text{PDE} \quad u_t = u_{xx} + \sin(\lambda_1 x) \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u_x(1,t) + u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 0 \quad 0 \leq x \leq 1$$

by the method of eigenfunction expansion where λ_1 is the first root of the equation $\tan \lambda = -\lambda$. What are the eigenfunctions $X_n(x)$ in this problem?

5. Solve the problem

$$\text{PDE} \quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = \cos t \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = x \quad 0 \leq x \leq 1$$

by

(a) transforming it to one with zero BCs.

(b) solving the resulting problem by expanding it in terms of eigenfunctions.

OTHER READING

Elementary Partial Differential Equations by P. W. Berg and J. L. McGregor. Holden-Day, 1966. One of the more popular texts on PDEs; slightly more advanced than this text; clearly written. An extensive section on nonhomogeneous problems (Chapter 5).

LESSON 10

Integral Transforms (Sine and Cosine Transforms)

PURPOSE OF LESSON: To introduce the idea of integral transforms and show how they transform PDEs in n variables into differential equations in $n-1$ variables.

It is also shown that these transforms can be interpreted as resolving the input of the problem into simple parts (frequency resolution), finding the solution for each subpart, and adding the results.

In summary, integral transforms change differentiation to multiplication, and, hence, certain partial derivatives are changed into algebraic expressions.

The sine and cosine transforms are introduced and are used to solve an infinite-diffusion problem. The solution is interesting in that it involves the complementary-error function.

An integral transformation is merely a transformation that assigns to one function $f(t)$ a new function $F(s)$ by means of a formula like

$$F(s) = \int_A^B K(s,t)f(t)\,dt$$

Note that we *start* with a function of t and *end* with a function of s. The function $K(s,t)$ is called the **kernel of the transformation** and is the major ingredient that distinguishes one transform from another; it is chosen so that the transform has certain desirable properties. The limits of integration A and B also change from transformation to transformation.

The general philosophy behind integral transformations is that they eliminate *partial derivatives* with respect to one of the variables; hence, the new equation has one less variable. For example, if we apply a transform to the PDE

$$u_t = u_{xx}$$

for the purpose of eliminating the time derivative, then we would arrive at an ODE in x. On the other hand, if we had the PDE

$$u_{xx} + u_{yy} + u_{zz} = 0$$

and applied the Fourier transform to the x-variable, then we would eliminate the derivative u_{xx} and would have a new PDE in y and z. We could, of course, apply the Fourier transform again to eliminate one of the other variables (and arrive at an ODE in the last remaining variable). In other words, integral transforms change problems into easier ones. The transformed problem is then solved, and its *inverse* is obtained to find the solution to the original problem; this general strategy is illustrated in Figure 10.1.

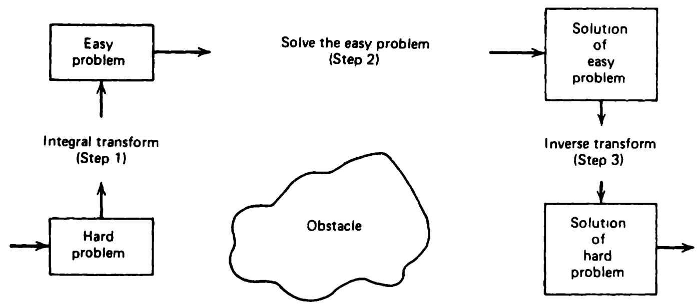


FIGURE 10.1 General philosophy of transforms.

In Figure 10.1, we see that along with every integral transform, there is an *inverse transform* that will reproduce that original function from its transform. The transform and its inverse together form what is called a **transform pair**. Table 10.1 lists several common transform pairs that we will use to solve PDEs.

TABLE 10.1 Some Common Transform Pairs

Transform pairs

1. $$\begin{cases} \mathcal{F}_s[f] = F(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(t)\sin(\omega t)\,dt & \text{(Fourier sine transform)} \\ \mathcal{F}_s^{-1}[F] = f(t) = \displaystyle\int_0^\infty F(\omega)\sin(\omega t)\,d\omega & \text{(inverse sine transform)} \end{cases}$$

2. $$\begin{cases} \mathcal{F}_c[f] = F(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(t)\cos(\omega t)\,dt & \text{(Fourier cosine transform)} \\ \mathcal{F}_c^{-1}[F] = f(t) = \displaystyle\int_0^\infty F(\omega)\cos(\omega t)\,d\omega & \text{(inverse cosine transform)} \end{cases}$$

3. $$\begin{cases} \mathcal{F}[f] = F(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^\infty f(x)\,e^{-i\omega x}dx & \text{(Fourier transform)} \\ \mathcal{F}^{-1}[F] = f(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^\infty F(\omega)\,e^{i\omega x}d\omega & \text{(inverse Fourier transform)} \end{cases}$$

4. $$\begin{cases} \mathcal{F}_s[f] = S_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin(n\pi x/L)\,dx & \text{(finite-sine transform)} \\ \mathcal{F}_s^{-1}[F_n] = f(x) = \displaystyle\sum_{n=1}^{\infty} S_n \sin(n\pi x/L) & \text{(inverse finite-sine transform)} \end{cases}$$

5. $$\begin{cases} \mathcal{F}_c[f] = C_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos(n\pi x/L)\,dx & \text{(finite-cosine transform)} \\ \mathcal{F}_c^{-1}[F_n] = f(x) = \dfrac{C_0}{2} + \displaystyle\sum_{n=1}^{\infty} C_n \cos(n\pi x/L) & \text{(inverse finite-cosine transform)} \end{cases}$$

6. $$\begin{cases} \mathcal{L}[f] = F(s) = \displaystyle\int_0^\infty f(t)\,e^{-st}\,dt & \text{(Laplace transform)} \\ \mathcal{L}^{-1}[F] = f(t) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds & \text{(inverse Laplace transform)} \end{cases}$$

7. $$\begin{cases} H[f] = F_n(\xi) = \displaystyle\int_0^\infty rJ_n(\xi r)f(r)\,dr & \text{(Hankel transform)} \\ H^{-1}[F_n] = f(r) = \displaystyle\int_0^\infty J_n(\xi r)F_n(\xi)\,d\xi & \text{(inverse Hankel transform)} \end{cases}$$

Note that in these transforms we have alternative notations. For instance, in the case of the Laplace transform, the notation $\mathcal{L}[f]$ indicates that we are taking the transform of f, whereas the alternative notation $F(s)$ indicates a function of s.

The current lesson does not attempt to study all of these transform pairs—only the sine and cosine transforms (1. and 2.); later, we will study several of the others. Questions about the relationship between the transforms, when to apply them, advantages and disadvantages of each, will be answered as we go along. However, before we begin the study of integral transforms, it will be useful to study what is called the *spectrum of a function* (or the *spectral resolution* of a function).

The Spectrum of a Function

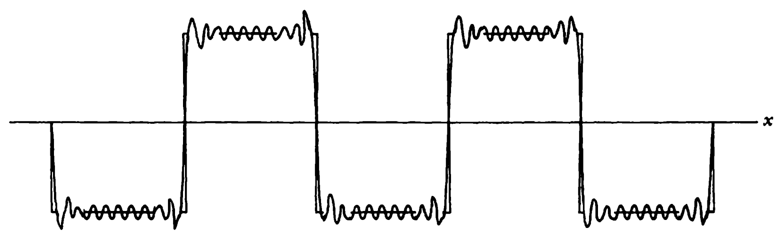
Integral transforms and the spectrum of a function are closely related; in fact, an **integral transformation** can be thought of as a resolution of a function into a certain spectrum of components. How the transform actually resolves the function changes from transform to transform, but the function is being resolved into something nevertheless.

For instance, let's consider the resolution of a periodic funciton $f(x)$ into sines and cosines (Fourier series)*

* Fourier series will be discussed in detail in Chapter 11.

$$f(x) = \sum_{n=0}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]$$

(Figure 10.2).



A square wave approximated by sines and cosines

FIGURE 10.2 Expansion of a periodic function into sines and cosines.

Here, the coefficients A_n and B_n represent the amount of the function $f(x)$ made up by cos (nx) and sin (nx), respectively, while the square root

$$\sqrt{A_n^2 + B_n^2}$$

(called the **spectrum of the function**) measures the amount of $f(x)$ with frequency n.

For example, if the function $f(x)$ were a simple sum of sines and cosines

$$f(x) = 1 + \sin x + \frac{1}{5}\sin(3x) + \cos x + \frac{1}{2}\cos(2x) + \frac{1}{4}\cos(4x)$$

then its spectrum (discrete) would be as given in Figure 10.3.

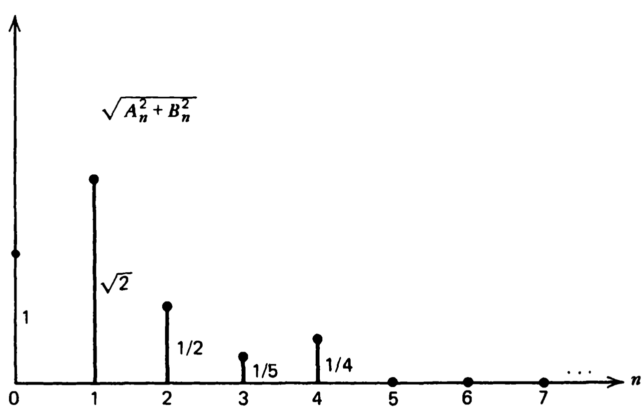


FIGURE 10.3 Discrete spectrum of $f(x)$.

By reading off the values of $\sqrt{A^2{}_n + B^2{}_n}$, we can tell the magnitude of the component in $f(x)$ with frequency n.

Functions that are *periodic* can be resolved into *infinite series* (they have discrete spectrums), whereas functions that are *not periodic* must be resolved into a *continuous spectrum* of values (of course, if a function is defined only on a *finite interval*, we could extend the function outside the interval in a periodic way, so that a Fourier series representation could be obtained for the function inside the interval).

For example, although a nonperiodic function $f(x)$ cannot be represented by an infinite series of sines and cosines, we might be tempted to write it as a *continuous analog* of the Fourier series; that is,

$$f(x) = \int_{-\infty}^{\infty} [C(\omega) \cos(\omega x) + S(\omega) \sin(\omega x)]\, d\omega$$

where the functions $S(\omega)$ and $C(\omega)$ measure the sine and cosine component of $f(x)$ and

$$\sqrt{S^2(\omega) + C^2(\omega)}$$

measures the ω frequency component of $f(x)$ and is called the spectrum(continuous spectrum) of $f(x)$.

With this intuitive explanation of the spectrum of a function, we now get to the nuts and bolts of integral transforms. The first step would be to list a few properties of these transforms that make them work.

Sine and Cosine Transforms of Derivatives

(10.1)

1. $\mathscr{F}_s[f'] = -\omega\, \mathscr{F}_c[f]$ (proved by integration by parts)
2. $\mathscr{F}_s[f''] = \frac{2}{\pi}\omega f(0) - \omega^2 \mathscr{F}_s[f]$
3. $\mathscr{F}_c[f'] = \frac{-2}{\pi} f(0) + \omega \mathscr{F}_s[f]$
4. $\mathscr{F}_c[f''] = -\frac{2}{\pi} f'(0) - \omega^2 \mathscr{F}_c[f]$

Several other sine and cosine transforms and their inverses can be found in the tables at the end of the lessons. We now show how the sine transform can solve an important initial-boundary-value problem.

Solution of an Infinite-Diffusion Problem via the Sine Transform

The problem we are interested in is the *infinite diffusion problem*

$$
\begin{array}{llll}
\text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < \infty & 0 < t < \infty \\
\text{BC} & u(0, t) = A & 0 < t < \infty & \\
\text{IC} & u(x, 0) = 0 & 0 \leq x < \infty &
\end{array}
$$

(Figure 10.4).

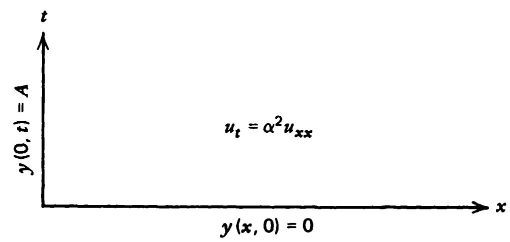


FIGURE 10.4 Diffusion problem in a semi-infinite medium.

STEP 1 To solve this, we break it into three simple steps. First our strategy is to transform the *x-variable* via the Fourier sine transform so that we get an ODE in time. We start by transforming each side of the PDE; in other words

$$\mathcal{F}_s[u_t] = \alpha^2 \mathcal{F}_s[u_{xx}]$$

Let's consider each term individually:

$\mathcal{F}_s[u_t]$: The partial derivative u_t in this problem is what we could call the *off derivative*, since our transform is *with respect to x*. In this case, we can write

$$
\begin{aligned}
\mathcal{F}_s[u_t] &= \frac{2}{\pi} \int_0^\infty u_t(x,t) \sin(\omega x)\, dx \\
&= \frac{\partial}{\partial t} \left[\frac{2}{\pi} \int_0^\infty u(x,t) \sin(\omega x)\, dx \right] \\
&= \frac{d}{dt} \mathcal{F}_s[u] \\
&= \frac{d}{dt} U(t)
\end{aligned}
$$

The fact that we took the derivative *outside* the integral is a property from calculus. Note that u is a function of x and t, whereas its transform

$$\mathcal{F}_s[u] = U(\omega,t)$$

is a function of ω and t. The new variable ω will be treated like a parameter in the new problem, and, hence, we call the sine transform a function of t alone; that is,

$$\mathcal{F}_s[u] = U(t)$$

$\mathcal{F}_s[u_{xx}]$: For this one, we have the identity

$$\begin{aligned}\mathcal{F}_s[u_{xx}] &= \frac{2}{\pi}\,\omega u(0,t) - \omega^2\mathcal{F}_s[u] \\ &= \frac{2}{\pi}\,\omega u(0,t) - \omega^2 U(t) \\ &= \frac{2A\omega}{\pi} - \omega^2 U(t)\end{aligned}$$

Note here that when you proved these identities (10.1), you did it for functions of *one* variable $f(x)$. We now have a slight modification, since $u(x,t)$ depends on x and t. You should use the formulas according to which variable is being transformed and treat the others as constants. In this case, the transform is with respect to x, and, hence, t is just carried along as a constant. Also note that the BC $u(0,t) = A$ is used at this point in our operation.

Substituting these expressions into our PDE, we arrive at the ODE

$$\frac{dU}{dt} = \alpha^2\left[-\omega^2 U(t) + \frac{2A\omega}{\pi}\right]$$

The only thing missing is an IC for $U(t)$; we get this by transforming the IC $u(x,0) = 0$ to get

$$\mathcal{F}_s[u(x,0)] = U(0) = 0$$

This completes the first step in the transform process—we have changed the original problem into an initial-value problem

$$\text{(10.2)}\qquad \begin{aligned}&\text{ODE} \quad \frac{dU}{dt} + \omega^2\alpha^2 U = \frac{2A\omega\alpha^2}{\pi} \\ &\text{IC} \quad U(0) = 0\end{aligned}$$

STEP 2 To solve this IVP, we could use a variety of elementary techniques from ordinary differential equations (integrating factor, homogeneous and particular solution); in any case, the solution is

$$U(t) = \frac{2A}{\pi\omega}\left(1 - e^{-\omega^2\alpha^2 t}\right)$$

We have now found the sine transform for the answer $u(x,t)$. The last step is to find the inverse transform of $U(t)$; that is,

$$u(x,t) = \mathcal{F}_s^{-1}[U]$$

STEP 3 To find the solution, we can either evaluate the inverse transform directly from the integral or else resort to the tables. Using the tables, we see that

$$u(x,t) = A\ erfc\ (x/2\alpha\sqrt{t})$$

where $erfc(x)$, $0 < x < \infty$, is called the **complementary-error function** and is given by

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

See Figure 10.5 for its graph.

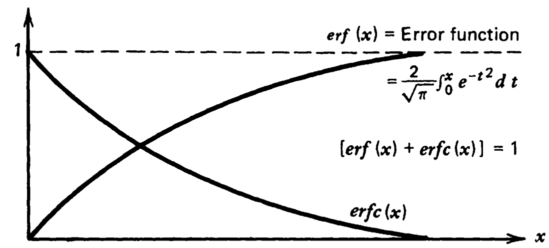


FIGURE 10.5 Graphs of *erf(x)* and *erfc(x)*.

The exact values of these well-known functions can be found in most tables for physics and chemistry. It should be noted that these integrals cannot be integrated by the usual elementary tricks of calculus.

Interpretation of the Solution

The solution

$$u(x,t) = A\ erfc\ [x/2\alpha\sqrt{t}]$$

makes a lot of sense. For different values of time, we have the graph of a complementary-error function with different scale factors on the x-axis. As time increases, the error function gets pulled out; for a graph of the solution at different values of time, see Figure 10.6.

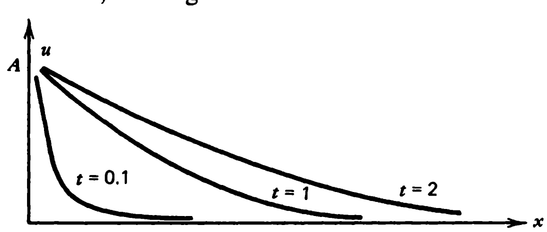


FIGURE 10.6 Solution to semi-infinite rod with fixed temperature A at the end.

PROBLEMS

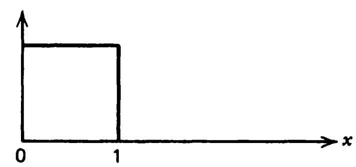
1. Prove the identities (10.1). What assumptions do you need to assume about the function f?
2. Solve the ordinary-differential equation problem (10.2).
3. Solve by means of the sine *or* cosine transform

$$\begin{array}{rll} \text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < \infty \\ \text{BC} & u_x(0, t) = 0 & 0 < t < \infty \\ \text{IC} & u(x,0) = H(1 - x) & 0 \leq x < \infty \end{array}$$

where $H(x)$ is the *Heaviside function*:

$$H(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

In other words, the IC looks like



What does the graph of the solution look like for various values of time?

OTHER READING

1. *Operational Mathematics* by R. V. Churchill. McGraw-Hill, 1958. An excellent text covering many of the integral transforms; good problems and many tables.

2. *Tables of Integral Transform* by A. Erdelyi. McGraw-Hill, 1954. One of the most comprehensive tables of integral transform.

3. *Integral Transforms in Mathematical Physics* by C. J. Tranter. Chapman and Hall (Science Paperbacks), 1971. A small, but concise paperback; easy to read with many examples.

LESSON 11

The Fourier Series and Transform

PURPOSE OF LESSON: To introduce the Fourier series and to show how it can represent certain periodic functions $f(x)$ by sums of sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$$

In the case of nonperiodic functions on $(-\infty, \infty)$, to show also how the Fourier series is replaced by the *Fourier transform* and how a function $f(x)$ can be represented by a continuous resolution of simple functions. This resolution (the Fourier integral) can be written in the complex form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \right] e^{i\xi x}\, d\xi$$

which gives rise to the Fourier and inverse Fourier transforms

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \qquad \text{(Fourier transform)}$$

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x}\, d\xi \qquad \text{(inverse Fourier transform)}$$

The importance of the Fourier series in PDE theory is that periodic functions $f(x)$ defined on $(-\infty, \infty)$ *or* functions defined on *finite intervals* can be represented by infinite series of sines and cosines, and in this way, problems can be resolved into simple ones. For example, the so-called **sawtooth wave**

$$\begin{aligned} &f(x) = x \qquad -L < x < L \\ &f(x + 2L) = f(x) \qquad \text{(periodic condition)} \end{aligned}$$

shown in Figure 11.1

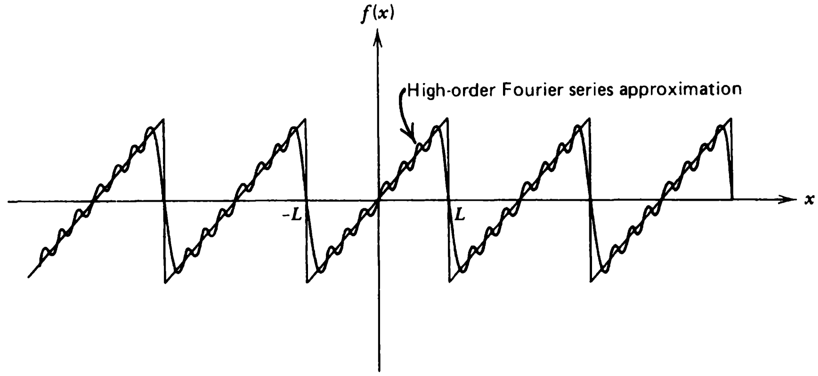


FIGURE 11.1 Sawtooth wave represented by a Fourier series.

can be represented by the Fourier series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]} \tag{11.1}$$

where the **Fourier coefficients** a_n and b_n are given by the **Euler formulas**

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx = 0 \qquad n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx = -(2L/n\pi)(-1)^n \qquad n = 1, 2, \ldots \end{aligned} \tag{11.2}$$

These integrations are routine calculus evaluations. To find Euler's formulas for a_n and b_n, respectively, we multiply each side of equation (11.1) by sin (nx) or cos (nx) and integrate the resulting equation from $-L$ to L. The *orthogonality* of the functions $\{\sin(n\pi x/L)\}$ and $\{\cos(n\pi x/L)\}$ allows us to solve for the coefficients a_n and b_n; see problem 6. For the sawtooth wave, the Fourier representation is given by

$$f(x) = \frac{2L}{\pi}\left[\sin(\pi x/L) - \frac{1}{2}\sin(2\pi x/L) + \frac{1}{3}\sin(3\pi x/L) - \ldots\right] \tag{11.3}$$

where each term (called **harmonic**) has a larger frequency than the previous term, and all frequencies are *multiples* of a fundamental frequency that has the same period as the function $f(x)$

One of the drawbacks of the Fourier series is that in order for a function to have a Fourier series representation, the function must be periodic. Of course, if we want to expand a function (say, $f(x) = x$ for $0 \leq x \leq 1$) defined on a *finite interval*, we could use expansion (11.1). The fact that the Fourier series is periodic *outside* the interval [0,1] doesn't concern us, since we're only interested in the function *inside* the interval. As a matter of fact, we can represent a function inside an interval with many different types of Fourier series by considering different types of extensions outside the interval (some converge faster than others).

The reader shouldn't get the idea that every periodic function can be represented by a Fourier series expansion. What we do know is that if a function $f(x)$ can be represented by a Fourier series (11.1), then the coefficients a_n and b_n are given by the *Euler formulas* (11.2). What's more, even if a function $f(x)$ can be represented by a Fourier series, it isn't always true that the *derivative* $f'(x)$ can be found by differentiating the series term by term. In fact, we can easily see that the derivative of $f(x) = x$ (the sawtooth function) cannot be found by differentiating each term of the Fourier series (11.3). Indeed, the differentiated series will not even converge for any x (the reader can verify this himself or herself).

The *exact conditions* that insure that a function $f(x)$ will have a Fourier series representation and that the representation can be differentiated term by term are found in the recommended reading for this lesson. For our purposes, we are content to know the important result of P. G. Dirichlet's (Deer-ish-lay) theorem, which states

> *Dirichlet's theorem* (sufficient conditions for a function to have a Fourier series representation):
>
> If $f(x)$ is a bounded periodic function that contains a finite number of maximum points, minimum points, and points of discontinuity in each period, then the Fourier series of $f(x)$ converges to $f(x)$ at each point x where $f(x)$ is continuous and to the *average* of the left- and right-hand limits of $f(x)$ at those points where $f(x)$ is discontinuous.

For example, in Figure 11.1, the Fourier series converges to the function $f(x)$ for all except $x = \pm L, \pm 3L, \ldots$ (points of discontinuity), in which case it converges to zero (the average of $+L$ and $-L$).

We are now almost ready to introduce the *Fourier transform*. Before we do, however, it will be useful to introduce the idea of the *frequency spectrum* of a periodic function.

Discrete Frequency Spectrum of a Periodic Function

For periodic functions, we can interpret the Fourier series as the replacement of a periodic function $f(x)$ by a sequence $\{c_n\}$ of numbers

$$c_n = \sqrt{a_n^2 + b_n^2} \qquad n = 0, 1, 2, \ldots .$$

where the numbers c_n can be taken as measuring the contributions of the various frequency components of the function $f(x)$. For example, the sawtooth wave $f(x)$ has the Fourier series representation

$$f(x) = \frac{2L}{\pi}\left[\sin(\pi x/L) - \frac{1}{2}\sin(2\pi x/L) + \frac{1}{3}\sin(3\pi x/\mathrm{L}) - \dots\right]$$

and, hence, the frequency spectrum $\{c_n\}$ is $c_n = 2L/n\pi$ for $n = 1, 2, \dots$ and $c_0 = \left|\frac{a_0}{2}\right| = 0$ (Figure 11.2).

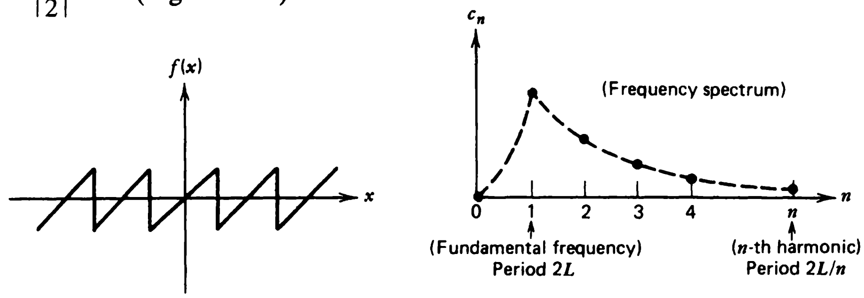


FIGURE 11.2 Discrete frequency spectrum of the sawtooth wave.

The sequence $\{c_n\}$ is somewhat similar to the decomposition of white light into the frequency spectrum of colors obtained with a spectroscope.
We now introduce the Fourier transform.

The Fourier Transform

The major difficulty with Fourier series representation is that nonperiodic functions defined on $(-\infty,\infty)$ cannot be represented. It is possible, however, to find an analogous representation for some of these functions. Without going through the details of the proof, we can show that the Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]$$

changes to the *Fourier integral* representation (continuous frequency resolution)

$$f(x) = \int_0^\infty a(\xi)\cos(\xi x)\,d\xi + \int_0^\infty b(\xi)\sin(\xi x)\,d\xi \tag{11.4}$$

where

$$\begin{aligned} a(\xi) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\xi x)\,dx \\ b(\xi) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\xi x)\,dx \end{aligned} \tag{11.5}$$

for nonperiodic functions defined on $(-\infty,\infty)$. Here, we see that the Fourier integral representation has resolved the function $f(x)$ into *all* frequencies $0 < \xi < \infty$ (and not just multiples of *one basic frequency*, as with periodic functions). As we did in the Fourier series, we define the **frequency spectrum**

$$C(\xi) = \sqrt{a^2(\xi) + b^2(\xi)}$$

which measures the composition of the function $f(x)$ in terms of its frequencies. Some examples of functions $f(x)$ and their frequency spectrums are given in Figure 11.3.

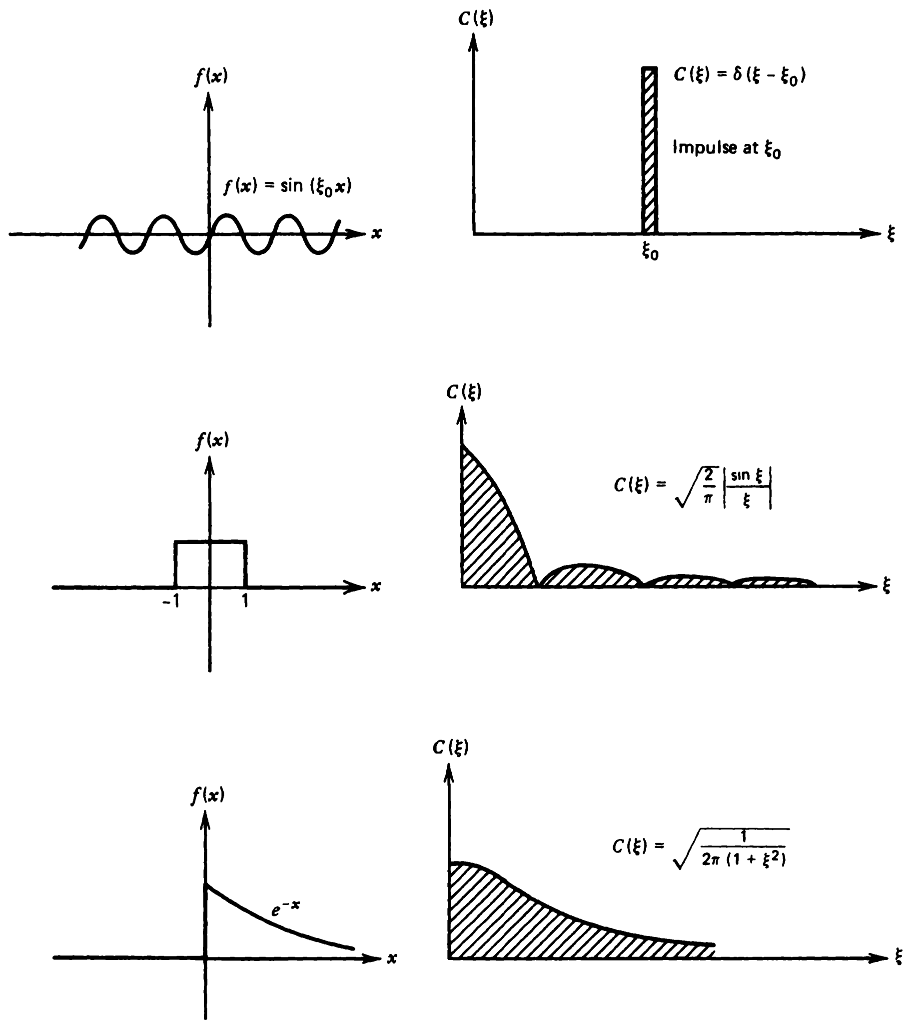


FIGURE 11.3 Frequency spectra for various functions.

Note that functions $f(x)$ that have sharp corners give rise to frequency spectra with large frequencies, since sharp corners require high-frequency components to represent them. On the other hand, the simple periodic function $f(x) = \sin(\xi_0 x)$ obviously has a frequency spectrum that is zero everywhere except at $\xi = \xi_0$.

We are now in a position to define what is generally known as the *exponential Fourier transform* (Equations 11.5 are known as the **Fourier sine and cosine transforms**). By use of Euler's (Oy-ler) equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we can rewrite equation (11.4) after a little effort as

$$(11.6) \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \right] e^{i\xi x}\, d\xi$$

which is known as the **Fourier integral representation**. From this, we can write the two equations

$$(11.7) \qquad \boxed{\begin{aligned} \mathcal{F}[f] &\equiv F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \qquad \text{(Fourier transform)} \\ \mathcal{F}^{-1}[F] &\equiv f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x}\, d\xi \qquad \text{(inverse Fourier transform)} \end{aligned}}$$

which are the Fourier and inverse Fourier transforms. Properties of this transform pair will be discussed in the next lesson along with problems using these transforms.

NOTES

1. The Fourier transform $F(\xi)$ of $f(x)$ can be a *complex function*; for example, the Fourier transform of

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x} & x > 0 \end{cases}$$

   is $F(\xi) = \dfrac{1}{\sqrt{2\pi}} \dfrac{1 - i\xi}{1 + \xi^2}$

2. The absolute value of the Fourier transform $F(\xi)$ is the frequency spectrum of $f(x)$. For example, in note 1, the frequency spectrum of $f(x)$ is

$$|F(\xi)| = \sqrt{\frac{1}{2\pi\,(1 + \xi^2)}}$$

(the reader should be able to find the magnitude of a complex number).

3. Not all functions have Fourier transforms [the integral (11.7) may not exist]; in fact, $f(x) = c$, $\sin x$, e^x, x^2, do *not* have Fourier transforms. Only functions that go to zero sufficiently fast as $|x| \to \infty$ have transforms. In applications, we apply the Fourier transform to temperature functions, wave functions, and other physical phenomena that go to zero as $|x| \to \infty$.

PROBLEMS

1. What is the Fourier series expansion of the square sine wave

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 \leq x < 1 \end{cases}$$

$$f(x + 2) = f(x) \qquad \text{(periodic condition)}$$

Graph the first 2, 3, 4 terms of the series to see how it is converging to $f(x)$. Also graph the frequency spectrum of $f(x)$.

2. Show that if we differentiate the Fourier series expansion (11.3) of the sawtooth wave term by term, we arrive at an infinite series that clearly does not represent the derivative of the sawtooth curve.
3. Graph the frequency spectrum of the following periodic functions:
   (*a*) $f(x) = \sin x$
   (*b*) $f(x) = \sin x + \cos 2x$
   (*c*) $f(x) = \sin x + \cos x + 0.5 \sin 3x$
4. What is the Fourier transform $F(\xi)$ and the frequency spectrum $C(\xi) = |F(\xi)|$ of the function

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

5. Show that the absolute value of the function $F(\xi) = 1/(1 + i\xi)$ is $|F(\xi)| = \sqrt{1/(1 + \xi^2)}$.

HINT First multiply the numerator and denominator by $1 - i\xi$ to get rid of the complex number i in the denominator.

6. Verify the *orthogonality* properties of sines and cosines on the interval $[-L, L]$

$$\int_{-L}^{L} \sin(m\pi x/L)\sin(n\pi x/L)\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \cos(m\pi x/L)\cos(n\pi x/L)\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \sin(m\pi x/L)\cos(n\pi x/L)\,dx = 0 \qquad \text{all } m, n = 1, 2, 3, \ldots$$

OTHER READING

Partial Differential Equations of Mathematical Physics by Tyn Myint-U. Elsevier, 1973. A well-written text with a fairly extensive section on the Fourier series and transform (Chapters 5, 11). Most of the important questions dealing with whether a function actually has a Fourier series or integral representation, whether the representation can be differentiated term by term or under the integral to get the derivative of the function, and so forth, are answered in these chapters.

LESSON 12

The Fourier Transform and Its Application to PDEs

PURPOSE OF LESSON: To illustrate several useful properties of the Fourier transform and to show how these properties can be used to solve PDEs. In particular, it is shown how the Fourier transform changes *differentiation* to *multiplication,* so differential equations change into algebraic equations. Also, the idea of the *infinite convolution* is introduced.

The Fourier transform of the function $f(x)$ for $-\infty < x < \infty$ is given by the formula

$$\boxed{\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx} \tag{12.1}$$

That is, we start with a function $f(x)$ defined on the real x-axis, substitute it into equation (12.1), and arrive at the new function $F(\xi)$ for $-\infty < \xi < \infty$. For example, Table 12.1 lists some common Fourier transforms.

TABLE 12.1 Some Common Fourier Transforms

	Function $f(x)$	Fourier Transform $F(\xi)$
1.	$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ -e^{x} & x < 0 \end{cases}$	$F(\xi) = -i\sqrt{\frac{2}{\pi}}\frac{\xi}{1+\xi^2}$ (complex function)
2.	$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$	$F(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin \xi}{\xi}$ (real function)
3.	$f(x) = e^{-x^2}$	$F(\xi) = \frac{1}{\sqrt{2}} e^{-(\xi/2)^2}$ (real function)

The reader can refer to the tables in the appendix for additional transforms. We can see from the examples that the transformed function $F(\xi)$ may or may not be a complex-valued function of ξ. In the first example, the transformed function $F(\xi)$ contains the complex number i, so we call it a **complex-valued function** of the *real variable* ξ (ξ ranges from $-\infty$ to ∞). In other words, the argument ξ is real, but the value of the function is complex.

The usefulness of the Fourier transform (as with most integral transforms) comes from the fact that it changes the operation of differentiation into multiplication; that is, differential equations are changed into algebraic equations. There are also a host of other properties that make the Fourier transform a useful operational tool; we list a few of the more important ones.

Useful Properties of the Fourier Transform

Property 1 (Fourier Transform Pair)

The Fourier transform of $f(x)$, $-\infty < x < \infty$, produces a new function $F(\xi)$ defined by the formula

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$$

and the *inverse* Fourier transform of $F(\xi)$, $-\infty < \xi < \infty$ will reproduce the original function $f(x)$ according to

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x}\, d\xi$$

For example,

$$e^{-|x|} \xrightarrow{\mathcal{F}} \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2} \xrightarrow{\mathcal{F}^{-1}} e^{-|x|}$$

See Figure 12.1.

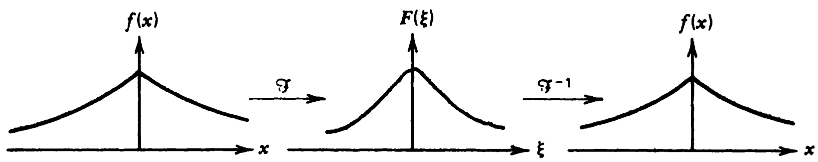


FIGURE 12.1 Graph of a function and its transform.

Property 2 (Linear Transformation)

The Fourier transform is a linear transformation; that is

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$$

This is easy to see. The reader can spend a few minutes to verify this property, which is used over and over again. For example, the Fourier transform of the expression

$$\frac{1}{x^2 + 1} + 3e^{-x^2}$$

would be

$$\mathcal{F}\left[\frac{1}{x^2 + 1}\right] + 3\mathcal{F}[e^{-x^2}]$$

Property 3 (Transformation of Partial Derivatives)

When we discuss how derivatives transform, we must distinguish partial derivatives with respect to various variables. For instance, if the Fourier transform transforms the x-variable (the variable of integration in the transform) and *if* the function being transformed is a partial derivative of a function $u(x,t)$ *with respect to* x, then the **rules of transformation are**

$$\mathcal{F}[u_x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_x(x,t)e^{-i\xi x}\, dx = i\xi\mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx}(x,t)e^{-i\xi x}\, dx = -\xi^2\mathcal{F}[u]$$

On the other hand, if we transform the partial derivative $u_t(x,t)$ (and if the variable of integration in the transform is x), then the transform is given by

$$\mathcal{F}[u_t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t)\, e^{-i\xi x}dx = \frac{\partial}{\partial t}\mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{tt}(x,t)e^{-i\xi x}\, dx = \frac{\partial^2}{\partial t^2}\mathcal{F}[u]$$

Property 4 (Convolution Property)

Every integral transform has what is called a *convolution* property. The general idea is that the transform of a product of two functions $f(x)g(x)$ is *not* the product of the individual transforms; that is,

$$\mathcal{F}[f(x)g(x)] \neq \mathcal{F}[f]\mathcal{F}[g]$$

However, in transform theory there is something called the convolution $f * g$ of two functions that more or less plays the role of the product. What is true about this convolution $f * g$ is that

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g] \tag{12.2}$$

So what is this mysterious convolution $f * g$? It's given by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi)g(\xi)\, d\xi \tag{12.3}$$

and it can be shown without too much trouble that (12.2) holds. We see from the definition of the convolution that given two functions $f(x)$ and $g(x)$, the convolution $(f * g)(x)$ is a new function.

Example of a Convolution of Two Functions

Given the two functions

$$f(x) = x$$
$$g(x) = e^{-x^2}$$

the convolution is given by

$$\begin{aligned}(f * g)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \xi)e^{-\xi^2}\, d\xi \\ &= x/\sqrt{2} \qquad \text{(a new function)}\end{aligned}$$

We have used the formula

$$\int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \sqrt{\pi}$$

to arrive at this value.

The importance of the convolution (12.3) in applications is due to the fact that quite often, the final step in solving a PDE boils down to finding the inverse transform of some expression that we can interpret as the product of two transforms $\mathcal{F}[f]\mathcal{F}[g]$; that is, we must find

$$\mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\} \tag{12.4}$$

By taking the inverse of each side of (12.2), we arrive at the result

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\} \tag{12.5}$$

Hence, to find (12.4), all we have to do is find the inverse transform of *each* factor to get f and g and *then* compute their convolution. We are now in a position to work an important problem in PDE theory.

Solution of an Initial-Value Problem

Consider the heat flow in an *infinite* rod where the initial temperature is $u(x,0) = \phi(x)$. In other words, we look for the solution to the *initial-value problem* (IVP), sometimes called a *Cauchy problem*

$$\text{(12.6)} \qquad \begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} & -\infty < x < \infty \qquad 0 < t < \infty \\ \text{IC} & u(x,0) = \phi(x) & -\infty < x < \infty \end{array}$$

There are three basic steps in solving this problem.

STEP 1 (Transforming the problem)
Since the space variable x ranges from $-\infty$ to ∞, we take the Fourier transform of the PDE and IC with respect to this variable x (the variable of integration in the transform is x). Doing this, we get

$$\mathcal{F}[u_t] = \alpha^2 \mathcal{F}[u_{xx}]$$
$$\mathcal{F}[u(x,0)] = \mathcal{F}[\phi(x)]$$

and using the properties of the Fourier transform, we have

$$\text{(12.7)} \qquad \begin{array}{l} \dfrac{dU(t)}{dt} = -\alpha^2\xi^2 U(t) \\ U(0) = \Phi(\xi) \qquad (\Phi \text{ is the Fourier transform of } \phi) \end{array}$$

where $U(t) = \mathcal{F}[u(x,t)]$. The reader should note here that the function U actually depends on *both* t and the new transformed variable ξ, but, for simplicity, since ξ is a constant insofar as the differential equation (12.7) is concerned, we will drop the notation and just call $U = U(t)$.

STEP 2 (Solving the transformed problem)

Remember the new variable ξ is nothing more than a constant in this differential equation, so the solution to this problem is

$$\text{(12.8)} \qquad U(t) = \Phi(\xi)\, e^{-\alpha^2\xi^2 t}$$

STEP 3 (Finding the inverse transform)

To find the solution $u(x,t)$, we merely compute

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[U(\xi,t)] \\ &= \mathcal{F}^{-1}[\Phi(\xi)e^{-\alpha^2\xi^2 t}] \end{aligned}$$

Here is where the convolution theorem (12.5) comes to the rescue. Using this property, we can write

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[\Phi(\xi)e^{-\alpha^2\xi^2 t}] \\ &= \mathcal{F}^{-1}[\Phi(\xi)] * \mathcal{F}^{-1}[e^{-\alpha^2\xi^2 t}] \\ &= \phi(x) * \left[\frac{1}{\alpha\sqrt{2t}}\, e^{-(x^2/4\alpha^2 t)}\right] \qquad \text{(using tables)} \\ &= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(\xi)\, e^{-(x-\xi)^2/4\alpha^2 t}\, d\xi \end{aligned} \tag{12.9}$$

We're done; equation (12.9) is the solution to our problem.

Before stopping, however, let's analyze this result. Note that the integrand is made up of two terms

1. The initial temperature $\phi(x)$
2. The function $G(x,t) = \dfrac{1}{2\alpha\sqrt{\pi t}}\, e^{-(x-\xi)^2/4\alpha^2 t}$ (which is called **Green's function** or the **impulse-response function)**

It can be shown that this impulse-response function $G(x,t)$ is the *temperature response* to an initial temperature impulse at $x = \xi$. In other words, $G(x,t)$ is the temperature along the rod at time t due to an initial *unit* impulse of heat at $x = \xi$ (Figure 12.2).

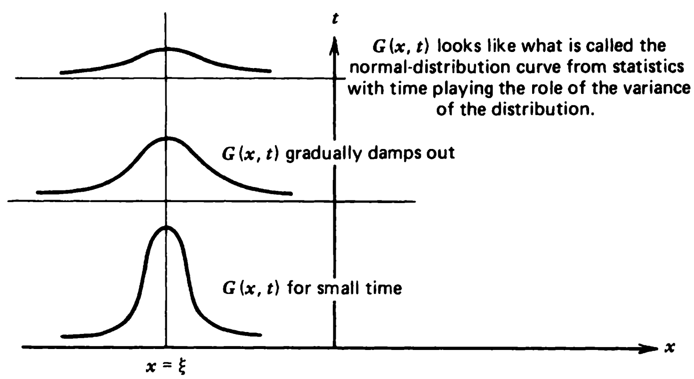


FIGURE 12.2 Impulse response $G(x,t)$ from a temperature impulse at $x = \xi$.

Hence, the interpretation of solution (12.9) is that the initial temperature $u(x,0) = \phi(x)$ is *decomposed* into a continuum of impulses of magnitude $\Phi(\xi)$ (at each point $x = \xi$) and the resulting temperature $\Phi(\xi)G(x,t)$ is found. These resulting temperatures are then added (integrated) to obtain solution (12.9). Later, we will see that this general idea is known as **superposition**.

From a practical point of view, solution (12.9) can often be integrated for some particular initial temperature $\phi(x)$. If this integration cannot be carried out *analytically*, the solution can be found at any point (x,t) by *numerically* integrating the integral.

NOTES

The major drawback of the Fourier transform is that all functions can not be transformed; for example, even simple functions like

$$f(x) = \text{constant}$$
$$f(x) = e^x$$
$$f(x) = \sin x$$

cannot be transformed, since the integral

$$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx$$

does not exist. Only functions that damp to zero sufficiently fast as $|x| \to \infty$ have transforms. Also, the Fourier transform could not be used to transform the time variable in the previous initial value problem, since $0 < t < \infty$.

PROBLEMS

1. Find the Fourier transform of

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x} & 0 < x \end{cases}$$

Check your answer by using the tables in the appendix.

2. Verify that the Fourier and inverse Fourier transforms are linear transforms.
3. Solve the initial-value problem

$$\begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} & -\infty < x < \infty \\ \text{IC} & u(x,0) = e^{-x^2} & -\infty < x < \infty \end{array}$$

by using the Fourier transform.

4. Verify the properties

$$\mathcal{F}[u_x] = i\xi\, \mathcal{F}[u]$$
$$\mathcal{F}[u_{xx}] = -\xi^2\, \mathcal{F}[u]$$

HINT Use integration by parts.

5. Verify that the convolution of two functions f and g can be written as either

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi)g(\xi)\, d\xi$$

or

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)g(x - \xi)\, d\xi$$

OTHER READING

Fourier Series and Orthogonal Functions by H. Davis. Allyn and Bacon, 1963; Dover, 1989. An excellent book gives the reader an intuitive as well as rigorous viewpoint of Fourier series and transforms.

LESSON 13

The Laplace Transform

PURPOSE OF LESSON: To introduce the important transform pair

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \text{(Laplace transform)}$$

$$\mathcal{L}^{-1}[F] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,e^{st}\,ds \qquad \text{(inverse Laplace transform)}$$

and illustrate useful properties. The Laplace transform has an advantage over the Fourier transform because it contains the damping factor e^{-st} that allow us to transform a wider class of functions. Inasmuch as the transform operates on functions defined on $[0,\infty)$, it is mostly applied to the time variable t.

After discussing some useful properties of the Laplace transform, we will solve an important problem in PDE theory.

Of all the integral transforms we will study in this book, the Laplace transform

$$\mathcal{L}[f] = \int_0^\infty f(t)e^{-st}\,dt \tag{13.1}$$

is probably the only one the reader has seen before, since it is a very powerful tool for transforming initial-value problems in ODE into algebraic equations. Not only is the Laplace transform useful in transforming ODEs into algebraic equations, but now we will use the Laplace transform to transform PDEs into ODEs. In particular, we will attempt to apply the Laplace transform to any variable x, y, z, t, . . . that ranges from 0 to ∞ (although it will generally be applied to time). The major difference in applying the Laplace transform to PDEs in contrast to ODEs is that now when the original PDE is transformed, the new resulting equation will be either a new PDE with one less independent variable or else an ODE in one variable. We must then decide how to solve this new problem (maybe by *another* transform, by separation of variables, and so on). Before actually solving a very interesting problem, we enumerate some useful properties of this transform.

Properties of the Laplace Transform

Property 1 (Transform Pair)

The Laplace transform and its inverse are given by

$$(13.2)\qquad \begin{aligned} \mathcal{L}[f] &= F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \text{(Laplace transform)} \\ \mathcal{L}^{-1}[F] &= f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds \qquad \text{(inverse Laplace transform)} \end{aligned}$$

The Laplace transform has one major advantage over the Fourier transform in that the damping factor e^{-st} in the integrand allows us to transform a wider class of functions (the factor $e^{i\xi x}$ in the Fourier transform doesn't do any damping, since its absolute value is one). In fact, the exact conditions that insure that a function $f(t)$ has a Laplace transform are given by the following theorem:

Sufficient Conditions to Insure the Existence of a Laplace Transform

If

1. f is piecewise continuous on the interval $0 \le t \le A$ for any positive A
2. we can find constants M and a such that $|f(t)| \le Me^{at}$ for all values of t greater than some number T

then

$$\text{the Laplace transform } \mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \text{ exists for } s > a$$

We now list a few functions that have Laplace transforms and graph them on the s-axis.

1. $f(t) = 1 \quad 0 < t < \infty$ (pick $M = 1 \quad a = 0$) $\qquad F(s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$ (see Figure 13.1a)

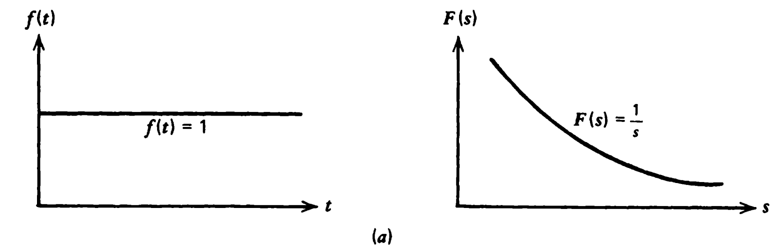
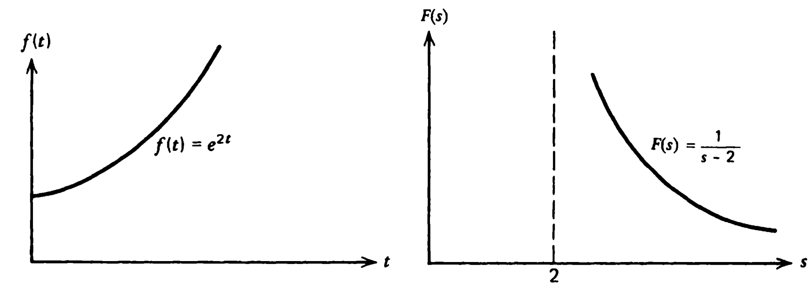


FIGURE 13.1a–13.1c Graphs of a few Laplace transforms.

2. $f(t) = e^{2t} \quad 0 < t < \infty$
(pick $M = 1 \quad a = 2$)

$$F(s) = \frac{1}{s - 2} \qquad s > 2$$

(see Figure 13.1*b*)

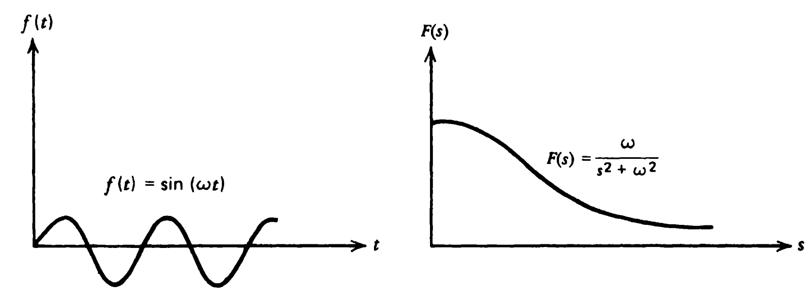


(*b*)

3. $f(t) = \sin(\omega t)$
(pick $M = 1 \quad a = 0$)

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

(see Figure 13.1*c*)



(*c*)

4. $f(t) = e^{t^2}$ (doesn't have a Laplace transform)

In the definition of the Laplace transform, the variable s is taken to be a real variable $0 < s < \infty$. It is possible, however (often desirable), to extend this definition to *complex* values of s and, in fact, to evaluate the *inverse Laplace transform*

$$\mathcal{L}^{-1}[F] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)e^{st}\, ds$$

We must often resort to *contour integration* in the complex plane and the theory of residues. We won't bother ourselves with this topic here but will use the tables in the appendix for finding inverse transforms.

Property 2 (Transforms of Partial Derivatives)

Suppose we have a function $u(x,t)$ of two variables and wish to transform various partial derivatives u_t, u_{tt}, u_x, u_{xx}, ... Since the Laplace transform transforms the *t-variable* (variable of integration), the rules of transformation for partial derivatives are

$$\mathcal{L}[u_t] = \int_0^\infty u_t(x,t)e^{-st}\,dt = sU(x,s) - u(x,0)$$

$$\mathcal{L}(u_{tt}) = \int_0^\infty u_{tt}(x,t)e^{-st}\,dt = s^2U(x,s) - su(x,0) - u_t(x,0)$$

$$\mathcal{L}[u_x] = \int_0^\infty u_x(x,t)e^{-st}\,dt = \frac{\partial U}{\partial x}(x,s)$$

$$\mathcal{L}[u_{xx}] = \int_0^\infty u_{xx}(x,s)e^{-st}\,dt = \frac{\partial^2 U}{\partial x^2}(x,s)$$

where $U(x,s) = \mathcal{L}[u(x,t)]$. The transform rules for u_x and u_{xx} are a result of a basic rule in calculus

$$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b \frac{\partial f}{\partial x}(x,y)\,dy$$

while the rules for u_t and u_{tt} can be proven by using the integration by parts formula.

Property 3 (Convolution Property)

Convolution plays the same role here as it did in the Fourier transform, but now the convolution is defined slightly differently.

Definition of the Finite Convolution

The **finite convolution** of two functions f and g is defined by

$$\begin{aligned}(f * g)(t) &= \int_0^t f(\tau)g(t - \tau)\,d\tau \\ &= \int_0^t f(t - \tau)g(\tau)\,d\tau\end{aligned}$$

(these two integrals are the same). In other words, in the *finite* convolution, we integrate from 0 to t instead of from $-\infty$ to ∞, as we did in the *infinite* convolution. An example of the finite convolution of two functions

$$f(t) = t$$
$$g(t) = t$$

would be

$$(f * g)(t) = \int_0^t \tau(t - \tau)\, d\tau = t^3/6$$

As in the case of the infinite convolution, the important property of this new convolution is that

$$\mathscr{L}[f * g] = \mathscr{L}[f]\, \mathscr{L}[g] \tag{13.4}$$

or the equivalent formula

$$\mathscr{L}^{-1}\{\mathscr{L}[f]\, \mathscr{L}[g]\} = f * g \tag{13.5}$$

This property will allow us to find the inverse Laplace transform of a product of two functions (which we interpret as $\mathscr{L}[f]\mathscr{L}[g]$) by finding the inverses of each factor $\mathscr{L}[f]$ and $\mathscr{L}[g]$ to get f and g and then finding their convolution. For example

$$\mathscr{L}^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2 + 1}\right] = \int_0^t \sin \tau\, d\tau = 1 - \cos t$$

$$F(s) = \frac{1}{s} \xrightarrow{\mathscr{L}^{-1}} f(t) = 1 \qquad G(s) \xrightarrow{\mathscr{L}^{-1}} g(t) = \sin t$$

We are now ready to work an important initial-boundary-value problem.

Heat Conduction in a Semi Infinite Medium

Consider a large (deep) container of liquid that is insulated on the sides. Suppose the liquid has an initial temperature of u_0 and that the temperature of the air above the liquid is zero (some reference temperature). Our goal is to find the temperature of the liquid at various depths of the container at different values of time. To do so, we must solve the problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx} & 0 < x < \infty \quad 0 < t < \infty \\ \text{BC} & u_x(0,t) - u(0,t) = 0 & 0 < t < \infty \\ \text{IC} & u(x,0) = u_0 & 0 < x < \infty \end{array} \tag{13.6}$$

See Figure 13.2.

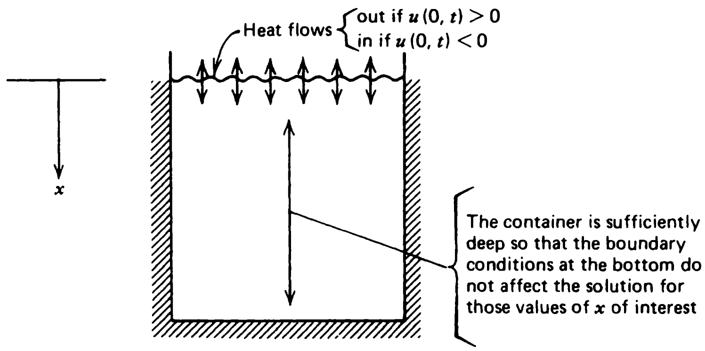


FIGURE 13.2 Diagram illustrating the heat-flow problem.

To solve this problem, we transform the *time variable t* by means of the Laplace transform (conceivably, we could also transform x by means of the Laplace transform, since x also ranges from 0 to ∞). Transforming our problem, we arrive at an ODE in x

$$
(13.7) \qquad \begin{array}{ll} \text{ODE} & sU(x) - u_0 = \dfrac{d^2U}{dx^2}(x) \qquad 0 < x < \infty \\ \text{BC} & \dfrac{dU}{dx}(0) = U(0) \end{array}
$$

(we transform the PDE and the BC—not the IC). This is a second-order ODE with one BC at $x = 0$ [for physical reasons, we *really* have a second, implied BC that says $U(x)$ is bounded]. Note that we have dropped the s-notation in $U(x,s)$ in favor of the simpler notation $U(x)$, since the differential equation in (13.7) depends only on x.

To solve (13.7), we first find the general solution (homogeneous + a particular solution), which is

$$
U(x) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{u_0}{s}
$$

Substituting this expression into the BCs of (13.7) allows us to find the constants c_1 and c_2 (first note that $c_1 = 0$ or else the temperature will go to infinity as x gets large). Finding c_2 from the BC at $x = 0$ gives us the answer for $U(x)$

$$
(13.8) \qquad U(x) = -u_0 \left\{ \frac{e^{-\sqrt{s}\,x}}{s(\sqrt{s} + 1)} \right\} + \frac{u_0}{s}
$$

Now for the last step. To find the temperature $u(x,t)$, we compute

$$
u(x,t) = \mathcal{L}^{-1}[U(x,s)]
$$

[we now put back s in $U(x,s)$]. To find this inverse transform, we must resort to the tables of inverse Laplace transforms in the appendix; they will give us

$$(13.9) \qquad u(x,t) = u_0 - u_0 \left[erfc(x/2\sqrt{t}) - erfc\left(\sqrt{t} + x/2\sqrt{t}\right) e^{(x+t)}\right]$$

where

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\xi^2}\, d\xi$$

is the *complementary-error function* whose graph is given in Figure 13.3.

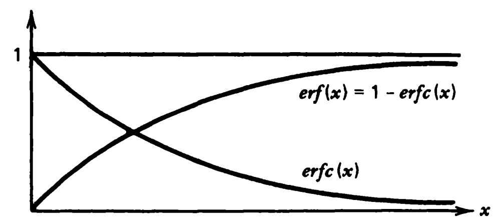


FIGURE 13.3 Graphs of the error (*erf*) and complementary-error (*erfc*) functions.

If we spend a little time analyzing this equation and graphing it by means of a computer with a plotter attachment, we will see that it looks like Figure 13.4.

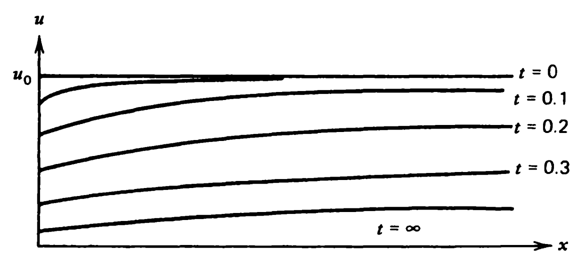


FIGURE 13.4 Temperatures inside the infinite medium for different values of time.

NOTES

1. The Laplace transform can also be applied to problems where the PDE is nonhomogeneous (in separation of variables, the equation had to be homogeneous), but the Laplace transform will generally work only if the equation has constant coefficients (in separation of variables, we could have variable coefficients). The following table lists the types of problems the two methods will handle.

TABLE 13.2 Comparison of Laplace Transform and Separation of Variables

	Method	
	Laplace Transform	Separation of Variables
Nonhomogeneous PDE	yes	no
Nonhomogeneous BC	yes	no
Variable coefficients	no	yes
Nonlinear equations	no	no

2. The Hankel and Mellin transforms are also used to solve IBVPs and BVPs but differ from the Laplace transform in one regard. The Laplace transform converts derivatives to multiplication operations by means of a formula like

$$\mathscr{L}[y'] = s\mathscr{L}[y] - y(0)$$

whereas the Hankel and Mellin transforms convert *differential operators* to multiplication; for example, the Hankel transform

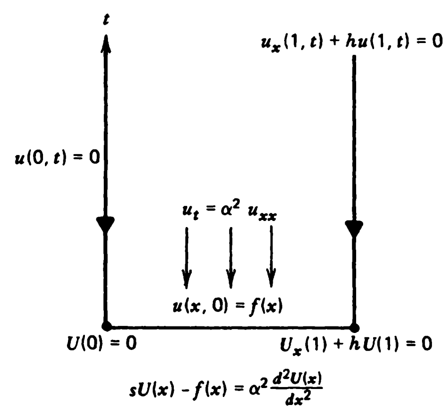
$$H[y] = \int_0^\infty rJ_0(\xi r)y(r)\, dr$$

transforms the differential operator

$$H[y''(r) + \frac{1}{r}y'(r)] = -\xi^2 H[y]$$

In this way, specific differential equations with variable coefficients (Bessel's equation) can be solved.

3. The Laplace transform (which transforms t) can be interpreted as projecting the xt-plane onto the x-axis, and the original BCs, PDE, and IC are transformed into a new differential equation and BCs. See the following diagram.



PROBLEMS

1. Verify the following formula for the transform of the partial derivative u_t

$$\mathcal{L}[u_t(x,t)] = sU(x,s) - u(x,0)$$

2. Solve the following initial-value problem by means of the Laplace transform

$$\begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} & -\infty < x < \infty \quad 0 < t < \infty \\ \text{IC} & u(x,0) = \sin x & -\infty < x < \infty \end{array}$$

3. Solve the problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx} & 0 < x < \infty \quad 0 < t < \infty \\ \text{BC} & u(0,t) = \sin t & 0 < t < \infty \\ \text{IC} & u(x,0) = 0 & 0 \leqslant x < \infty \end{array}$$

by means of the Laplace transform (transform t). What is the physical interpretation of this problem?

4. Solve the boundary-value problem

$$\text{ODE} \qquad \frac{d^2U}{dx^2} - sU = A \qquad 0 < x < 1$$

$$\text{BCs} \qquad \begin{cases} \dfrac{dU}{dx}(0) = 0 \\ U(1) = 0 \end{cases}$$

OTHER READING

1. *A First Course in PDE* by H. Weinberger. Ginn and Co., 1965. This text contains an extensive section on contour integration, which is the tool used for evaluating the inverse Laplace transform.

2. Almost any beginning text in ODE will contain a chapter on the Laplace transform.

LESSON 14

Duhamel's Principle

PURPOSE OF LESSON: To show how the Laplace transform can bring out interesting underlying phenomena concerning solutions of differential equations, in particular, by algebraically manipulating the Laplace transform of the solution of a PDE, we discover an interesting idea known as *Duhamel's principle.* This principle has interpretations in ODE, but we will illustrate how it works in the context of a specific initial-boundary-value problem.

In addition to providing a powerful tool for solving PDEs, the Laplace transform also provides insight into the nature of solutions to physical problems. With the help of the Laplace transform, we illustrate a very important and interesting concept known as *Duhamel's principle* in this lesson. Before getting to this principle, however, let's discuss a problem that occurs frequently in engineering.

Heat Flow within a Rod with Temperature Fixed on the Boundaries

Quite often, it is important to find the temperature inside a medium due to *time-varying boundary conditions*. For example, consider an insulated rod with temperature specified as $f(t)$ on the right end

$$\text{PDE} \quad u_t = u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$(14.1) \qquad \text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = f(t) \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 0 \qquad 0 \leq x \leq 1$$

See Figure 14.1.

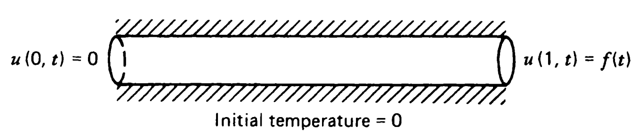


FIGURE 14.1 Time-varying boundary conditions.

We may think that the solution to problem (14.1) can be easily found once we know the solution to the simpler version (constant temperature on the boundaries)

$$
(14.2) \qquad
\begin{array}{lll}
\text{PDE} & w_t = w_{xx} & 0 < x < 1 \quad 0 < t < \infty \\[2ex]
\text{BCs} & \begin{cases} w(0,t) = 0 \\ w(1,t) = 1 \end{cases} & 0 < t < \infty \\[2ex]
\text{IC} & w(x,0) = 0 & 0 \leqslant x \leqslant 1
\end{array}
$$

In fact, if we solve problems (14.1) and (14.2) side by side by the Laplace transform, we will see a striking result (Duhamel's principle) that will give us the solution to (14.1) *in terms of the solution of (14.2).*

So, solving (14.1) and (14.2) at the same time, we have

Easy Problem (14.2)
(constant BC)

$\boxed{\text{Transform problem (14.2) by the Laplace transform}}$

$\downarrow$

$$\frac{d^2W}{dx^2} - sW(x) = 0$$
$$W(0) = 0$$
$$W(1) = 1/s$$

$\boxed{\text{Solve the ODE}}$

$\downarrow$

$$W(x,s) = \frac{1}{s}\left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\right]$$

$\boxed{\text{Find the inverse transform}}$

$\downarrow$

$$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2t}\sin(n\pi x)$$

(solution to the constant BC problem)

Hard Problem (14.1)
(time-varying BC)

$\boxed{\text{Transform problem (14.1) by the Laplace transform}}$

$\downarrow$

$$\frac{d^2U}{dx^2} - sU(x) = 0$$
$$U(0) = 0$$
$$U(1) = F(s)$$

$\boxed{\text{Solve the ODE}}$

$\downarrow$

$$U(x,s) = F(s)\left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\right]$$

$\boxed{\text{Don't invert yet—first multiply and divide by } s}$

$\downarrow$

$$U(x,s) = F(s)\left\{s\left[\frac{\sinh(x\sqrt{s})}{s\sinh(\sqrt{s})}\right]\right\}$$

Easy Problem (14.2) (Cont.) (constant BC)	***Hard Problem*** (14.1) (Cont.) (time-varying BC)
	Using the relationship $$\mathcal{L}[w_t] = sW - w(x,0)$$ we have

$$\downarrow$$

$$U(x,s) = F(s)\,\mathcal{L}[w_t]$$

Hence

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}\{F(s)\mathcal{L}[w_t]\} \\ &= \mathcal{L}^{-1}[F(s)] * [w_t] \\ &= f(t) * w_t(t) \\ &= \int_0^t f(\tau)\, w_\tau(x,t-\tau)\, d\tau \end{aligned}$$

(or by integration by parts)

$$= \int_0^t w(x,t-\tau)\, f'(\tau)\, d\tau + f(0)w(x,t)$$

(solution to the time-varying problem in terms of the solution of the constant BC problem)

In other words, we have found the solution $u(x,t)$ to the *time-varying* problem in terms of the solution to the easy problem (constant BCs); that is,

$$\begin{aligned} u(x,t) &= \int_0^t w_t(x,t-\tau)f(\tau)\, d\tau \\ &= \int_0^t w(x,t-\tau)f'(\tau)\, d\tau + f(0)w(x,t) \end{aligned} \tag{14.3}$$

Equations (14.3) are known as **Duhamel's principle.** We can now take the solution $w(x,t)$ to the constant BC problem

$$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(n\pi)^2 t} \sin(n\pi x)$$

and substitute it into equation (14.3) to obtain the solution to the time-varying problem [we must use the second equation in (14.3), since if we differentiate the infinite series representation for $w(x,t)$ term by term (with respect to t), it results in a *divergent* series].

There is another aspect of Duhamel's formulas that makes them very useful.

The Importance of Duhamel's Principle

In the problem just discussed, we were able to solve the easy problem with constant BCs, so we used Duhamel's formulas (14.3) to obtain the solution to the time-varying BCs. Quite often, however, even the easy problem (constant BCs) cannot be solved analytically. What we can do, however, is observe the solution *experimentally*; in other words, we can rig a device that has constant BCs and experimentally measure the response. We can then use Duhamel's principle to find the solution for *any* time-varying BC. In fact, we have only to observe the response $w(x,t)$ to the constant BC problem *once*. When we have this data, we can then solve the problem with *arbitrary* BC $f(t)$ by substituting into Duhamel's formulas (14.3).

NOTES

There is another interesting version of Duhamel's principle that gives the answer to the problem

$$(14.4)\qquad \begin{array}{lll} \text{PDE} & u_t = u_{xx} \quad 0 < x < 1 & 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = f(t) \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = 0 \quad 0 \le x \le 1 & \end{array}$$

in terms of the solution $w(x,t)$ of the *alternative simple problem*

$$(14.5)\qquad \begin{array}{lll} \text{PDE} & w_t = w_{xx} \quad 0 < x < 1 & 0 < t < \infty \\ \text{BCs} & \begin{cases} w(0,t) = 0 \\ w(1,t) = \delta(t) \end{cases} & \text{(temperature impulse at } t = 0 \\ \text{IC} & w(x,0) = 0 \quad 0 \le x \le 1 & \end{array}$$

Knowing this formula, which is

$$(14.6)\qquad u(x,t) = \int_0^t w(x,t-\tau)f(\tau)\,d\tau$$

allows us to find the temperature response $u(x,t)$ to an arbitrary boundary temperature $f(t)$ once we have carried out an experiment to determine the temperature response $w(x,t)$ from an impulse temperature.

PROBLEMS

1. Prove the Duhamel principle (14.6) by transforming both problems (14.4) and (14.5) and using an argument similar to the one for finding (14.3) in the lesson.
   HINT The Laplace transform of the impulse function $\delta(t)$ is $\mathcal{L}[\delta(t)] = 1$.

2. Show that the partial derivative w_t of

$$w(x,t) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(n\pi)^2 t} \sin (n\pi x)$$

   diverges for all x if we differentiate the series term by term.
3. Suppose we have a metal rod (laterally insulated) and we supply an *initial impulse of heat* at the right-hand side (the left-hand side is fixed at zero). Suppose the initial temperature of the rod is zero (some reference temperature) and the temperature at the midpoint $x = 0.5$ is measured at various values of time, so that we have the following table:

Values of Time	**Midpoint Temperature**
t_1	w_1
$t_2 = 2t_1$	w_2
$t_3 = 3t_1$	w_3
.	.
.	.
.	.
$t_n = nt_1$	w_n

   Using this data, how could we approximate the temperature response at the point u(0.5, t_n) due to the BCs

   (a) $u(1,t) = \sin t$
   (b) $u(1,t) = f(t)$ (arbitrary $f(t)$)
4. Using Duhamel's principle, what is the solution of the IBVP

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = \sin t \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = 0 \qquad 0 \leq x \leq 1$$

OTHER READING

1. *Differential Equations* by C. Wylie. McGraw-Hill, 1979. Duhamel's principle is discussed in conjunction with problems in ODE in Chapter 6 of this text.

2. *Equations of Mathematical Physics* by A. N. Tikhonov and A. A. Samarskii. Macmillan, 1963; Dover, 1990. An excellent source of all kinds of applied problems; the Duhamel principle is discussed on page 261.

LESSON 15

The Convection Term u_x in the Diffusion Problems

PURPOSE OF LESSON: To show how the term u_x in the diffusion equation

$$u_t = \underbrace{Du_{xx}}_{\substack{\text{Diffusion}\\ \text{term}}} - \underbrace{Vu_x}_{\substack{\text{Convection}\\ \text{term}}}$$

represents the phenomenon of convection. Phenomena described by this convection-diffusion equation exhibit both diffusion and convection properties and are common in many situations. How much diffusion and convection takes place depends on the relative size of the two coefficients D and V.

Inasmuch as the convection of a substance represents material moving with the medium, it is possible to pick a moving coordinate system that moves with the medium. In this way, the convection term is eliminated and the equation can be solved in terms of the moving coordinate and then transformed back into the original variable x.

So far, we have been concerned with heat flow (or diffusion of some kind) in a one-dimensional domain. Suppose now we consider the problem of finding the *concentration* of a substance upwards from the surface of the earth where the substance both diffuses through the air and is *carried upward* (convected) by moving currents (moving with velocity V). Clearly, it is possible for the convection of the substance to contribute more of a movement in the substance than the diffusion itself. (It would depend on the relative size of the diffusion coefficient and the velocity of the air.) **Diffusion** is mixing the substance through the air, while **convection** is the movement of the substance *by means* of the air (the movement of the medium); in any case, it is our purpose here to solve the diffusion-convection equation

$$\boxed{u_t = Du_{xx} - Vu_x}$$

and to show how it is derived.

To verify this equation for a concentration $u(x,t)$ of a substance, we use *two basic facts*

1. *Flux due to convection*
   The flux of material (from left to right) across a point due to *convection* is given by $Vu(x,t)$, where V is the velocity of the medium (cm/sec) and $u(x,t)$ is the linear concentration (g/cm) (Figure 15.1).

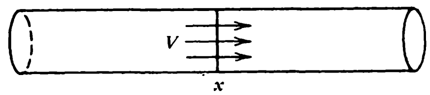


FIGURE 15.1 Amount of material across x (per second) due to convection is $Vu(x,t)$.

2. *Flux due to diffusion*
   The *flux* of material (from left to right) across a point x due to *diffusion* is given by $-Du_x(x,t)$, where D is the diffusion coefficient.

If we substitute these two expressions into the *conservation equation* in Lesson 4, we can prove that the basic PDE is

$$\boxed{u_t = Du_{xx} - Vu_x}$$

To get an idea of what solutions look like or how they behave with the convection term included, let's first work a problem that is pure convection (the diffusion term is zero). A typical problem would be dumping a substance into a clean river (moving with velocity V) and observing the concentration of the substance downstream. For example, if x measures the distance downstream from where the substance is added and if the substance *does not diffuse* with the running water, then the concentration of the substance $u(x,t)$ can be found by solving the following mathematical model:

$$(15.1)\quad \begin{array}{lll} \text{PDE} & u_t = -Vu_x & 0 < x < \infty \qquad 0 < t < \infty \\ \text{BC} & u(0,t) = P & \longleftarrow \text{Constant input of the substance} \\ \text{IC} & u(x,0) = 0 & \longleftarrow \text{Initially a clean river} \end{array}$$

This problem is illustrated in Figure 15.2.

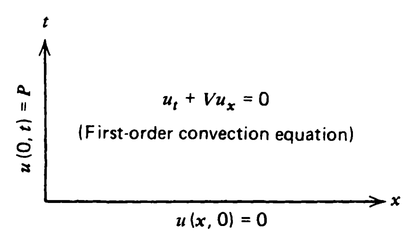


FIGURE 15.2 Pure convection problem.

Before solving this problem, we should think a little about what the solution should be. It's obvious that the pollutant (substance in the river) will initially be zero, but once it is added at a constant rate at $x = 0$, it will move downstream with velocity V. To see this mathematically, let's solve (15.1). Since it is a linear PDE with linear BCs, we should think in terms of separation of variables and integral transforms; however, since the x-variable is unbounded, separation of variables is out. Let's use the Laplace transform on t.

Laplace Transform Solution to the Convection Problem

The convection problem (15.1) can be replaced by

$$\begin{aligned} sU(x) &= -V\frac{dU}{dx} \qquad 0 < x < \infty \\ U(0) &= \frac{P}{s} \end{aligned}$$

by using the Laplace transform

$$U(x) = \int_0^\infty u(x,t)e^{-st}\,dt$$

Solving this very simple initial-value problem, we get

$$U(x) = \frac{P}{s}e^{-(sx/V)}$$

Looking up the inverse transform in the tables, we see

$$u(x,t) = \mathscr{L}^{-1}[U] = PH(t - x/V)$$

where $H(\xi)$ is the Heaviside function (step function)

$$H(\xi) = \begin{cases} 0 & \xi < 0 \\ 1 & \xi \geqslant 0 \end{cases}$$

Hence, the solution of our problem is just

$$u(x,t) = \begin{cases} 0 & t < x/V \\ P & t \geqslant x/V \end{cases}$$

This was pretty simple; certainly, it isn't any more complicated than dumping something on a conveyor belt and watching it move along. It does, however, become more interesting when the solute (pollutant) *diffuses* with the medium.

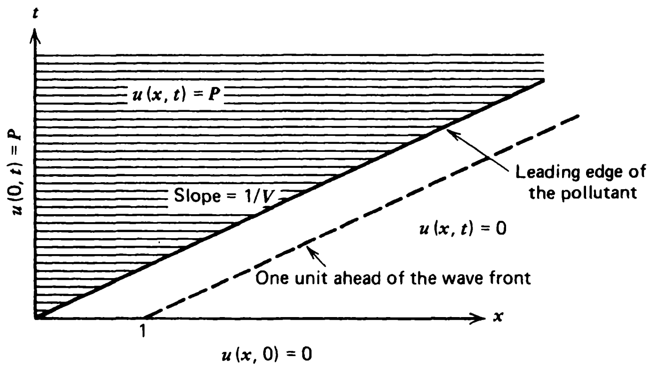


FIGURE 15.3 Pure convection wave.

To see what happens when a moving wave diffuses, we solve the following problem:

$$
\begin{array}{llll}
(15.2) & \text{PDE} & u_t = Du_{xx} - Vu_x & -\infty < x < \infty \\
 & \text{IC} & u(x,0) = 1 - H(x) & -\infty < x < \infty
\end{array}
$$

where, as usual, $H(x)$ is the Heaviside function. The initial concentration is shown in Figure 15.4.

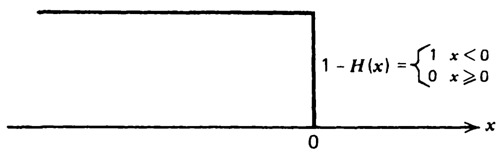


FIGURE 15.4 Initial condition for the diffusion-convection equation.

Note that in the new problem (15.2), we have moved the boundary to $-\infty$ (we now have an *initial-value problem*), so that it doesn't confuse the real issue of measuring the relative effects of D versus V (just a technicality). To solve (15.2), we could use the Laplace transform on t or the Fourier transform on x; however, in this case, there is another alternative that is very interesting. That is, instead of measuring the concentration $u(x,t)$ as a function of x, we introduce a new coordinate ξ, which moves along the x-axis with velocity V. In other words, instead of placing our coordinate system along the bank of the river (so to speak), we now place our coordinate system so that it moves with the *wave front* (of course, now when we have diffusion in addition to convection, we won't have a sharp wave front). Mathematically this says that we change our space coordinate x to $\xi = x - Vt$. It's now clear that

when $\xi = 0$	we are on the wave front
when $\xi = 1$	we are one unit ahead of the front
when $\xi = -1$	we are one unit behind the front

So our goal is to transform the initial-value problem (IVP)

$$\begin{array}{rll} \text{PDE} & u_t = Du_{xx} - Vu_x & -\infty < x < \infty \\ \text{IC} & u(x,0) = 1 - H(x) & -\infty < x < \infty \end{array}$$

into a *new one* in the moving coordinate system, solve it, and then transform back to get the solution in terms of the original coordinates (x,t). To begin, we make what is called a *change of variables* (change of *independent* variables). In place of the old coordinates (x,t), we introduce new ones (ξ,τ)

$$\begin{aligned} \xi &= x - Vt \\ \tau &= t \end{aligned}$$

The reader should note that τ is the same as t, but it is less confusing if we give it a new name. To rewrite the PDE in terms of (ξ,τ), we use the chain rule

$$\begin{aligned} u_t &= u_\xi \xi_t + u_\tau \tau_t = -Vu_\xi + u_\tau \\ u_x &= u_\xi \xi_x = u_\xi \\ u_{xx} &= (u_\xi)_x = u_{\xi\xi}\xi_x = u_{\xi\xi} \end{aligned}$$

Using *functional diagrams*, as in Figure 15.5, makes these chain-rule arguments clearer.

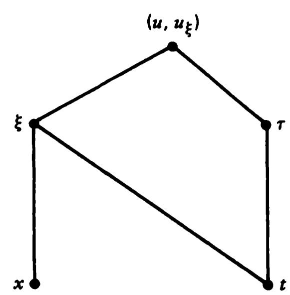


FIGURE 15.5 Diagram illustrating functional dependence of variables.

The diagram in Figure 15.5 is useful for computing the partial derivatives of u, u_ξ with respect to x and t, since it shows exactly how u and u_ξ depend, in general, on ξ and τ and that ξ depends, in turn, on both x and t. The variable τ, on the other hand, depends only on t.

So much for the transformation. We now substitute our computed u_t, u_x, and u_{xx} into the PDE to get

$$-Vu_\xi + u_\tau = Du_{\xi\xi} - Vu_\xi$$

or

$$u_\tau = Du_{\xi\xi}$$

Hence, our new IVP in terms of ξ and τ is

$$\begin{array}{lll} \text{PDE} & u_\tau = Du_{\xi\xi} & -\infty < \xi < \infty \\ \text{IC} & u(\xi,0) = 1 - H(\xi) & -\infty < \xi < \infty \end{array}$$

(Note that $\xi = x$ when $t = 0$, so our ICs are both the same.) This problem has already been solved in Lesson 12 by the Fourier transform and has the solution

$$u(\xi,\tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{\infty} \phi(\beta) e^{-(\xi-\beta)^2/4D\tau}\, d\beta$$

where $\phi(\beta)$ is the initial condition. Hence, in our case, we have

$$u(\xi,\tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{0} e^{-(\xi-\beta)^2/4D\tau}\, d\beta$$

By letting

$$\bar{\beta} = \frac{\xi-\beta}{2\sqrt{D\tau}} \qquad d\bar{\beta} = \frac{-1}{2\sqrt{D\tau}}\, d\beta$$

we get the interesting result

$$(15.3) \quad u(\xi,\tau) = \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{\xi}{2\sqrt{D\tau}}}^{\infty} e^{-\bar{\beta}^2}\, d\bar{\beta}\right]$$

$$= \begin{cases} \dfrac{1}{2}\left[1 + erf\left(\dfrac{-\xi}{2\sqrt{D\tau}}\right)\right] & \xi < 0 \\ \dfrac{1}{2}\, erfc\left(\dfrac{\xi}{2\sqrt{D\tau}}\right) & \xi \geq 0 \end{cases}$$

The graph of this function is plotted for various values of t in Figure 15.6.

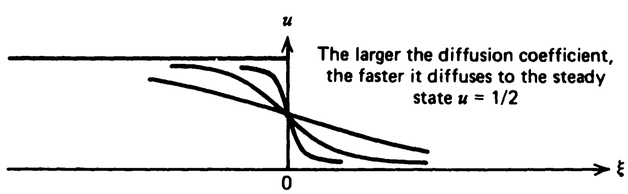


FIGURE 15.6 Simple diffusion from high to low concentrations.

Finally, to get the solution of our problem in terms of the coordinates x and t, we substitute

$$\xi = x - Vt$$
$$\tau = t$$

into equation (15.3) to get

$$(15.4) \qquad u(x,t) = \begin{cases} \dfrac{1}{2}\left[1 + erf\left(\dfrac{Vt - x}{2\sqrt{D\tau}}\right)\right] & Vt > x \\ erfc\left(\dfrac{x - Vt}{2\sqrt{D\tau}}\right) & Vt \leq x \end{cases}$$

This is the solution of our diffusion-convection problem (15.2), and it is really very easy to interpret; it's just a moving version of Figure 15.6. In other words, depending on the relative size of D (diffusion coefficient) and V (velocity of the stream), the solution moves to the right with velocity V while, at the same time, the leading edge is diffusing at a rate defined by D (Figure 15.7 shows the break up of the leading edge).

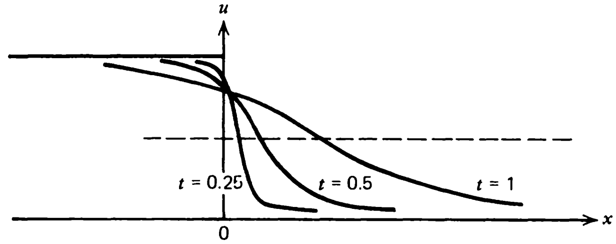


FIGURE 15.7 Diffusion-convection solution moving and diffusing at the same time.

NOTES

Changing coordinates is a very important technique in PDEs. By looking at physical systems with different coordinates, the equations are sometimes simplified.

PROBLEMS

1. Solve the initial-value problem

$$u_t = u_{xx} - 2u_x \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$u(x,0) = \sin x \qquad -\infty < x < \infty$$

2. Solve the following diffusion-convection problem by making a transformation as shown in Lesson 8:

$$u_t = u_{xx} - 2u_x \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$u(x,0) = e^x \sin x \qquad -\infty < x < \infty$$

3. What is the solution of the following *convection* problem:

$$\text{PDE} \quad u_t = -2u_x \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$\text{IC} \quad u(x,0) = e^{-x^2}$$

Check your answer.

4. Solve

$$u_t = Du_{xx} - Vu_x \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$u(x,0) = e^{-x^2} \qquad -\infty < x < \infty$$

Does the solution check? What does the solution look like for various values of time?

HINT Note that our transformation to moving coordinates allows us to essentially neglect the term Vu_x in the PDE. After solving the *new* problem,

$$\text{PDE} \quad u_\tau = Du_{\xi\xi} \qquad -\infty < \xi < \infty \qquad 0 < \tau < \infty$$
$$\text{IC} \quad u(\xi,0) = e^{-\xi^2} \qquad -\infty < \xi < \infty$$

we merely set $\xi = x - Vt$ and $\tau = t$. In this particular problem, it is possible to *evaluate* the integral

$$u(\xi,\tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{\infty} e^{-\beta^2} e^{-(\xi-\beta)^2/4D\tau} \, d\beta$$

This is the Fourier transform solution from Lesson 12. It may be more convenient for the reader to rewrite this integrand and then look it up in a table of integrals.

Proof. 1. Choose $w \in \mathcal{A}$. Then (46) implies

$$0 = \int_U (-\Delta u - f)(u - w)\, dx.$$

An integration by parts yields

$$0 = \int_U Du \cdot D(u - w) - f(u - w)\, dx,$$

and there is no boundary term since $u - w = g - g = 0$ on ∂U. Hence

$$\begin{aligned}\int_U |Du|^2 - uf\, dx &= \int_U Du \cdot Dw - wf\, dx \\ &\leq \int_U \frac{1}{2}|Du|^2\, dx + \int_U \frac{1}{2}|Dw|^2 - wf\, dx,\end{aligned}$$

where we employed the estimates

$$|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2,$$

following from the Cauchy–Schwarz and Cauchy inequalities (§B.2). Rearranging, we conclude

$$I[u] \leq I[w] \quad (w \in \mathcal{A}). \tag{48}$$

Since $u \in \mathcal{A}$, (47) follows from (48).

2. Now, conversely, suppose (47) holds. Fix any $v \in C_c^\infty(U)$ and write

$$i(\tau) := I[u + \tau v] \quad (\tau \in \mathbb{R}).$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\cdot)$ has a minimum at zero, and thus

$$i'(0) = 0 \quad \left(' = \frac{d}{d\tau}\right),$$

provided this derivative exists. But

$$\begin{aligned} i(\tau) &= \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v)f\, dx \\ &= \int_U \frac{1}{2}|Du|^2 + \tau Du \cdot Dv + \frac{\tau^2}{2}|Dv|^2 - (u + \tau v)f\, dx.\end{aligned}$$

Consequently

$$0 = i'(0) = \int_U Du \cdot Dv - vf\, dx = \int_U (-\Delta u - f)v\, dx.$$

This identity is valid for each function $v \in C_c^\infty(U)$ and so $-\Delta u = f$ in U. □

Dirichlet's principle is an instance of the *calculus of variations* applied to Laplace's equation. See Chapter 8 for more.

2.3. HEAT EQUATION

Next we study the *heat equation*

$$u_t - \Delta u = 0 \tag{1}$$

and the *nonhomogeneous heat equation*

$$u_t - \Delta u = f, \tag{2}$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0,\infty) \to \mathbb{R}$, $u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, \dots, x_n)$: $\Delta u = \Delta_x u = \sum_{i=1}^n u_{x_i x_i}$. In (2) the function $f : U \times [0,\infty) \to \mathbb{R}$ is given.

A guiding principle is that any assertion about harmonic functions yields an analogous (but more complicated) statement about solutions of the heat equation. Accordingly our development will largely parallel the corresponding theory for Laplace's equation.

Physical interpretation. The heat equation, also known as the *diffusion equation*, describes in typical applications the evolution in time of the density u of some quantity such as heat, chemical concentration, etc. If $V \subset U$ is any smooth subregion, the rate of change of the total quantity within V equals the negative of the net flux through ∂V:

$$\frac{d}{dt}\int_V u\,dx = -\int_{\partial V} \mathbf{F}\cdot\boldsymbol{\nu}\,dS,$$

$\mathbf{F}$ being the flux density. Thus

$$u_t = -\operatorname{div}\mathbf{F}, \tag{3}$$

as V was arbitrary. In many situations $\mathbf{F}$ is proportional to the gradient of u, but points in the opposite direction (since the flow is from regions of higher to lower concentration):

$$\mathbf{F} = -aDu \quad (a > 0).$$

Substituting into (3), we obtain the PDE

$$u_t = a\operatorname{div}(Du) = a\Delta u,$$

which for $a = 1$ is the heat equation.

The heat equation appears as well in the study of Brownian motion.

□

2.3.1. Fundamental solution.

a. Derivation of the fundamental solution.

As noted in §2.2.1 an important first step in studying any PDE is often to come up with some specific solutions.

We observe that the heat equation involves one derivative with respect to the time variable t, but two derivatives with respect to the space variables x_i $(i = 1, \dots, n)$. Consequently we see that if u solves (1), then so does $u(\lambda x, \lambda^2 t)$ for $\lambda \in \mathbb{R}$. This scaling indicates the ratio $\frac{r^2}{t}$ $(r = |x|)$ is important for the heat equation and suggests that we search for a solution of (1) having the form $u(x,t) = v(\frac{r^2}{t}) = v(\frac{|x|^2}{t})$ $(t > 0,\ x \in \mathbb{R}^n)$, for some function v as yet undetermined.

Although this approach eventually leads to what we want (see Problem 11), it is quicker to seek a solution u having the special structure

$$u(x,t) = \frac{1}{t^\alpha} v\Big(\frac{x}{t^\beta}\Big) \quad (x \in \mathbb{R}^n,\ t > 0), \tag{4}$$

where the constants α, β and the function $v : \mathbb{R}^n \to \mathbb{R}$ must be found. We come to (4) if we look for a solution u of the heat equation invariant under the *dilation scaling*

$$u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t).$$

That is, we ask

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

for all $\lambda > 0$, $x \in \mathbb{R}^n$, $t > 0$. Setting $\lambda = t^{-1}$, we derive (4) for $v(y) := u(y, 1)$.

Let us insert (4) into (1), and thereafter compute

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \tag{5}$$

for $y := t^{-\beta} x$. In order to transform (5) into an expression involving the variable y alone, we take $\beta = \frac{1}{2}$. Then the terms with t are identical, and so (5) reduces to

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0. \tag{6}$$

We simplify further by guessing v to be radial; that is, $v(y) = w(|y|)$ for some $w : \mathbb{R} \to \mathbb{R}$. Thereupon (6) becomes

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0,$$

for $r = |y|$, $' = \frac{d}{dr}$. Now if we set $\alpha = \frac{n}{2}$, this simplifies to read

$$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0.$$

Thus

$$r^{n-1}w' + \frac{1}{2}r^n w = a$$

for some constant a. Assuming $\lim_{r\to\infty} w,\ w' = 0$, we conclude $a = 0$; whence

$$w' = -\frac{1}{2}rw.$$

But then for some constant b

$$w = be^{-\frac{r^2}{4}}. \tag{7}$$

Combining (4), (7) and our choices for α, β, we conclude that $\frac{b}{t^{n/2}}e^{-\frac{|x|^2}{4t}}$ solves the heat equation (1).

This computation motivates the following

DEFINITION. *The function*

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n,\ t > 0) \\ 0 & (x \in \mathbb{R}^n,\ t < 0) \end{cases}$$

is called the fundamental solution of the heat equation.

Notice that Φ is singular at the point $(0,0)$. We will sometimes write $\Phi(x,t) = \Phi(|x|,t)$ to emphasize that the fundamental solution is radial in the variable x. The choice of the normalizing constant $(4\pi)^{-n/2}$ is dictated by the following

LEMMA (Integral of fundamental solution). *For each time $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1.$$

Proof. We calculate

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)\, dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz \\ &= \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-z_i^2}\, dz_i = 1. \end{aligned}$$

□

A different derivation of the fundamental solution of the heat equation appears in §4.3.2.

b. Initial-value problem.

We now employ Φ to fashion a solution to the *initial-value* (or *Cauchy*) *problem*

$$
\text{(8)} \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\; u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}
$$

Let us note the function $(x,t) \mapsto \Phi(x,t)$ solves the heat equation away from the singularity at $(0,0)$, and thus so does $(x,t) \mapsto \Phi(x-y,t)$ for each fixed $y \in \mathbb{R}^n$. Consequently the convolution

$$
\text{(9)} \qquad \begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n,\ t>0) \end{aligned}
$$

should also be a solution.

THEOREM 1 (Solution of initial-value problem). *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define u by (9). Then*

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t(x,t) - \Delta u(x,t) = 0 \quad (x \in \mathbb{R}^n,\, t > 0)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0)$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. 1. Since the function $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$ is infinitely differentiable, with uniformly bounded derivatives of all orders, on $\mathbb{R}^n \times [\delta,\infty)$ for each $\delta > 0$, we see that $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$. Furthermore

$$
\text{(10)} \qquad \begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x-y,t)] g(y)\, dy \\ &= 0 \quad (x \in \mathbb{R}^n,\, t>0), \end{aligned}
$$

since Φ itself solves the heat equation.

2. Fix $x^0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ such that

$$
\text{(11)} \qquad |g(y) - g(x^0)| < \varepsilon \quad \text{if } |y - x^0| < \delta,\ y \in \mathbb{R}^n.
$$

Then if $|x - x^0| < \frac{\delta}{2}$, we have, according to the lemma,

$$\begin{aligned}
|u(x,t) - g(x^0)| &= \left|\int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x^0)]\,dy\right| \\
&\le \int_{B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\,dy \\
&\quad + \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\,dy \\
&=: I + J.
\end{aligned}$$

Now

$$I \le \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t)\,dy = \varepsilon,$$

owing to (11) and the lemma. Furthermore, if $|x - x^0| \le \frac{\delta}{2}$ and $|y - x^0| \ge \delta$, then

$$|y - x^0| \le |y - x| + \frac{\delta}{2} \le |y - x| + \frac{1}{2}|y - x^0|.$$

Thus $|y - x| \ge \frac{1}{2}|y - x^0|$. Consequently

$$\begin{aligned}
J &\le 2\|g\|_{L^\infty} \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)\,dy \\
&\le \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0,\delta)} e^{-\frac{|x-y|^2}{4t}}\,dy \\
&\le \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0,\delta)} e^{-\frac{|y-x^0|^2}{16t}}\,dy \\
&= \frac{C}{t^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{16t}} r^{n-1}\,dr \to 0 \qquad \text{as } t \to 0^+.
\end{aligned}$$

Hence if $|x - x^0| < \frac{\delta}{2}$ and $t > 0$ is small enough, $|u(x,t) - g(x^0)| < 2\varepsilon$. $\square$

Remarks. (i) In view of Theorem 1 we sometimes write

$$\begin{cases} \Phi_t - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi = \delta_0 & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

δ_0 denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0.

(ii) Notice that if g is bounded, continuous, $g \ge 0$, $g \not\equiv 0$, then

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy$$

is in fact positive for *all* points $x \in \mathbb{R}^n$ and times $t > 0$. We interpret this observation by saying the heat equation forces *infinite propagation speed*

for disturbances. If the initial temperature is nonnegative and is positive somewhere, the temperature at any later time (no matter how small) is everywhere positive. (We will learn in §2.4.3 that the wave equation in contrast supports finite propagation speed for disturbances.) □

c. Nonhomogeneous problem.

Now let us turn our attention to the *nonhomogeneous* initial-value problem

$$\text{(12)} \qquad \begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\; u = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

How can we produce a formula for the solution? If we recall the motivation leading up to (9), we should note further that the mapping $(x,t) \mapsto \Phi(x-y,t-s)$ is a solution of the heat equation (for given $y \in \mathbb{R}^n$, $0 < s < t$). Now for fixed s, the function

$$u = u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy$$

solves

$$\text{(12}_s\text{)} \qquad \begin{cases} u_t(\cdot;s) - \Delta u(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ \qquad\qquad\; u(\cdot;s) = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{t=s\}, \end{cases}$$

which is just an initial-value problem of the form (8), with the starting time $t=0$ replaced by $t=s$, and g replaced by $f(\cdot,s)$. Thus $u(\cdot;s)$ is certainly not a solution of (12).

However *Duhamel's principle** asserts that we can build a solution of (12) out of the solutions of (12_s), by integrating with respect to s. The idea is to consider

$$u(x,t) = \int_0^t u(x,t;s)\,ds \quad (x \in \mathbb{R}^n,\ t \geq 0).$$

Rewriting, we have

$$\text{(13)} \qquad \begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dyds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dyds, \end{aligned}$$

for $x \in \mathbb{R}^n,\ t > 0$.

To confirm that formula (13) works, let us for simplicity assume $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and f has compact support.

*Duhamel's principle has wide applicability to linear ODE and PDE, and does not depend on the specific structure of the heat equation. It yields, for example, the solution of the nonhomogeneous transport equation, obtained by different means in §2.1.2. We will invoke Duhamel's principle for the wave equation in §2.4.2.

THEOREM 2 (Solution of nonhomogeneous problem). *Define u by* (13). *Then*

(i) $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t(x,t) - \Delta u(x,t) = f(x,t) \quad (x \in \mathbb{R}^n,\, t > 0)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = 0$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. 1. Since Φ has a singularity at $(0,0)$, we cannot directly justify differentiating under the integral sign. We instead proceed somewhat as in the proof of Theorem 1 in §2.2.1.

First we change variables, to write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\, dyds.$$

As $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ has compact support and $\Phi = \Phi(y,s)$ is smooth near $s = t > 0$, we compute

$$\begin{aligned} u_t(x,t) = &\int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\, dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \end{aligned}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial^2}{\partial x_i \partial x_j} f(x-y,t-s)\, dyds \quad (i,j = 1,\dots,n).$$

Thus $u_t, D_x^2 u$, and likewise $u, D_x u$, belong to $C(\mathbb{R}^n \times (0,\infty))$.

2. We then calculate

(14)

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)[(\frac{\partial}{\partial t} - \Delta_x) f(x-y,t-s)]\, dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]\, dyds \\ &\quad + \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]\, dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy. \\ &=: I_\varepsilon + J_\varepsilon + K. \end{aligned}$$

Now

$$|J_\varepsilon| \leq (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\,dyds \leq \varepsilon C, \tag{15}$$

by the lemma. Integrating by parts, we also find

$$\begin{aligned} I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Big[\Big(\frac{\partial}{\partial s} - \Delta_y\Big)\Phi(y,s)\Big] f(x-y,t-s)\,dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy \\ &\quad - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy \\ &= \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy - K, \end{aligned} \tag{16}$$

since Φ solves the heat equation. Combining (14)–(16), we ascertain

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy \\ &= f(x,t) \quad (x \in \mathbb{R}^n,\ t > 0), \end{aligned}$$

the limit as $\varepsilon \to 0$ being computed as in the proof of Theorem 1. Finally note $\|u(\cdot,t)\|_{L^\infty} \leq t\|f\|_{L^\infty} \to 0$. □

Remark. We can of course combine Theorems 1 and 2 to discover that

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dyds \tag{17}$$

is, under the hypotheses on g and f as above, a solution of

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{18}$$

□

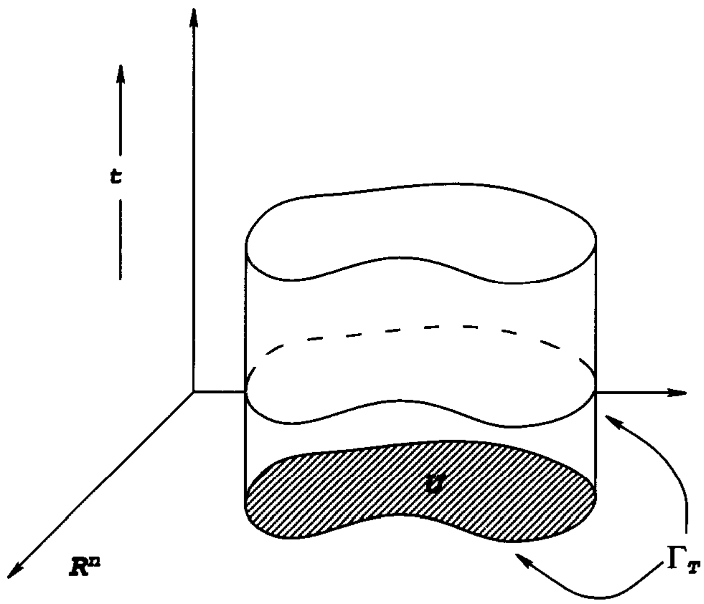
2.3.2. Mean-value formula.

First we recall some useful notation from §A.2. Assume $U \subset \mathbb{R}^n$ is open and bounded, and fix a time $T > 0$.

DEFINITIONS.

(i) *We define the* parabolic cylinder

$$U_T := U \times (0,T].$$



The region $\mathbf{U}_T$

(ii) *The* parabolic boundary *of U_T is*

$$\Gamma_T := \bar{U}_T - U_T.$$

We interpret U_T as being the *parabolic interior* of $\bar{U} \times [0, T]$: note carefully that U_T includes the top $U \times \{t = T\}$. The parabolic boundary Γ_T comprises the bottom and vertical sides of $U \times [0, T]$, but not the top.

We want next to derive a kind of analogue to the mean-value property for harmonic functions, as discussed in §2.2.2. There is no such simple formula. However let us observe that for fixed x the spheres $\partial B(x, r)$ are level sets of the fundamental solution $\Phi(x-y)$ for Laplace's equation. This suggests that perhaps for fixed (x, t) the level sets of fundamental solution $\Phi(x - y, t - s)$ for the heat equation may be relevant.

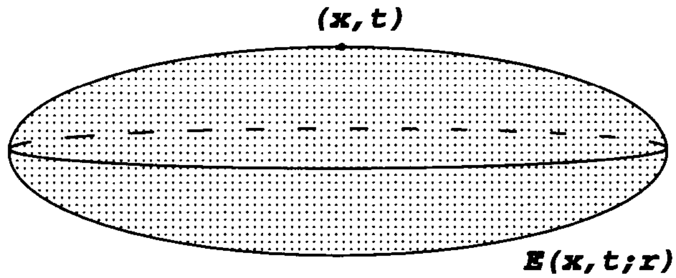
DEFINITION. *For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define*

$$E(x,t;r) := \left\{ (y,s) \in \mathbb{R}^{n+1} \mid s \leq t, \ \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}.$$

This is a region in space-time, the boundary of which is a level set of $\Phi(x-y, t-s)$. Note that the point (x, t) is at the center of the top. $E(x, t; r)$ is sometimes called a "heat ball".

THEOREM 3 (A mean-value property for the heat equation). *Let $u \in C_1^2(U_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dyds \tag{19}$$



A "heat ball"

for each $E(x,t;r) \subset U_T$.

Formula (19) is a sort of analogue for the heat equation of the mean-value formulas for Laplace's equation. Observe that the right hand side involves only $u(y,s)$ for times $s \le t$. This is reasonable, as the value $u(x,t)$ should not depend upon future times.

Proof. We may as well assume upon translating the space and time coordinates that $x = 0$ and $t = 0$. Write $E(r) = E(0,0;r)$ and set

$$
\begin{aligned}
\phi(r) &:= \frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dyds \\
&= \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dyds.
\end{aligned} \tag{20}
$$

We compute

$$
\begin{aligned}
\phi'(r) &= \iint_{E(1)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2r u_s \frac{|y|^2}{s}\, dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s}\, dyds \\
&=: A + B.
\end{aligned}
$$

Also, let us introduce the useful function

$$
\psi := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n \log r, \tag{21}
$$

and observe $\psi = 0$ on $\partial E(r)$, since $\Phi(y,-s) = r^{-n}$ on $\partial E(r)$. We utilize (21) to write

$$
\begin{aligned}
B &= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i}\, dyds \\
&= -\frac{1}{r^{n+1}} \iint_{E(r)} 4n u_s \psi + 4 \sum_{i=1}^{n} u_{s y_i} y_i \psi\, dyds;
\end{aligned}
$$

there is no boundary term since $\psi = 0$ on $\partial E(r)$. Integrating by parts with respect to s, we discover

$$\begin{aligned}
B &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i \psi_s \, dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2}\right) dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i \, dyds - A.
\end{aligned}$$

Consequently, since u solves the heat equation,

$$\begin{aligned}
\phi'(r) &= A + B \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} -4n\Delta u\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i \, dyds \\
&= \sum_{i=1}^{n} \frac{1}{r^{n+1}} \iint_{E(r)} 4nu_{y_i}\psi_{y_i} - \frac{2n}{s} u_{y_i} y_i \, dyds \\
&= 0, \quad \text{according to (21).}
\end{aligned}$$

Thus ϕ is constant, and therefore

$$\phi(r) = \lim_{t\to 0} \phi(t) = u(0,0) \left(\lim_{t\to 0} \frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds\right) = 4u(0,0),$$

as

$$\frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds = \iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4.$$

We omit the details of this last computation. □

2.3.3. Properties of solutions.

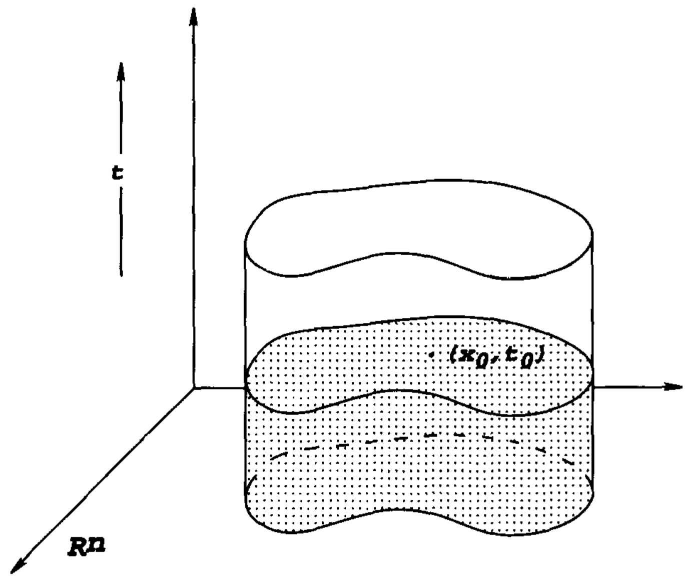
a. Strong maximum principle, uniqueness.

First we employ the mean-value property to give a quick proof of the strong maximum principle.

THEOREM 4 (Strong maximum principle for the heat equation). *Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation in U_T.*

(i) *Then*

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$



Strong maximum principle for the heat equation

(ii) *Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that*

$$u(x_0, t_0) = \max_{\bar{U}_T} u,$$

then

$$u \text{ is constant in } \bar{U}_{t_0}.$$

Assertion (i) is the *maximum principle* for the heat equation and (ii) is the *strong maximum principle.* Similar assertions are valid with "min" replacing "max".

Remark. *So if u attains its maximum (or minimum) at an interior point, then u is constant at all earlier times.* This accords with our strong intuitive interpretation of the variable t as denoting time: the solution will be constant on the time interval $[0, t_0]$ provided the initial and boundary conditions are constant. However, the solution may change at times $t > t_0$, provided the boundary conditions alter after t_0. The solution will however not respond to changes in boundary conditions until these changes happen.

Take note that whereas all this is obvious on intuitive, physical grounds, such insights do not constitute a proof. The task is to *deduce* such behavior from the PDE. □

Proof. 1. Suppose there exists a point $(x_0, t_0) \in U_T$ with $u(x_0, t_0) = M := \max_{\bar{U}_T} u$. Then for all sufficiently small $r > 0$, $E(x_0, t_0; r) \subset U_T$; and we

employ the mean-value property to deduce

$$M = u(x_0, t_0) = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dyds \leq M,$$

since

$$1 = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dyds.$$

Equality holds only if u is identically equal to M within $E(x_0, t_0; r)$. Consequently

$$u(y,s) = M \qquad \text{for all } (y,s) \in E(x_0, t_0; r).$$

Draw any line segment L in U_T connecting (x_0, t_0) with some other point $(y_0, s_0) \in U_T$, with $s_0 < t_0$. Consider

$$r_0 := \min\{s \geq s_0 \mid u(x,t) = M \text{ for all points } (x,t) \in L,\ s \leq t \leq t_0\}.$$

Since u is continuous, the minimum is attained. Assume $r_0 > s_0$. Then $u(z_0, r_0) = M$ for some point (z_0, r_0) on $L \cap U_T$ and so $u \equiv M$ on $E(z_0, r_0; r)$ for all sufficiently small $r > 0$. Since $E(z_0, r_0; r)$ contains $L \cap \{r_0 - \sigma \leq t \leq r_0\}$ for some small $\sigma > 0$, we have a contradiction. Thus $r_0 = s_0$, and hence $u \equiv M$ on L.

2. Now fix any point $x \in U$ and any time $0 \leq t < t_0$. There exist points $\{x_0, x_1, \ldots, x_m = x\}$ such that the line segments in $\mathbb{R}^n$ connecting x_{i-1} to x_i lie in U for $i = 1, \ldots, m$. (This follows since the set of points in U which can be so connected to x_0 by a polygonal path is nonempty, open and relatively closed in U.) Select times $t_0 > t_1 > \cdots > t_m = t$. Then the line segments in $\mathbb{R}^{n+1}$ connecting (x_{i-1}, t_{i-1}) to (x_i, t_i) $(i = 1, \ldots, m)$ lie in U_T. According to Step 1, $u \equiv M$ on each such segment and so $u(x,t) = M$. □

Remark. The strong maximum principle implies that if U is connected and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

where $g \geq 0$, then u is positive *everywhere* within U_T if g is positive *somewhere* on U. This is another illustration of infinite propagation speed for disturbances. □

An important application of the maximum principle is the following uniqueness assertion.

THEOREM 5 (Uniqueness on bounded domains). *Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of the initial/boundary-value problem*

$$
(22) \qquad \begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T. \end{cases}
$$

Proof. If u and $\tilde{u}$ are two solutions of (22), apply Theorem 4 to $w := \pm(u - \tilde{u})$. □

We next extend our uniqueness assertion to the *Cauchy problem*, that is, the initial value problem for $U = \mathbb{R}^n$. As we are no longer on a bounded region, we must introduce some control on the behavior of solutions for large $|x|$.

THEOREM 6 (Maximum principle for the Cauchy problem). *Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves*

$$
(23) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}
$$

and satisfies the growth estimate

$$
(24) \qquad u(x,t) \leq Ae^{a|x|^2} \quad (x \in \mathbb{R}^n,\ 0 \leq t \leq T)
$$

for constants $A, a > 0$. Then

$$
\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.
$$

Proof. 1. First assume

$$
(25) \qquad 4aT < 1;
$$

in which case

$$
(26) \qquad 4a(T + \varepsilon) < 1
$$

for some $\varepsilon > 0$. Fix $y \in \mathbb{R}^n$, $\mu > 0$, and define

$$
v(x,t) := u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \quad (x \in \mathbb{R}^n,\ t > 0).
$$

A direct calculation (cf. §2.3.1) shows

$$v_t - \Delta v = 0 \quad \text{in } \mathbb{R}^n \times (0,T].$$

Fix $r > 0$ and set $U := B^0(y,r)$, $U_T = B^0(y,r) \times (0,T]$. Then according to Theorem 4,

$$\max_{\bar{U}_T} v = \max_{\Gamma_T} v. \tag{27}$$

2. Now if $x \in \mathbb{R}^n$,

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \\ &\leq u(x,0) = g(x); \end{aligned} \tag{28}$$

and if $|x-y| = r$, $0 \leq t \leq T$, then

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\ &\leq Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \qquad \text{by (24)} \\ &\leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}}. \end{aligned}$$

Now according to (26), $\frac{1}{4(T+\varepsilon)} = a+\gamma$ for some $\gamma > 0$. Thus we may continue the calculation above to find

$$v(x,t) \leq Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2} \leq \sup_{\mathbb{R}^n} g, \tag{29}$$

for r selected sufficiently large. Thus (27)–(29) imply

$$v(y,t) \leq \sup_{\mathbb{R}^n} g$$

for all $y \in \mathbb{R}^n$, $0 \leq t \leq T$, provided (25) is valid. Let $\mu \to 0$.

3. In the general case that (25) fails, we repeatedly apply the result above on the time intervals $[0,T_1]$, $[T_1, 2T_1,]$, etc., for $T_1 = \frac{1}{8a}$. □

THEOREM 7 (Uniqueness for Cauchy problem). *Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of the initial-value problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{30}$$

satisfying the growth estimate

$$|u(x,t)| \leq Ae^{a|x|^2} \quad (x \in \mathbb{R}^n,\ 0 \leq t \leq T) \tag{31}$$

for constants A, $a > 0$.

Proof. If u and $\tilde{u}$ both satisfy (30), (31), we apply Theorem 6 to $w := \pm(u - \tilde{u})$. □

Remark. There are in fact infinitely many solutions of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}; \end{cases} \tag{32}$$

see for instance John [**J**, Chapter 7]. Each of the solutions besides $u \equiv 0$ grows very rapidly as $|x| \to \infty$.

There is an interesting point here: although $u \equiv 0$ is certainly the "physically correct" solution of (32), this initial-value problem in fact admits other, "nonphysical" solutions. Theorem 7 provides a criterion which excludes the "wrong" solutions. We will encounter somewhat analogous situations in our study of Hamilton–Jacobi equations and conservation laws, in Chapters 3, 10 and 11. □

b. Regularity.

We next demonstrate that solutions of the heat equation are automatically smooth.

THEOREM 8 (Smoothness). *Suppose $u \in C^2_1(U_T)$ solves the heat equation in U_T. Then*

$$u \in C^\infty(U_T).$$

This regularity assertion is valid even if u attains nonsmooth boundary values on Γ_T.

Proof. 1. Recall from §A.2 that we write

$$C(x,t;r) = \{(y,s) \mid |x-y| \le r,\ t - r^2 \le s \le t\}$$

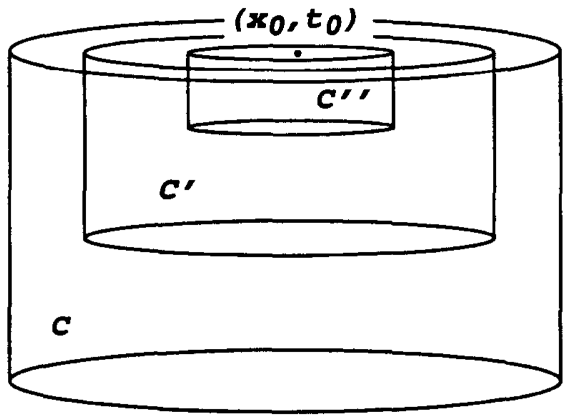
to denote the closed circular cylinder of radius r, height r^2, and top center point (x,t).

Fix $(x_0,t_0) \in U_T$ and choose $r > 0$ so small that $C := C(x_0,t_0;r) \subset U_T$. Define also the smaller cylinders $C' := C(x_0,t_0;\frac{3}{4}r)$, $C'' := C(x_0,t_0;\frac{1}{2}r)$, which have the same top center point (x_0,t_0).

Choose a smooth cutoff function $\zeta = \zeta(x,t)$ such that

$$\begin{cases} 0 \le \zeta \le 1,\ \zeta \equiv 1 \text{ on } C', \\ \zeta \equiv 0 \text{ near the parabolic boundary of } C. \end{cases}$$

Extend $\zeta \equiv 0$ in $(\mathbb{R}^n \times [0,t_0]) - C$.



2. Assume temporarily that $u \in C^\infty(U_T)$ and set

$$v(x,t) := \zeta(x,t)u(x,t) \quad (x \in \mathbb{R}^n,\ 0 \leq t \leq t_0).$$

Then

$$v_t = \zeta u_t + \zeta_t u, \ \ \Delta v = \zeta \Delta u + 2D\zeta \cdot Du + u\Delta\zeta.$$

Consequently

$$v = 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{33}$$

and

$$v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta =: \tilde{f} \tag{34}$$

in $\mathbb{R}^n \times (0, t_0)$. Now set

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds.$$

According to Theorem 2

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} & \text{in } \mathbb{R}^n \times (0, t_0) \\ \tilde{v} = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{35}$$

Since $|v|, |\tilde{v}| \leq A$ for some constant A, Theorem 7 implies $v \equiv \tilde{v}$; that is,

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds. \tag{36}$$

Now suppose $(x,t) \in C''$. As $\zeta \equiv 0$ off the cylinder C, (34) and (36) imply

$$\begin{aligned} u(x,t) = \iint_C \Phi(x-y, t-s)[(\zeta_s(y,s) - \Delta\zeta(y,s))u(y,s) \\ - 2D\zeta(y,s) \cdot Du(y,s)]\, dyds. \end{aligned}$$

Note in this expression that the expression in the square brackets vanishes in some region *near* the singularity of Φ. Integrate the last term by parts:

$$
\begin{aligned}
(37) \qquad u(x,t) = \iint_C [&\Phi(x-y,t-s)(\zeta_s(y,s)+\Delta\zeta(y,s)) \\
&+ 2D_y\Phi(x-y,t-s)\cdot D\zeta(y,s)]u(y,s)\,dyds.
\end{aligned}
$$

We have proved this formula assuming $u \in C^\infty$. If u satisfies only the hypotheses of the theorem, we derive (37) with $u^\varepsilon = \eta_\varepsilon * u$ replacing u, η_ε being the standard mollifier in the variables x and t, and let $\varepsilon \to 0$.

3. Formula (37) has the form

$$
(38) \qquad u(x,t) = \iint_C K(x,t,y,s)u(y,s)\,dyds \quad ((x,t) \in C''),
$$

where

$$
K(x,t,y,s) = 0 \quad \text{for all points } (y,s) \in C',
$$

since $\zeta \equiv 1$ on C'. Note also K is smooth on $C - C'$. In view of expression (38), we see u is C^∞ within $C'' = C(x_0,t_0;\frac{1}{2}r)$. □

c. Local estimates for solutions of the heat equation.

Next we record some estimates on the derivatives of solutions to the heat equation, paying attention to the differences between derivatives with respect to x_i $(i = 1,\dots,n)$ and with respect to t.

THEOREM 9 (Estimates on derivatives). *There exists for each pair of integers $k,l = 0,1,\dots,$ a constant $C_{k,l}$ such that*

$$
\max_{C(x,t;\,r/2)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t;r))}
$$

for all cylinders $C(x,t;r/2) \subset C(x,t;r) \subset U_T$, and all solutions u of the heat equation in U_T.

Proof. 1. Fix some point in U_T. Upon shifting the coordinates, we may as well assume the point is $(0,0)$. Suppose first that the cylinder $C(1) := C(0,0;1)$ lies in U_T. Let $C\left(\frac{1}{2}\right) := C\left(0,0;\frac{1}{2}\right)$. Then, as in the proof of Theorem 8,

$$
u(x,t) = \iint_{C(1)} K(x,t,y,s)u(y,s)\,dyds \quad ((x,t) \in C(\tfrac{1}{2}))
$$

for some smooth function K. Consequently

$$\begin{aligned} |D_x^k D_t^l u(x,t)| &\leq \iint_{C(1)} |D_t^l D_x^k K(x,t,y,s)||u(y,s)|\,dyds \\ &\leq C_{kl}\|u\|_{L^1(C(1))} \end{aligned} \tag{39}$$

for some constant C_{kl}.

2. Now suppose the cylinder $C(r) := C(0,0;r)$ lies in U_T. Let $C(r/2) = C(0,0;r/2)$. We rescale by defining

$$v(x,t) := u(rx, r^2 t).$$

Then $v_t - \Delta v = 0$ in the cylinder $C(1)$. According to (39),

$$|D_x^k D_t^l v(x,t)| \leq C_{kl}\|v\|_{L^1(C(1))} \quad ((x,t) \in C(\tfrac{1}{2})).$$

But $D_x^k D_t^l v(x,t) = r^{2l+k} D_x^k D_t^l u(rx, r^2 t)$ and $\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}}\|u\|_{L^1(C(r))}$. Therefore

$$\max_{C(r/2)} |D_x^k D_t^l u| \leq \frac{C_{kl}}{r^{2l+k+n+2}}\|u\|_{L^1(C(r))}.$$

□

Remark. If u solves the heat equation within U_T, then for each fixed time $0 < t \leq T$, the mapping $x \mapsto u(x,t)$ is analytic. (See Mikhailov [**M**].) However the mapping $t \mapsto u(x,t)$ is not in general analytic. □

2.3.4. Energy methods.

a. Uniqueness.

Let us investigate again the initial/boundary-value problem

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T. \end{cases} \tag{40}$$

We earlier invoked the maximum principle to show uniqueness, and now—by analogy with §2.2.5—provide an alternative argument based upon integration by parts. We assume as usual that $U \subset \mathbb{R}^n$ is open, bounded and that ∂U is C^1. The terminal time $T > 0$ is given.

THEOREM 10 (Uniqueness). *There exists at most one solution* $u \in C_1^2(\bar{U}_T)$ *of* (40).

Proof. 1. If $\tilde{u}$ is another solution, $w := u - \tilde{u}$ solves

$$\left\{\begin{aligned} w_t - \Delta w &= 0 \quad \text{in } U_T \\ w &= 0 \quad \text{on } \Gamma_T. \end{aligned}\right. \tag{41}$$

2. Set

$$e(t) := \int_U w^2(x,t)\,dx \quad (0 \leq t \leq T).$$

Then

$$\begin{aligned} \dot{e}(t) &= 2\int_U w w_t\,dx \quad \left(\cdot = \frac{d}{dt}\right) \\ &= 2\int_U w\Delta w\,dx \\ &= -2\int_U |Dw|^2 dx \leq 0, \end{aligned}$$

and so

$$e(t) \leq e(0) = 0 \quad (0 \leq t \leq T).$$

Consequently $w = u - \tilde{u} \equiv 0$ in U_T. □

Observe that the foregoing is a time-dependent variant of the proof of Theorem 16 in §2.2.5.

b. Backwards uniqueness.

A rather more subtle question concerns uniqueness *backwards in time* for the heat equation. For this, suppose u and $\tilde{u}$ are both smooth solutions of the heat equation in U_T, with the same boundary conditions on ∂U:

$$\left\{\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } U_T \\ u &= g \quad \text{on } \partial U \times [0,T], \end{aligned}\right. \tag{42}$$

$$\left\{\begin{aligned} \tilde{u}_t - \Delta \tilde{u} &= 0 \quad \text{in } U_T \\ \tilde{u} &= g \quad \text{on } \partial U \times [0,T], \end{aligned}\right. \tag{43}$$

for some function g. Note carefully that we are *not* supposing $u = \tilde{u}$ at time $t = 0$.

THEOREM 11 (Backwards uniqueness). *Suppose* $u, \tilde{u} \in C^2(\bar{U}_T)$ *solve* (42), (43). *If*

$$u(x,T) = \tilde{u}(x,T) \quad (x \in U),$$

then

$$u \equiv \tilde{u} \quad \textit{within } U_T.$$

In other words, if two temperature distributions on U agree at some time $T > 0$, and have had the same boundary values for times $0 \leq t \leq T$, then these temperatures must have been identically equal within U at all earlier times. This is not at all obvious.

Proof. 1. Write $w := u - \tilde{u}$ and, as in the proof of Theorem 10, set

$$e(t) := \int_U w^2(x,t)\,dx \quad (0 \leq t \leq T).$$

As before

$$\dot{e}(t) = -2 \int_U |Dw|^2 dx \quad \left(\cdot = \frac{d}{dt}\right). \tag{44}$$

Furthermore

$$\begin{aligned} \ddot{e}(t) &= -4 \int_U Dw \cdot Dw_t\,dx \\ &= 4 \int_U \Delta w w_t\,dx \\ &= 4 \int_U (\Delta w)^2 dx \quad \text{by (41).} \end{aligned} \tag{45}$$

Now since $w = 0$ on ∂U,

$$\begin{aligned} \int_U |Dw|^2 dx &= -\int_U w \Delta w\,dx \\ &\leq \left(\int_U w^2 dx\right)^{1/2} \left(\int_U (\Delta w)^2 dx\right)^{1/2}. \end{aligned}$$

Thus (44) and (45) imply

$$\begin{aligned} (\dot{e}(t))^2 &= 4 \left(\int_U |Dw|^2 dx\right)^2 \\ &\leq \left(\int_U w^2 dx\right) \left(4 \int_U (\Delta w)^2 dx\right) \\ &= e(t)\ddot{e}(t). \end{aligned}$$

Hence

$$\ddot{e}(t)e(t) \geq (\dot{e}(t))^2 \quad (0 \leq t \leq T). \tag{46}$$

2. Now if $e(t) = 0$ for all $0 \leq t \leq T$, we are done. Otherwise there exists an interval $[t_1, t_2] \subset [0, T]$, with

$$e(t) > 0 \quad \text{for } t_1 \leq t < t_2, \quad e(t_2) = 0. \tag{47}$$

3. Now write

$$f(t) := \log e(t) \quad (t_1 \leq t < t_2). \tag{48}$$

Then

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geq 0 \quad \text{by (46);}$$

and so f is convex on the interval (t_1, t_2). Consequently if $0 < \tau < 1$, $t_1 < t < t_2$, we have

$$f((1-\tau)t_1 + \tau t) \leq (1-\tau)f(t_1) + \tau f(t).$$

Recalling (48), we deduce

$$e((1-\tau)t_1 + \tau t) \leq e(t_1)^{1-\tau} e(t)^{\tau},$$

and so

$$0 \leq e((1-\tau)t_1 + \tau t_2) \leq e(t_1)^{1-\tau} e(t_2)^{\tau} \quad (0 < \tau < 1).$$

But in view of (47) this inequality implies $e(t) = 0$ for all times $t_1 \leq t \leq t_2$, a contradiction. $\square$

2.4. WAVE EQUATION

In this section we investigate the *wave equation*

$$u_{tt} - \Delta u = 0 \tag{1}$$

and the *nonhomogeneous wave equation*

$$u_{tt} - \Delta = f, \tag{2}$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables

Proof. 1. Choose $w \in \mathcal{A}$. Then (46) implies

$$0 = \int_U (-\Delta u - f)(u - w)\,dx.$$

An integration by parts yields

$$0 = \int_U Du \cdot D(u - w) - f(u - w)\,dx,$$

and there is no boundary term since $u - w = g - g = 0$ on ∂U. Hence

$$\begin{aligned} \int_U |Du|^2 - uf\,dx &= \int_U Du \cdot Dw - wf\,dx \\ &\leq \int_U \frac{1}{2}|Du|^2\,dx + \int_U \frac{1}{2}|Dw|^2 - wf\,dx, \end{aligned}$$

where we employed the estimates

$$|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2,$$

following from the Cauchy–Schwarz and Cauchy inequalities (§B.2). Rearranging, we conclude

$$I[u] \leq I[w] \quad (w \in \mathcal{A}). \tag{48}$$

Since $u \in \mathcal{A}$, (47) follows from (48).

2. Now, conversely, suppose (47) holds. Fix any $v \in C_c^\infty(U)$ and write

$$i(\tau) := I[u + \tau v] \quad (\tau \in \mathbb{R}).$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\cdot)$ has a minimum at zero, and thus

$$i'(0) = 0 \quad \left({}' = \frac{d}{d\tau}\right),$$

provided this derivative exists. But

$$\begin{aligned} i(\tau) &= \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v)f\,dx \\ &= \int_U \frac{1}{2}|Du|^2 + \tau Du \cdot Dv + \frac{\tau^2}{2}|Dv|^2 - (u + \tau v)f\,dx. \end{aligned}$$

Consequently

$$0 = i'(0) = \int_U Du \cdot Dv - vf\,dx = \int_U (-\Delta u - f)v\,dx.$$

This identity is valid for each function $v \in C_c^\infty(U)$ and so $-\Delta u = f$ in U. □

Dirichlet's principle is an instance of the *calculus of variations* applied to Laplace's equation. See Chapter 8 for more.

2.3. HEAT EQUATION

Next we study the *heat equation*

$$(1) \qquad u_t - \Delta u = 0$$

and the *nonhomogeneous heat equation*

$$(2) \qquad u_t - \Delta u = f,$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, \dots, x_n)$: $\Delta u = \Delta_x u = \sum_{i=1}^n u_{x_i x_i}$. In (2) the function $f : U \times [0, \infty) \to \mathbb{R}$ is given.

A guiding principle is that any assertion about harmonic functions yields an analogous (but more complicated) statement about solutions of the heat equation. Accordingly our development will largely parallel the corresponding theory for Laplace's equation.

Physical interpretation. The heat equation, also known as the *diffusion equation*, describes in typical applications the evolution in time of the density u of some quantity such as heat, chemical concentration, etc. If $V \subset U$ is any smooth subregion, the rate of change of the total quantity within V equals the negative of the net flux through ∂V:

$$\frac{d}{dt} \int_V u \, dx = - \int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS,$$

$\mathbf{F}$ being the flux density. Thus

$$(3) \qquad u_t = - \operatorname{div} \mathbf{F},$$

as V was arbitrary. In many situations $\mathbf{F}$ is proportional to the gradient of u, but points in the opposite direction (since the flow is from regions of higher to lower concentration):

$$\mathbf{F} = -aDu \quad (a > 0).$$

Substituting into (3), we obtain the PDE

$$u_t = a \operatorname{div}(Du) = a \Delta u,$$

which for $a = 1$ is the heat equation.

The heat equation appears as well in the study of Brownian motion.

□

2.3.1. Fundamental solution.

a. Derivation of the fundamental solution.

As noted in §2.2.1 an important first step in studying any PDE is often to come up with some specific solutions.

We observe that the heat equation involves one derivative with respect to the time variable t, but two derivatives with respect to the space variables x_i $(i = 1, \dots, n)$. Consequently we see that if u solves (1), then so does $u(\lambda x, \lambda^2 t)$ for $\lambda \in \mathbb{R}$. This scaling indicates the ratio $\frac{r^2}{t}$ $(r = |x|)$ is important for the heat equation and suggests that we search for a solution of (1) having the form $u(x,t) = v(\frac{r^2}{t}) = v(\frac{|x|^2}{t})$ $(t > 0,\ x \in \mathbb{R}^n)$, for some function v as yet undetermined.

Although this approach eventually leads to what we want (see Problem 11), it is quicker to seek a solution u having the special structure

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad (x \in \mathbb{R}^n,\ t > 0), \tag{4}$$

where the constants α, β and the function $v : \mathbb{R}^n \to \mathbb{R}$ must be found. We come to (4) if we look for a solution u of the heat equation invariant under the *dilation scaling*

$$u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t).$$

That is, we ask

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

for all $\lambda > 0$, $x \in \mathbb{R}^n$, $t > 0$. Setting $\lambda = t^{-1}$, we derive (4) for $v(y) := u(y, 1)$.

Let us insert (4) into (1), and thereafter compute

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \tag{5}$$

for $y := t^{-\beta} x$. In order to transform (5) into an expression involving the variable y alone, we take $\beta = \frac{1}{2}$. Then the terms with t are identical, and so (5) reduces to

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0. \tag{6}$$

We simplify further by guessing v to be radial; that is, $v(y) = w(|y|)$ for some $w : \mathbb{R} \to \mathbb{R}$. Thereupon (6) becomes

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0,$$

for $r = |y|$, $' = \frac{d}{dr}$. Now if we set $\alpha = \frac{n}{2}$, this simplifies to read

$$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0.$$

Thus

$$r^{n-1}w' + \frac{1}{2}r^n w = a$$

for some constant a. Assuming $\lim_{r\to\infty} w,\ w' = 0$, we conclude $a = 0$; whence

$$w' = -\frac{1}{2}rw.$$

But then for some constant b

$$w = be^{-\frac{r^2}{4}}. \tag{7}$$

Combining (4), (7) and our choices for α, β, we conclude that $\frac{b}{t^{n/2}}e^{-\frac{|x|^2}{4t}}$ solves the heat equation (1).

This computation motivates the following

DEFINITION. *The function*

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n,\ t > 0) \\ 0 & (x \in \mathbb{R}^n,\ t < 0) \end{cases}$$

is called the fundamental solution of the heat equation.

Notice that Φ is singular at the point $(0,0)$. We will sometimes write $\Phi(x,t) = \Phi(|x|,t)$ to emphasize that the fundamental solution is radial in the variable x. The choice of the normalizing constant $(4\pi)^{-n/2}$ is dictated by the following

LEMMA (Integral of fundamental solution). *For each time $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1.$$

Proof. We calculate

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)\, dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz \\ &= \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-z_i^2}\, dz_i = 1. \end{aligned}$$

□

A different derivation of the fundamental solution of the heat equation appears in §4.3.2.

b. Initial-value problem.

We now employ Φ to fashion a solution to the *initial-value* (or *Cauchy*) *problem*

$$
(8) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}
$$

Let us note the function $(x,t) \mapsto \Phi(x,t)$ solves the heat equation away from the singularity at $(0,0)$, and thus so does $(x,t) \mapsto \Phi(x-y,t)$ for each fixed $y \in \mathbb{R}^n$. Consequently the convolution

$$
(9) \qquad \begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n,\ t > 0) \end{aligned}
$$

should also be a solution.

THEOREM 1 (Solution of initial-value problem). *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define u by (9). Then*

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t(x,t) - \Delta u(x,t) = 0 \quad (x \in \mathbb{R}^n,\ t > 0)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0)$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. 1. Since the function $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$ is infinitely differentiable, with uniformly bounded derivatives of all orders, on $\mathbb{R}^n \times [\delta,\infty)$ for each $\delta > 0$, we see that $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$. Furthermore

$$
(10) \qquad \begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x-y,t)] g(y)\, dy \\ &= 0 \quad (x \in \mathbb{R}^n,\ t > 0), \end{aligned}
$$

since Φ itself solves the heat equation.

2. Fix $x^0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ such that

$$
(11) \qquad |g(y) - g(x^0)| < \varepsilon \quad \text{if } |y - x^0| < \delta,\ y \in \mathbb{R}^n.
$$

Then if $|x - x^0| < \frac{\delta}{2}$, we have, according to the lemma,

$$\begin{aligned}
|u(x,t) - g(x^0)| &= \Big|\int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x^0)]\,dy\Big| \\
&\leq \int_{B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\,dy \\
&\qquad + \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\,dy \\
&=: I + J.
\end{aligned}$$

Now

$$I \leq \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t)\,dy = \varepsilon,$$

owing to (11) and the lemma. Furthermore, if $|x - x^0| \leq \frac{\delta}{2}$ and $|y - x^0| \geq \delta$, then

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|.$$

Thus $|y - x| \geq \frac{1}{2}|y - x^0|$. Consequently

$$\begin{aligned}
J &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)\,dy \\
&\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0,\delta)} e^{-\frac{|x-y|^2}{4t}}\,dy \\
&\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0,\delta)} e^{-\frac{|y-x^0|^2}{16t}}\,dy \\
&= \frac{C}{t^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{16t}} r^{n-1}\,dr \to 0 \qquad \text{as } t \to 0^+.
\end{aligned}$$

Hence if $|x - x^0| < \frac{\delta}{2}$ and $t > 0$ is small enough, $|u(x,t) - g(x^0)| < 2\varepsilon$. $\square$

Remarks. (i) In view of Theorem 1 we sometimes write

$$\begin{cases} \Phi_t - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\;\; \Phi = \delta_0 & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

δ_0 denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0.

(ii) Notice that if g is bounded, continuous, $g \geq 0$, $g \not\equiv 0$, then

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy$$

is in fact positive for *all* points $x \in \mathbb{R}^n$ and times $t > 0$. We interpret this observation by saying the heat equation forces *infinite propagation speed*

for disturbances. If the initial temperature is nonnegative and is positive somewhere, the temperature at any later time (no matter how small) is everywhere positive. (We will learn in §2.4.3 that the wave equation in contrast supports finite propagation speed for disturbances.) □

c. Nonhomogeneous problem.

Now let us turn our attention to the *nonhomogeneous* initial-value problem

$$
(12) \qquad \begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\quad u = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}
$$

How can we produce a formula for the solution? If we recall the motivation leading up to (9), we should note further that the mapping $(x,t) \mapsto \Phi(x-y,t-s)$ is a solution of the heat equation (for given $y \in \mathbb{R}^n$, $0 < s < t$). Now for fixed s, the function

$$
u = u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy
$$

solves

$$
(12_s) \qquad \begin{cases} u_t(\cdot;s) - \Delta u(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ \qquad\qquad u(\cdot;s) = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{t=s\}, \end{cases}
$$

which is just an initial-value problem of the form (8), with the starting time $t=0$ replaced by $t=s$, and g replaced by $f(\cdot,s)$. Thus $u(\cdot;s)$ is certainly not a solution of (12).

However *Duhamel's principle** asserts that we can build a solution of (12) out of the solutions of (12_s), by integrating with respect to s. The idea is to consider

$$
u(x,t) = \int_0^t u(x,t;s)\,ds \quad (x \in \mathbb{R}^n,\ t \geq 0).
$$

Rewriting, we have

$$
(13) \qquad \begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dyds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dyds, \end{aligned}
$$

for $x \in \mathbb{R}^n$, $t > 0$.

To confirm that formula (13) works, let us for simplicity assume $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and f has compact support.

*Duhamel's principle has wide applicability to linear ODE and PDE, and does not depend on the specific structure of the heat equation. It yields, for example, the solution of the nonhomogeneous transport equation, obtained by different means in §2.1.2. We will invoke Duhamel's principle for the wave equation in §2.4.2.

THEOREM 2 (Solution of nonhomogeneous problem). *Define u by* (13). *Then*

(i) $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t(x,t) - \Delta u(x,t) = f(x,t) \quad (x \in \mathbb{R}^n,\ t > 0)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = 0$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. 1. Since Φ has a singularity at $(0,0)$, we cannot directly justify differentiating under the integral sign. We instead proceed somewhat as in the proof of Theorem 1 in §2.2.1.

First we change variables, to write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\,dyds.$$

As $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ has compact support and $\Phi = \Phi(y,s)$ is smooth near $s = t > 0$, we compute

$$\begin{aligned} u_t(x,t) = &\int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\,dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy \end{aligned}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial^2}{\partial x_i \partial x_j} f(x-y,t-s)\,dyds \quad (i,j = 1,\dots,n).$$

Thus $u_t, D_x^2 u$, and likewise $u, D_x u$, belong to $C(\mathbb{R}^n \times (0,\infty))$.

2. We then calculate

(14)
$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)[(\frac{\partial}{\partial t} - \Delta_x) f(x-y,t-s)]\,dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]\,dyds \\ &\quad + \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]\,dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy. \\ &=: I_\varepsilon + J_\varepsilon + K. \end{aligned}$$

Now

$$|J_\varepsilon| \leq (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\, dyds \leq \varepsilon C, \tag{15}$$

by the lemma. Integrating by parts, we also find

$$\begin{aligned} I_\varepsilon = &\int_\varepsilon^t \int_{\mathbb{R}^n} [(\frac{\partial}{\partial s} - \Delta_y)\Phi(y,s)] f(x-y,t-s)\, dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy \\ &- \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ = &\int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy - K, \end{aligned} \tag{16}$$

since Φ solves the heat equation. Combining (14)–(16), we ascertain

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy \\ &= f(x,t) \quad (x \in \mathbb{R}^n,\ t > 0), \end{aligned}$$

the limit as $\varepsilon \to 0$ being computed as in the proof of Theorem 1. Finally note $\|u(\cdot,t)\|_{L^\infty} \leq t\|f\|_{L^\infty} \to 0$. □

Remark. We can of course combine Theorems 1 and 2 to discover that

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dyds \tag{17}$$

is, under the hypotheses on g and f as above, a solution of

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\ \ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{18}$$

□

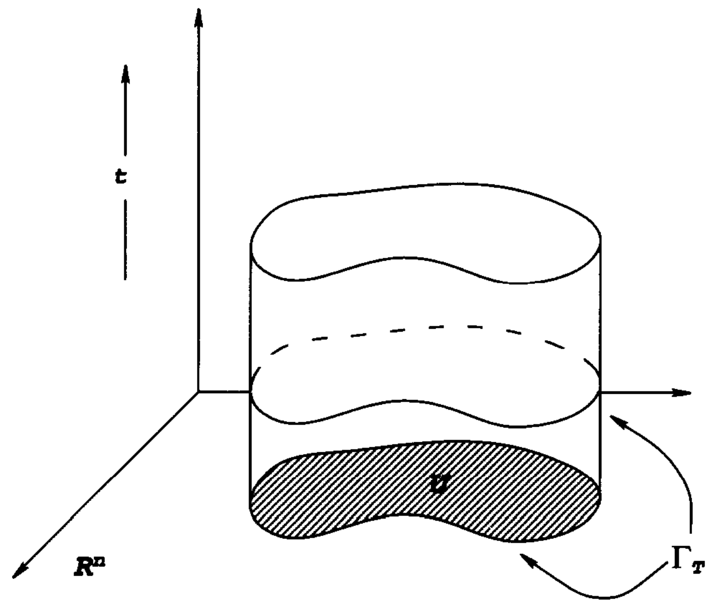
2.3.2. Mean-value formula.

First we recall some useful notation from §A.2. Assume $U \subset \mathbb{R}^n$ is open and bounded, and fix a time $T > 0$.

DEFINITIONS.

(i) *We define the* parabolic cylinder

$$U_T := U \times (0,T].$$



The region $\mathbf{U}_T$

(ii) *The* parabolic boundary *of U_T is*

$$\Gamma_T := \bar{U}_T - U_T.$$

We interpret U_T as being the *parabolic interior* of $\bar{U} \times [0, T]$: note carefully that U_T includes the top $U \times \{t = T\}$. The parabolic boundary Γ_T comprises the bottom and vertical sides of $U \times [0, T]$, but not the top.

We want next to derive a kind of analogue to the mean-value property for harmonic functions, as discussed in §2.2.2. There is no such simple formula. However let us observe that for fixed x the spheres $\partial B(x, r)$ are level sets of the fundamental solution $\Phi(x-y)$ for Laplace's equation. This suggests that perhaps for fixed (x, t) the level sets of fundamental solution $\Phi(x - y, t - s)$ for the heat equation may be relevant.

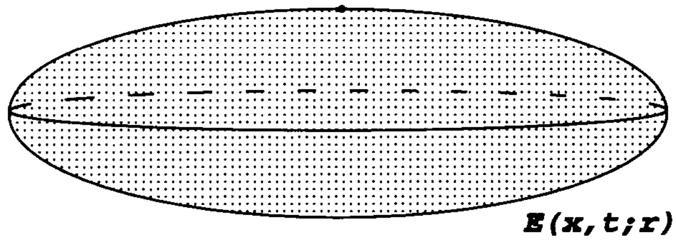
DEFINITION. *For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define*

$$E(x, t; r) := \left\{ (y, s) \in \mathbb{R}^{n+1} \mid s \leq t,\ \Phi(x - y, t - s) \geq \frac{1}{r^n} \right\}.$$

This is a region in space-time, the boundary of which is a level set of $\Phi(x-y, t-s)$. Note that the point (x, t) is at the center of the top. $E(x, t; r)$ is sometimes called a "heat ball".

THEOREM 3 (A mean-value property for the heat equation). *Let $u \in C_1^2(U_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dyds \tag{19}$$



A "heat ball"

for each $E(x,t;r) \subset U_T$.

Formula (19) is a sort of analogue for the heat equation of the mean-value formulas for Laplace's equation. Observe that the right hand side involves only $u(y,s)$ for times $s \le t$. This is reasonable, as the value $u(x,t)$ should not depend upon future times.

Proof. We may as well assume upon translating the space and time coordinates that $x = 0$ and $t = 0$. Write $E(r) = E(0,0;r)$ and set

$$\text{(20)} \qquad \begin{aligned} \phi(r) &:= \frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dyds \\ &= \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dyds. \end{aligned}$$

We compute

$$\begin{aligned} \phi'(r) &= \iint_{E(1)} \sum_{i=1}^n u_{y_i} y_i \frac{|y|^2}{s^2} + 2r u_s \frac{|y|^2}{s}\, dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} \sum_{i=1}^n u_{y_i} y_i \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s}\, dyds \\ &=: A + B. \end{aligned}$$

Also, let us introduce the useful function

$$\text{(21)} \qquad \psi := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n \log r,$$

and observe $\psi = 0$ on $\partial E(r)$, since $\Phi(y,-s) = r^{-n}$ on $\partial E(r)$. We utilize (21) to write

$$\begin{aligned} B &= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^n y_i \psi_{y_i}\, dyds \\ &= -\frac{1}{r^{n+1}} \iint_{E(r)} 4n u_s \psi + 4 \sum_{i=1}^n u_{s y_i} y_i \psi\, dyds; \end{aligned}$$

there is no boundary term since $\psi = 0$ on $\partial E(r)$. Integrating by parts with respect to s, we discover

$$\begin{aligned} B &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^n u_{y_i} y_i \psi_s \, dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^n u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2}\right) dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi - \frac{2n}{s}\sum_{i=1}^n u_{y_i} y_i \, dyds - A. \end{aligned}$$

Consequently, since u solves the heat equation,

$$\begin{aligned} \phi'(r) &= A + B \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4n\Delta u\psi - \frac{2n}{s}\sum_{i=1}^n u_{y_i} y_i \, dyds \\ &= \sum_{i=1}^n \frac{1}{r^{n+1}} \iint_{E(r)} 4nu_{y_i}\psi_{y_i} - \frac{2n}{s} u_{y_i} y_i \, dyds \\ &= 0, \quad \text{according to (21).} \end{aligned}$$

Thus ϕ is constant, and therefore

$$\phi(r) = \lim_{t\to 0} \phi(t) = u(0,0) \left(\lim_{t\to 0} \frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds\right) = 4u(0,0),$$

as

$$\frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds = \iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4.$$

We omit the details of this last computation. □

2.3.3. Properties of solutions.

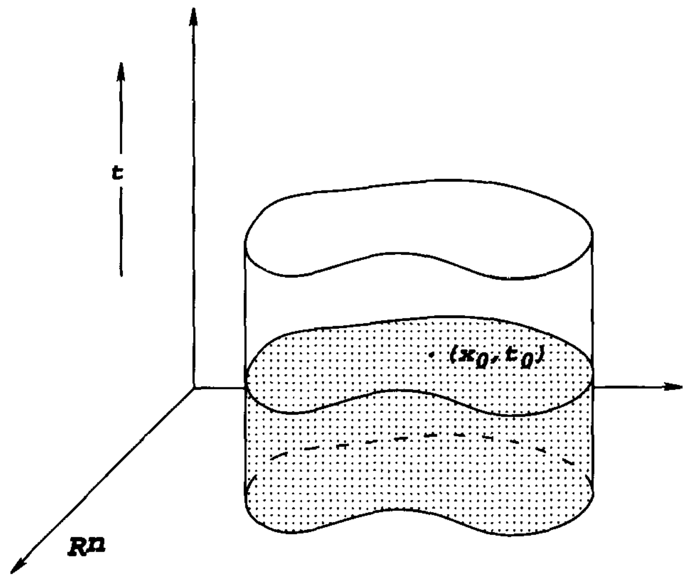
a. Strong maximum principle, uniqueness.

First we employ the mean-value property to give a quick proof of the strong maximum principle.

THEOREM 4 (Strong maximum principle for the heat equation). *Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation in U_T.*

(i) *Then*

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$



Strong maximum principle for the heat equation

(ii) *Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that*

$$u(x_0, t_0) = \max_{\bar{U}_T} u,$$

then

$$u \text{ is constant in } \bar{U}_{t_0}.$$

Assertion (i) is the *maximum principle* for the heat equation and (ii) is the *strong maximum principle.* Similar assertions are valid with "min" replacing "max".

Remark. *So if u attains its maximum (or minimum) at an interior point, then u is constant at all earlier times.* This accords with our strong intuitive interpretation of the variable t as denoting time: the solution will be constant on the time interval $[0, t_0]$ provided the initial and boundary conditions are constant. However, the solution may change at times $t > t_0$, provided the boundary conditions alter after t_0. The solution will however not respond to changes in boundary conditions until these changes happen.

Take note that whereas all this is obvious on intuitive, physical grounds, such insights do not constitute a proof. The task is to *deduce* such behavior from the PDE. □

Proof. 1. Suppose there exists a point $(x_0, t_0) \in U_T$ with $u(x_0, t_0) = M := \max_{\bar{U}_T} u$. Then for all sufficiently small $r > 0$, $E(x_0, t_0; r) \subset U_T$; and we

employ the mean-value property to deduce

$$M = u(x_0, t_0) = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dyds \leq M,$$

since

$$1 = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dyds.$$

Equality holds only if u is identically equal to M within $E(x_0, t_0; r)$. Consequently

$$u(y,s) = M \quad \text{ for all } (y,s) \in E(x_0, t_0; r).$$

Draw any line segment L in U_T connecting (x_0, t_0) with some other point $(y_0, s_0) \in U_T$, with $s_0 < t_0$. Consider

$$r_0 := \min\{s \geq s_0 \mid u(x,t) = M \text{ for all points } (x,t) \in L, \ s \leq t \leq t_0\}.$$

Since u is continuous, the minimum is attained. Assume $r_0 > s_0$. Then $u(z_0, r_0) = M$ for some point (z_0, r_0) on $L \cap U_T$ and so $u \equiv M$ on $E(z_0, r_0; r)$ for all sufficiently small $r > 0$. Since $E(z_0, r_0; r)$ contains $L \cap \{r_0 - \sigma \leq t \leq r_0\}$ for some small $\sigma > 0$, we have a contradiction. Thus $r_0 = s_0$, and hence $u \equiv M$ on L.

2. Now fix any point $x \in U$ and any time $0 \leq t < t_0$. There exist points $\{x_0, x_1, \ldots, x_m = x\}$ such that the line segments in $\mathbb{R}^n$ connecting x_{i-1} to x_i lie in U for $i = 1, \ldots, m$. (This follows since the set of points in U which can be so connected to x_0 by a polygonal path is nonempty, open and relatively closed in U.) Select times $t_0 > t_1 > \cdots > t_m = t$. Then the line segments in $\mathbb{R}^{n+1}$ connecting (x_{i-1}, t_{i-1}) to (x_i, t_i) $(i = 1, \ldots, m)$ lie in U_T. According to Step 1, $u \equiv M$ on each such segment and so $u(x,t) = M$. □

Remark. The strong maximum principle implies that if U is connected and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

where $g \geq 0$, then u is positive *everywhere* within U_T if g is positive *somewhere* on U. This is another illustration of infinite propagation speed for disturbances. □

An important application of the maximum principle is the following uniqueness assertion.

THEOREM 5 (Uniqueness on bounded domains). *Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of the initial/boundary-value problem*

$$
(22) \qquad \begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T. \end{cases}
$$

Proof. If u and $\tilde{u}$ are two solutions of (22), apply Theorem 4 to $w := \pm(u - \tilde{u})$. □

We next extend our uniqueness assertion to the *Cauchy problem*, that is, the initial value problem for $U = \mathbb{R}^n$. As we are no longer on a bounded region, we must introduce some control on the behavior of solutions for large $|x|$.

THEOREM 6 (Maximum principle for the Cauchy problem). *Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves*

$$
(23) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}
$$

and satisfies the growth estimate

$$
(24) \qquad u(x,t) \leq A e^{a|x|^2} \quad (x \in \mathbb{R}^n,\ 0 \leq t \leq T)
$$

for constants $A, a > 0$. Then

$$
\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.
$$

Proof. 1. First assume

$$
(25) \qquad 4aT < 1;
$$

in which case

$$
(26) \qquad 4a(T + \varepsilon) < 1
$$

for some $\varepsilon > 0$. Fix $y \in \mathbb{R}^n$, $\mu > 0$, and define

$$
v(x,t) := u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \quad (x \in \mathbb{R}^n,\ t > 0).
$$

A direct calculation (cf. §2.3.1) shows

$$v_t - \Delta v = 0 \quad \text{in } \mathbb{R}^n \times (0, T].$$

Fix $r > 0$ and set $U := B^0(y, r)$, $U_T = B^0(y, r) \times (0, T]$. Then according to Theorem 4,

$$\max_{\bar{U}_T} v = \max_{\Gamma_T} v. \tag{27}$$

2. Now if $x \in \mathbb{R}^n$,

$$\begin{aligned} v(x, 0) &= u(x, 0) - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \\ &\leq u(x, 0) = g(x); \end{aligned} \tag{28}$$

and if $|x - y| = r$, $0 \leq t \leq T$, then

$$\begin{aligned} v(x, t) &= u(x, t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\ &\leq Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \qquad \text{by (24)} \\ &\leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}}. \end{aligned}$$

Now according to (26), $\frac{1}{4(T+\varepsilon)} = a + \gamma$ for some $\gamma > 0$. Thus we may continue the calculation above to find

$$v(x, t) \leq Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2} \leq \sup_{\mathbb{R}^n} g, \tag{29}$$

for r selected sufficiently large. Thus (27)–(29) imply

$$v(y, t) \leq \sup_{\mathbb{R}^n} g$$

for all $y \in \mathbb{R}^n$, $0 \leq t \leq T$, provided (25) is valid. Let $\mu \to 0$.

3. In the general case that (25) fails, we repeatedly apply the result above on the time intervals $[0, T_1]$, $[T_1, 2T_1,]$, etc., for $T_1 = \frac{1}{8a}$. □

THEOREM 7 (Uniqueness for Cauchy problem). *Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0, T])$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ of the initial-value problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0, T) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{30}$$

satisfying the growth estimate

$$|u(x, t)| \leq Ae^{a|x|^2} \quad (x \in \mathbb{R}^n, \ 0 \leq t \leq T) \tag{31}$$

for constants A, $a > 0$.

Proof. If u and $\tilde{u}$ both satisfy (30), (31), we apply Theorem 6 to $w := \pm(u - \tilde{u})$. $\square$

Remark. There are in fact infinitely many solutions of

$$\left\{\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\ u &= 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\}; \end{aligned}\right. \tag{32}$$

see for instance John [**J**, Chapter 7]. Each of the solutions besides $u \equiv 0$ grows very rapidly as $|x| \to \infty$.

There is an interesting point here: although $u \equiv 0$ is certainly the "physically correct" solution of (32), this initial-value problem in fact admits other, "nonphysical" solutions. Theorem 7 provides a criterion which excludes the "wrong" solutions. We will encounter somewhat analogous situations in our study of Hamilton–Jacobi equations and conservation laws, in Chapters 3, 10 and 11. $\square$

b. Regularity.

We next demonstrate that solutions of the heat equation are automatically smooth.

THEOREM 8 (Smoothness). *Suppose $u \in C_1^2(U_T)$ solves the heat equation in U_T. Then*

$$u \in C^\infty(U_T).$$

This regularity assertion is valid even if u attains nonsmooth boundary values on Γ_T.

Proof. 1. Recall from §A.2 that we write

$$C(x,t;r) = \{(y,s) \mid |x-y| \leq r,\ t - r^2 \leq s \leq t\}$$

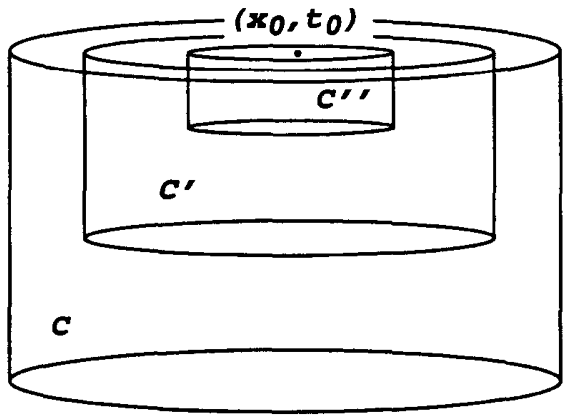
to denote the closed circular cylinder of radius r, height r^2, and top center point (x,t).

Fix $(x_0,t_0) \in U_T$ and choose $r > 0$ so small that $C := C(x_0,t_0;r) \subset U_T$. Define also the smaller cylinders $C' := C(x_0,t_0;\frac{3}{4}r)$, $C'' := C(x_0,t_0;\frac{1}{2}r)$, which have the same top center point (x_0,t_0).

Choose a smooth cutoff function $\zeta = \zeta(x,t)$ such that

$$\left\{\begin{array}{l} 0 \leq \zeta \leq 1,\ \zeta \equiv 1 \text{ on } C', \\ \zeta \equiv 0 \text{ near the parabolic boundary of } C. \end{array}\right.$$

Extend $\zeta \equiv 0$ in $(\mathbb{R}^n \times [0,t_0]) - C$.



2. Assume temporarily that $u \in C^\infty(U_T)$ and set

$$v(x,t) := \zeta(x,t)u(x,t) \quad (x \in \mathbb{R}^n,\, 0 \leq t \leq t_0).$$

Then

$$v_t = \zeta u_t + \zeta_t u, \;\; \Delta v = \zeta \Delta u + 2D\zeta \cdot Du + u\Delta\zeta.$$

Consequently

$$v = 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{33}$$

and

$$v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta =: \tilde{f} \tag{34}$$

in $\mathbb{R}^n \times (0, t_0)$. Now set

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds.$$

According to Theorem 2

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} & \text{in } \mathbb{R}^n \times (0, t_0) \\ \tilde{v} = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{35}$$

Since $|v|, |\tilde{v}| \leq A$ for some constant A, Theorem 7 implies $v \equiv \tilde{v}$; that is,

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds. \tag{36}$$

Now suppose $(x,t) \in C''$. As $\zeta \equiv 0$ off the cylinder C, (34) and (36) imply

$$\begin{aligned} u(x,t) = \iint_C \Phi(x-y, t-s)[(\zeta_s(y,s) - \Delta\zeta(y,s))u(y,s) \\ - 2D\zeta(y,s) \cdot Du(y,s)]\, dyds. \end{aligned}$$

Note in this expression that the expression in the square brackets vanishes in some region *near* the singularity of Φ. Integrate the last term by parts:

$$
\begin{aligned}
u(x,t) = \iint_C [&\Phi(x-y,t-s)(\zeta_s(y,s) + \Delta\zeta(y,s)) \\
&+ 2D_y\Phi(x-y,t-s)\cdot D\zeta(y,s)]u(y,s)\,dyds.
\end{aligned} \tag{37}
$$

We have proved this formula assuming $u \in C^\infty$. If u satisfies only the hypotheses of the theorem, we derive (37) with $u^\varepsilon = \eta_\varepsilon * u$ replacing u, η_ε being the standard mollifier in the variables x and t, and let $\varepsilon \to 0$.

3. Formula (37) has the form

$$
u(x,t) = \iint_C K(x,t,y,s)u(y,s)\,dyds \quad ((x,t) \in C''), \tag{38}
$$

where

$$
K(x,t,y,s) = 0 \quad \text{for all points } (y,s) \in C',
$$

since $\zeta \equiv 1$ on C'. Note also K is smooth on $C - C'$. In view of expression (38), we see u is C^∞ within $C'' = C(x_0,t_0;\frac{1}{2}r)$. □

c. Local estimates for solutions of the heat equation.

Next we record some estimates on the derivatives of solutions to the heat equation, paying attention to the differences between derivatives with respect to x_i $(i = 1,\dots,n)$ and with respect to t.

THEOREM 9 (Estimates on derivatives). *There exists for each pair of integers $k, l = 0, 1, \dots,$ a constant $C_{k,l}$ such that*

$$
\max_{C(x,t;\,r/2)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t;r))}
$$

for all cylinders $C(x,t;r/2) \subset C(x,t;r) \subset U_T$, and all solutions u of the heat equation in U_T.

Proof. 1. Fix some point in U_T. Upon shifting the coordinates, we may as well assume the point is $(0,0)$. Suppose first that the cylinder $C(1) := C(0,0;1)$ lies in U_T. Let $C\left(\frac{1}{2}\right) := C\left(0,0;\frac{1}{2}\right)$. Then, as in the proof of Theorem 8,

$$
u(x,t) = \iint_{C(1)} K(x,t,y,s)u(y,s)\,dyds \quad ((x,t) \in C(\tfrac{1}{2}))
$$

for some smooth function K. Consequently

$$
\begin{aligned}
(39) \qquad |D_x^k D_t^l u(x,t)| &\leq \iint_{C(1)} |D_t^l D_x^k K(x,t,y,s)||u(y,s)|\,dyds \\
&\leq C_{kl}\|u\|_{L^1(C(1))}
\end{aligned}
$$

for some constant C_{kl}.

2. Now suppose the cylinder $C(r) := C(0,0;r)$ lies in U_T. Let $C(r/2) = C(0,0;r/2)$. We rescale by defining

$$v(x,t) := u(rx, r^2 t).$$

Then $v_t - \Delta v = 0$ in the cylinder $C(1)$. According to (39),

$$|D_x^k D_t^l v(x,t)| \leq C_{kl}\|v\|_{L^1(C(1))} \quad ((x,t) \in C(\tfrac{1}{2})).$$

But $D_x^k D_t^l v(x,t) = r^{2l+k} D_x^k D_t^l u(rx, r^2 t)$ and $\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}}\|u\|_{L^1(C(r))}$. Therefore

$$\max_{C(r/2)} |D_x^k D_t^l u| \leq \frac{C_{kl}}{r^{2l+k+n+2}}\|u\|_{L^1(C(r))}.$$

□

Remark. If u solves the heat equation within U_T, then for each fixed time $0 < t \leq T$, the mapping $x \mapsto u(x,t)$ is analytic. (See Mikhailov [**M**].) However the mapping $t \mapsto u(x,t)$ is not in general analytic. □

2.3.4. Energy methods.

a. Uniqueness.

Let us investigate again the initial/boundary-value problem

$$
(40) \qquad \begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \qquad\; u = g & \text{on } \Gamma_T. \end{cases}
$$

We earlier invoked the maximum principle to show uniqueness, and now—by analogy with §2.2.5—provide an alternative argument based upon integration by parts. We assume as usual that $U \subset \mathbb{R}^n$ is open, bounded and that ∂U is C^1. The terminal time $T > 0$ is given.

THEOREM 10 (Uniqueness). *There exists at most one solution $u \in C_1^2(\bar{U}_T)$ of* (40).

Proof. 1. If $\tilde{u}$ is another solution, $w := u - \tilde{u}$ solves

$$\begin{cases} w_t - \Delta w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T. \end{cases} \tag{41}$$

2. Set

$$e(t) := \int_U w^2(x,t)\,dx \quad (0 \leq t \leq T).$$

Then

$$\begin{aligned} \dot{e}(t) &= 2\int_U w w_t\,dx \quad \left(\cdot = \frac{d}{dt}\right) \\ &= 2\int_U w\Delta w\,dx \\ &= -2\int_U |Dw|^2 dx \leq 0, \end{aligned}$$

and so

$$e(t) \leq e(0) = 0 \quad (0 \leq t \leq T).$$

Consequently $w = u - \tilde{u} \equiv 0$ in U_T. □

Observe that the foregoing is a time-dependent variant of the proof of Theorem 16 in §2.2.5.

b. Backwards uniqueness.

A rather more subtle question concerns uniqueness *backwards in time* for the heat equation. For this, suppose u and $\tilde{u}$ are both smooth solutions of the heat equation in U_T, with the same boundary conditions on ∂U:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = g & \text{on } \partial U \times [0,T], \end{cases} \tag{42}$$

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 & \text{in } U_T \\ \tilde{u} = g & \text{on } \partial U \times [0,T], \end{cases} \tag{43}$$

for some function g. Note carefully that we are *not* supposing $u = \tilde{u}$ at time $t = 0$.

THEOREM 11 (Backwards uniqueness). *Suppose* $u, \tilde{u} \in C^2(\bar{U}_T)$ *solve* (42), (43). *If*

$$u(x,T) = \tilde{u}(x,T) \quad (x \in U),$$

then

$$u \equiv \tilde{u} \quad \textit{within } U_T.$$

In other words, if two temperature distributions on U agree at some time $T > 0$, and have had the same boundary values for times $0 \leq t \leq T$, then these temperatures must have been identically equal within U at all earlier times. This is not at all obvious.

Proof. 1. Write $w := u - \tilde{u}$ and, as in the proof of Theorem 10, set

$$e(t) := \int_U w^2(x,t)\,dx \quad (0 \leq t \leq T).$$

As before

$$\dot{e}(t) = -2\int_U |Dw|^2 dx \quad \left(\dot{} = \frac{d}{dt} \right). \tag{44}$$

Furthermore

$$\begin{aligned} \ddot{e}(t) &= -4\int_U Dw \cdot Dw_t\,dx \\ &= 4\int_U \Delta w w_t\,dx \\ &= 4\int_U (\Delta w)^2 dx \quad \text{by (41).} \end{aligned} \tag{45}$$

Now since $w = 0$ on ∂U,

$$\begin{aligned} \int_U |Dw|^2 dx &= -\int_U w\Delta w\,dx \\ &\leq \left(\int_U w^2 dx\right)^{1/2} \left(\int_U (\Delta w)^2 dx\right)^{1/2}. \end{aligned}$$

Thus (44) and (45) imply

$$\begin{aligned} (\dot{e}(t))^2 &= 4\left(\int_U |Dw|^2 dx\right)^2 \\ &\leq \left(\int_U w^2 dx\right)\left(4\int_U (\Delta w)^2 dx\right) \\ &= e(t)\ddot{e}(t). \end{aligned}$$

Hence

$$\ddot{e}(t)e(t) \geq (\dot{e}(t))^2 \quad (0 \leq t \leq T). \tag{46}$$

2. Now if $e(t) = 0$ for all $0 \leq t \leq T$, we are done. Otherwise there exists an interval $[t_1, t_2] \subset [0, T]$, with

$$e(t) > 0 \quad \text{for } t_1 \leq t < t_2, \quad e(t_2) = 0. \tag{47}$$

3. Now write

$$f(t) := \log e(t) \quad (t_1 \leq t < t_2). \tag{48}$$

Then

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geq 0 \quad \text{by (46);}$$

and so f is convex on the interval (t_1, t_2). Consequently if $0 < \tau < 1$, $t_1 < t < t_2$, we have

$$f((1-\tau)t_1 + \tau t) \leq (1-\tau)f(t_1) + \tau f(t).$$

Recalling (48), we deduce

$$e((1-\tau)t_1 + \tau t) \leq e(t_1)^{1-\tau} e(t)^\tau,$$

and so

$$0 \leq e((1-\tau)t_1 + \tau t_2) \leq e(t_1)^{1-\tau} e(t_2)^\tau \quad (0 < \tau < 1).$$

But in view of (47) this inequality implies $e(t) = 0$ for all times $t_1 \leq t \leq t_2$, a contradiction. □

2.4. WAVE EQUATION

In this section we investigate the *wave equation*

$$u_{tt} - \Delta u = 0 \tag{1}$$

and the *nonhomogeneous wave equation*

$$u_{tt} - \Delta = f, \tag{2}$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables

Proof. 1. Choose $w \in \mathcal{A}$. Then (46) implies

$$0 = \int_U (-\Delta u - f)(u - w)\, dx.$$

An integration by parts yields

$$0 = \int_U Du \cdot D(u - w) - f(u - w)\, dx,$$

and there is no boundary term since $u - w = g - g = 0$ on ∂U. Hence

$$\begin{aligned}\int_U |Du|^2 - uf\, dx &= \int_U Du \cdot Dw - wf\, dx \\ &\leq \int_U \frac{1}{2}|Du|^2\, dx + \int_U \frac{1}{2}|Dw|^2 - wf\, dx,\end{aligned}$$

where we employed the estimates

$$|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2,$$

following from the Cauchy–Schwarz and Cauchy inequalities (§B.2). Rearranging, we conclude

$$I[u] \leq I[w] \quad (w \in \mathcal{A}). \tag{48}$$

Since $u \in \mathcal{A}$, (47) follows from (48).

2. Now, conversely, suppose (47) holds. Fix any $v \in C_c^\infty(U)$ and write

$$i(\tau) := I[u + \tau v] \quad (\tau \in \mathbb{R}).$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\cdot)$ has a minimum at zero, and thus

$$i'(0) = 0 \quad \left(' = \frac{d}{d\tau}\right),$$

provided this derivative exists. But

$$\begin{aligned}i(\tau) &= \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v)f\, dx \\ &= \int_U \frac{1}{2}|Du|^2 + \tau Du \cdot Dv + \frac{\tau^2}{2}|Dv|^2 - (u + \tau v)f\, dx.\end{aligned}$$

Consequently

$$0 = i'(0) = \int_U Du \cdot Dv - vf\, dx = \int_U (-\Delta u - f)v\, dx.$$

This identity is valid for each function $v \in C_c^\infty(U)$ and so $-\Delta u = f$ in U. □

Dirichlet's principle is an instance of the *calculus of variations* applied to Laplace's equation. See Chapter 8 for more.

2.3. HEAT EQUATION

Next we study the *heat equation*

$$u_t - \Delta u = 0 \tag{1}$$

and the *nonhomogeneous heat equation*

$$u_t - \Delta u = f, \tag{2}$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, \dots, x_n)$: $\Delta u = \Delta_x u = \sum_{i=1}^n u_{x_i x_i}$. In (2) the function $f : U \times [0, \infty) \to \mathbb{R}$ is given.

A guiding principle is that any assertion about harmonic functions yields an analogous (but more complicated) statement about solutions of the heat equation. Accordingly our development will largely parallel the corresponding theory for Laplace's equation.

Physical interpretation. The heat equation, also known as the *diffusion equation*, describes in typical applications the evolution in time of the density u of some quantity such as heat, chemical concentration, etc. If $V \subset U$ is any smooth subregion, the rate of change of the total quantity within V equals the negative of the net flux through ∂V:

$$\frac{d}{dt} \int_V u \, dx = - \int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS,$$

$\mathbf{F}$ being the flux density. Thus

$$u_t = - \operatorname{div} \mathbf{F}, \tag{3}$$

as V was arbitrary. In many situations $\mathbf{F}$ is proportional to the gradient of u, but points in the opposite direction (since the flow is from regions of higher to lower concentration):

$$\mathbf{F} = -aDu \quad (a > 0).$$

Substituting into (3), we obtain the PDE

$$u_t = a \operatorname{div}(Du) = a\Delta u,$$

which for $a = 1$ is the heat equation.

The heat equation appears as well in the study of Brownian motion.

□

2.3.1. Fundamental solution.

a. Derivation of the fundamental solution.

As noted in §2.2.1 an important first step in studying any PDE is often to come up with some specific solutions.

We observe that the heat equation involves one derivative with respect to the time variable t, but two derivatives with respect to the space variables x_i $(i = 1, \dots, n)$. Consequently we see that if u solves (1), then so does $u(\lambda x, \lambda^2 t)$ for $\lambda \in \mathbb{R}$. This scaling indicates the ratio $\frac{r^2}{t}$ $(r = |x|)$ is important for the heat equation and suggests that we search for a solution of (1) having the form $u(x,t) = v(\frac{r^2}{t}) = v(\frac{|x|^2}{t})$ $(t > 0,\ x \in \mathbb{R}^n)$, for some function v as yet undetermined.

Although this approach eventually leads to what we want (see Problem 11), it is quicker to seek a solution u having the special structure

$$u(x,t) = \frac{1}{t^\alpha} v\Big(\frac{x}{t^\beta}\Big) \quad (x \in \mathbb{R}^n,\ t > 0), \tag{4}$$

where the constants α, β and the function $v : \mathbb{R}^n \to \mathbb{R}$ must be found. We come to (4) if we look for a solution u of the heat equation invariant under the *dilation scaling*

$$u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t).$$

That is, we ask

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

for all $\lambda > 0$, $x \in \mathbb{R}^n$, $t > 0$. Setting $\lambda = t^{-1}$, we derive (4) for $v(y) := u(y,1)$.

Let us insert (4) into (1), and thereafter compute

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \tag{5}$$

for $y := t^{-\beta} x$. In order to transform (5) into an expression involving the variable y alone, we take $\beta = \frac{1}{2}$. Then the terms with t are identical, and so (5) reduces to

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0. \tag{6}$$

We simplify further by guessing v to be radial; that is, $v(y) = w(|y|)$ for some $w : \mathbb{R} \to \mathbb{R}$. Thereupon (6) becomes

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0,$$

for $r = |y|$, $' = \frac{d}{dr}$. Now if we set $\alpha = \frac{n}{2}$, this simplifies to read

$$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0.$$

Thus

$$r^{n-1}w' + \frac{1}{2}r^n w = a$$

for some constant a. Assuming $\lim_{r\to\infty} w,\ w' = 0$, we conclude $a = 0$; whence

$$w' = -\frac{1}{2}rw.$$

But then for some constant b

$$w = be^{-\frac{r^2}{4}}. \tag{7}$$

Combining (4), (7) and our choices for α, β, we conclude that $\frac{b}{t^{n/2}}e^{-\frac{|x|^2}{4t}}$ solves the heat equation (1).

This computation motivates the following

DEFINITION. *The function*

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n,\ t > 0) \\ 0 & (x \in \mathbb{R}^n,\ t < 0) \end{cases}$$

is called the fundamental solution of the heat equation.

Notice that Φ is singular at the point $(0,0)$. We will sometimes write $\Phi(x,t) = \Phi(|x|,t)$ to emphasize that the fundamental solution is radial in the variable x. The choice of the normalizing constant $(4\pi)^{-n/2}$ is dictated by the following

LEMMA (Integral of fundamental solution). *For each time $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1.$$

Proof. We calculate

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)\, dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz \\ &= \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-z_i^2}\, dz_i = 1. \end{aligned}$$

□

A different derivation of the fundamental solution of the heat equation appears in §4.3.2.

b. Initial-value problem.

We now employ Φ to fashion a solution to the *initial-value* (or *Cauchy*) *problem*

$$\left\{\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\}. \end{aligned}\right. \tag{8}$$

Let us note the function $(x,t) \mapsto \Phi(x,t)$ solves the heat equation away from the singularity at $(0,0)$, and thus so does $(x,t) \mapsto \Phi(x-y,t)$ for each fixed $y \in \mathbb{R}^n$. Consequently the convolution

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n,\ t > 0) \end{aligned} \tag{9}$$

should also be a solution.

THEOREM 1 (Solution of initial-value problem). *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define u by (9). Then*

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t(x,t) - \Delta u(x,t) = 0 \quad (x \in \mathbb{R}^n,\ t > 0)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0)$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. 1. Since the function $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$ is infinitely differentiable, with uniformly bounded derivatives of all orders, on $\mathbb{R}^n \times [\delta,\infty)$ for each $\delta > 0$, we see that $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$. Furthermore

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x-y,t)] g(y)\, dy \\ &= 0 \quad (x \in \mathbb{R}^n,\ t > 0), \end{aligned} \tag{10}$$

since Φ itself solves the heat equation.

2. Fix $x^0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ such that

$$|g(y) - g(x^0)| < \varepsilon \quad \text{if } |y - x^0| < \delta,\ y \in \mathbb{R}^n. \tag{11}$$

Then if $|x - x^0| < \frac{\delta}{2}$, we have, according to the lemma,

$$\begin{aligned}
|u(x,t) - g(x^0)| &= \left|\int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x^0)]\,dy\right| \\
&\leq \int_{B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\,dy \\
&\quad + \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|\,dy \\
&=: I + J.
\end{aligned}$$

Now

$$I \leq \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t)\,dy = \varepsilon,$$

owing to (11) and the lemma. Furthermore, if $|x - x^0| \leq \frac{\delta}{2}$ and $|y - x^0| \geq \delta$, then

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|.$$

Thus $|y - x| \geq \frac{1}{2}|y - x^0|$. Consequently

$$\begin{aligned}
J &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)\,dy \\
&\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0,\delta)} e^{-\frac{|x-y|^2}{4t}}\,dy \\
&\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n - B(x^0,\delta)} e^{-\frac{|y-x^0|^2}{16t}}\,dy \\
&= \frac{C}{t^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{16t}} r^{n-1}\,dr \to 0 \quad \text{as } t \to 0^+.
\end{aligned}$$

Hence if $|x - x^0| < \frac{\delta}{2}$ and $t > 0$ is small enough, $|u(x,t) - g(x^0)| < 2\varepsilon$. □

Remarks. (i) In view of Theorem 1 we sometimes write

$$\begin{cases} \Phi_t - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi = \delta_0 & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

δ_0 denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0.

(ii) Notice that if g is bounded, continuous, $g \geq 0$, $g \not\equiv 0$, then

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy$$

is in fact positive for *all* points $x \in \mathbb{R}^n$ and times $t > 0$. We interpret this observation by saying the heat equation forces *infinite propagation speed*

for disturbances. If the initial temperature is nonnegative and is positive somewhere, the temperature at any later time (no matter how small) is everywhere positive. (We will learn in §2.4.3 that the wave equation in contrast supports finite propagation speed for disturbances.) $\square$

c. Nonhomogeneous problem.

Now let us turn our attention to the *nonhomogeneous* initial-value problem

$$(12)\qquad \begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\quad u = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

How can we produce a formula for the solution? If we recall the motivation leading up to (9), we should note further that the mapping $(x,t) \mapsto \Phi(x-y,t-s)$ is a solution of the heat equation (for given $y \in \mathbb{R}^n$, $0 < s < t$). Now for fixed s, the function

$$u = u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy$$

solves

$$(12_s)\qquad \begin{cases} u_t(\cdot;s) - \Delta u(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ \qquad\qquad\quad u(\cdot;s) = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{t=s\}, \end{cases}$$

which is just an initial-value problem of the form (8), with the starting time $t=0$ replaced by $t=s$, and g replaced by $f(\cdot,s)$. Thus $u(\cdot;s)$ is certainly not a solution of (12).

However *Duhamel's principle** asserts that we can build a solution of (12) out of the solutions of (12_s), by integrating with respect to s. The idea is to consider

$$u(x,t) = \int_0^t u(x,t;s)\,ds \quad (x \in \mathbb{R}^n,\ t \geq 0).$$

Rewriting, we have

$$(13)\qquad \begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dyds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dyds, \end{aligned}$$

for $x \in \mathbb{R}^n,\ t > 0$.

To confirm that formula (13) works, let us for simplicity assume $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and f has compact support.

*Duhamel's principle has wide applicability to linear ODE and PDE, and does not depend on the specific structure of the heat equation. It yields, for example, the solution of the nonhomogeneous transport equation, obtained by different means in §2.1.2. We will invoke Duhamel's principle for the wave equation in §2.4.2.

THEOREM 2 (Solution of nonhomogeneous problem). *Define u by* (13). *Then*

(i) $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$,

(ii) $u_t(x,t) - \Delta u(x,t) = f(x,t) \quad (x \in \mathbb{R}^n,\, t > 0)$,

and

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = 0$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. 1. Since Φ has a singularity at $(0,0)$, we cannot directly justify differentiating under the integral sign. We instead proceed somewhat as in the proof of Theorem 1 in §2.2.1.

First we change variables, to write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dyds.$$

As $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ has compact support and $\Phi = \Phi(y,s)$ is smooth near $s = t > 0$, we compute

$$\begin{aligned} u_t(x,t) = &\int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\, dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \end{aligned}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial^2}{\partial x_i \partial x_j} f(x-y,t-s)\, dyds \quad (i,j = 1,\dots,n).$$

Thus $u_t, D_x^2 u$, and likewise $u, D_x u$, belong to $C(\mathbb{R}^n \times (0,\infty))$.

2. We then calculate

(14)
$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)[(\frac{\partial}{\partial t} - \Delta_x) f(x-y,t-s)]\, dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]\, dyds \\ &\quad + \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)[(-\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]\, dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy. \\ &=: I_\varepsilon + J_\varepsilon + K. \end{aligned}$$

Now

$$|J_\varepsilon| \leq (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\, dyds \leq \varepsilon C, \tag{15}$$

by the lemma. Integrating by parts, we also find

$$\begin{aligned} I_\varepsilon = & \int_\varepsilon^t \int_{\mathbb{R}^n} [(\frac{\partial}{\partial s} - \Delta_y)\Phi(y,s)] f(x-y,t-s)\, dyds \\ & + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy \\ & - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ = & \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy - K, \end{aligned} \tag{16}$$

since Φ solves the heat equation. Combining (14)–(16), we ascertain

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy \\ &= f(x,t) \quad (x \in \mathbb{R}^n,\ t > 0), \end{aligned}$$

the limit as $\varepsilon \to 0$ being computed as in the proof of Theorem 1. Finally note $\|u(\cdot,t)\|_{L^\infty} \leq t\|f\|_{L^\infty} \to 0$. □

Remark. We can of course combine Theorems 1 and 2 to discover that

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dyds \tag{17}$$

is, under the hypotheses on g and f as above, a solution of

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{18}$$

□

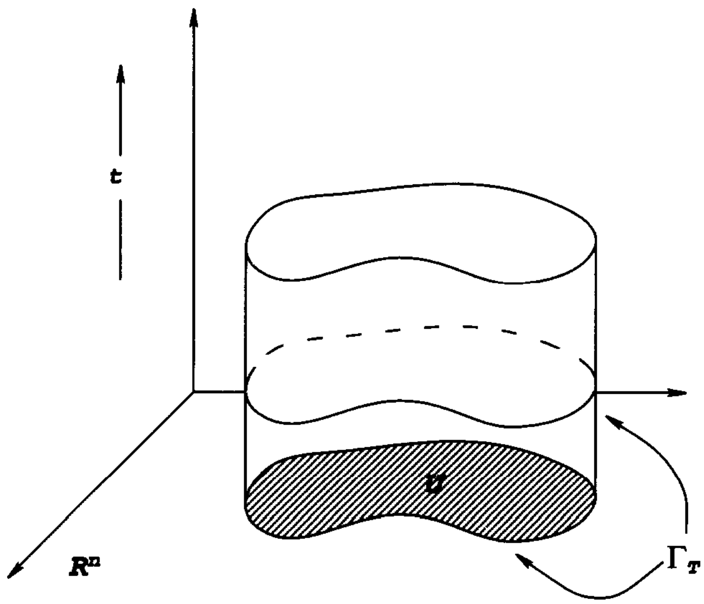
2.3.2. Mean-value formula.

First we recall some useful notation from §A.2. Assume $U \subset \mathbb{R}^n$ is open and bounded, and fix a time $T > 0$.

DEFINITIONS.

(i) *We define the* parabolic cylinder

$$U_T := U \times (0,T].$$



The region $\mathbf{U}_T$

(ii) *The* parabolic boundary *of U_T is*

$$\Gamma_T := \bar{U}_T - U_T.$$

We interpret U_T as being the *parabolic interior* of $\bar{U} \times [0,T]$: note carefully that U_T includes the top $U \times \{t = T\}$. The parabolic boundary Γ_T comprises the bottom and vertical sides of $U \times [0,T]$, but not the top.

We want next to derive a kind of analogue to the mean-value property for harmonic functions, as discussed in §2.2.2. There is no such simple formula. However let us observe that for fixed x the spheres $\partial B(x,r)$ are level sets of the fundamental solution $\Phi(x-y)$ for Laplace's equation. This suggests that perhaps for fixed (x,t) the level sets of fundamental solution $\Phi(x-y,t-s)$ for the heat equation may be relevant.

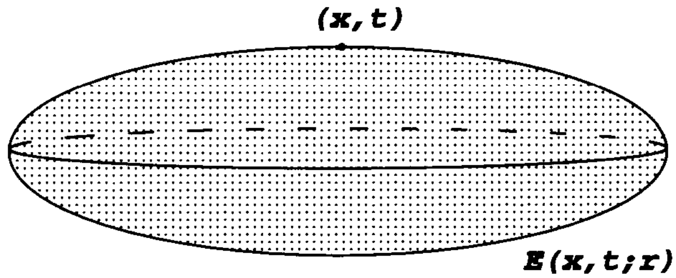
DEFINITION. *For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define*

$$E(x,t;r) := \left\{(y,s) \in \mathbb{R}^{n+1} \mid s \leq t,\ \Phi(x-y,t-s) \geq \frac{1}{r^n}\right\}.$$

This is a region in space-time, the boundary of which is a level set of $\Phi(x-y,t-s)$. Note that the point (x,t) is at the center of the top. $E(x,t;r)$ is sometimes called a "heat ball".

THEOREM 3 (A mean-value property for the heat equation). *Let $u \in C^2_1(U_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\,dyds \tag{19}$$



A "heat ball"

for each $E(x, t; r) \subset U_T$.

Formula (19) is a sort of analogue for the heat equation of the mean-value formulas for Laplace's equation. Observe that the right hand side involves only $u(y, s)$ for times $s \leq t$. This is reasonable, as the value $u(x, t)$ should not depend upon future times.

Proof. We may as well assume upon translating the space and time coordinates that $x = 0$ and $t = 0$. Write $E(r) = E(0, 0; r)$ and set

$$
\begin{aligned}
\phi(r) &:= \frac{1}{r^n} \iint_{E(r)} u(y, s) \frac{|y|^2}{s^2}\, dyds \\
&= \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dyds.
\end{aligned} \tag{20}
$$

We compute

$$
\begin{aligned}
\phi'(r) &= \iint_{E(1)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2ru_s \frac{|y|^2}{s}\, dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s}\, dyds \\
&=: A + B.
\end{aligned}
$$

Also, let us introduce the useful function

$$
\psi := -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r, \tag{21}
$$

and observe $\psi = 0$ on $\partial E(r)$, since $\Phi(y, -s) = r^{-n}$ on $\partial E(r)$. We utilize (21) to write

$$
\begin{aligned}
B &= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i}\, dyds \\
&= -\frac{1}{r^{n+1}} \iint_{E(r)} 4nu_s \psi + 4 \sum_{i=1}^{n} u_{sy_i} y_i \psi\, dyds;
\end{aligned}
$$

there is no boundary term since $\psi = 0$ on $\partial E(r)$. Integrating by parts with respect to s, we discover

$$\begin{aligned}
B &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i \psi_s \, dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2}\right) dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i \, dyds - A.
\end{aligned}$$

Consequently, since u solves the heat equation,

$$\begin{aligned}
\phi'(r) &= A + B \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} -4n\Delta u\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i \, dyds \\
&= \sum_{i=1}^{n} \frac{1}{r^{n+1}} \iint_{E(r)} 4nu_{y_i}\psi_{y_i} - \frac{2n}{s} u_{y_i} y_i \, dyds \\
&= 0, \quad \text{according to (21).}
\end{aligned}$$

Thus ϕ is constant, and therefore

$$\phi(r) = \lim_{t\to 0} \phi(t) = u(0,0) \left(\lim_{t\to 0} \frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds\right) = 4u(0,0),$$

as

$$\frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds = \iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4.$$

We omit the details of this last computation. □

2.3.3. Properties of solutions.

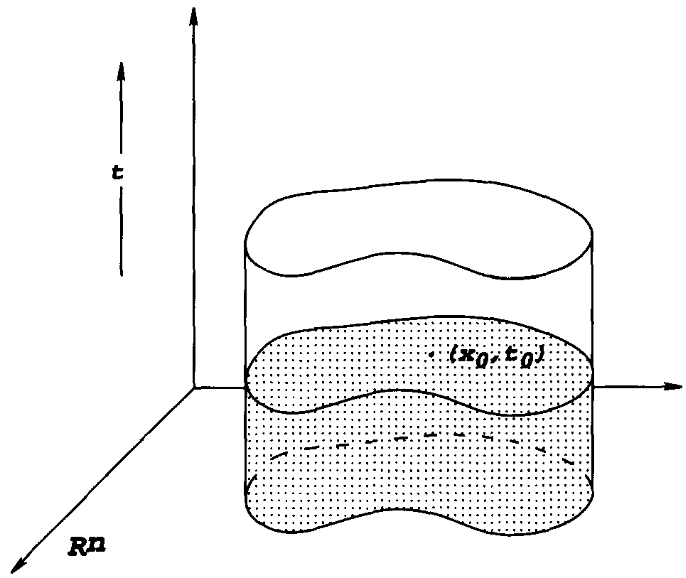
a. Strong maximum principle, uniqueness.

First we employ the mean-value property to give a quick proof of the strong maximum principle.

THEOREM 4 (Strong maximum principle for the heat equation). *Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation in U_T.*

(i) *Then*

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$



Strong maximum principle for the heat equation

(ii) *Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that*

$$u(x_0, t_0) = \max_{\bar{U}_T} u,$$

then

$$u \text{ is constant in } \bar{U}_{t_0}.$$

Assertion (i) is the *maximum principle* for the heat equation and (ii) is the *strong maximum principle*. Similar assertions are valid with "min" replacing "max".

Remark. *So if u attains its maximum (or minimum) at an interior point, then u is constant at all earlier times.* This accords with our strong intuitive interpretation of the variable t as denoting time: the solution will be constant on the time interval $[0, t_0]$ provided the initial and boundary conditions are constant. However, the solution may change at times $t > t_0$, provided the boundary conditions alter after t_0. The solution will however not respond to changes in boundary conditions until these changes happen.

Take note that whereas all this is obvious on intuitive, physical grounds, such insights do not constitute a proof. The task is to *deduce* such behavior from the PDE. □

Proof. 1. Suppose there exists a point $(x_0, t_0) \in U_T$ with $u(x_0, t_0) = M := \max_{\bar{U}_T} u$. Then for all sufficiently small $r > 0$, $E(x_0, t_0; r) \subset U_T$; and we

employ the mean-value property to deduce

$$M = u(x_0, t_0) = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dyds \le M,$$

since

$$1 = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dyds.$$

Equality holds only if u is identically equal to M within $E(x_0, t_0; r)$. Consequently

$$u(y,s) = M \qquad \text{for all } (y,s) \in E(x_0, t_0; r).$$

Draw any line segment L in U_T connecting (x_0, t_0) with some other point $(y_0, s_0) \in U_T$, with $s_0 < t_0$. Consider

$$r_0 := \min\{s \ge s_0 \mid u(x,t) = M \text{ for all points } (x,t) \in L, \ s \le t \le t_0\}.$$

Since u is continuous, the minimum is attained. Assume $r_0 > s_0$. Then $u(z_0, r_0) = M$ for some point (z_0, r_0) on $L \cap U_T$ and so $u \equiv M$ on $E(z_0, r_0; r)$ for all sufficiently small $r > 0$. Since $E(z_0, r_0; r)$ contains $L \cap \{r_0 - \sigma \le t \le r_0\}$ for some small $\sigma > 0$, we have a contradiction. Thus $r_0 = s_0$, and hence $u \equiv M$ on L.

2. Now fix any point $x \in U$ and any time $0 \le t < t_0$. There exist points $\{x_0, x_1, \dots, x_m = x\}$ such that the line segments in $\mathbb{R}^n$ connecting x_{i-1} to x_i lie in U for $i = 1, \dots, m$. (This follows since the set of points in U which can be so connected to x_0 by a polygonal path is nonempty, open and relatively closed in U.) Select times $t_0 > t_1 > \dots > t_m = t$. Then the line segments in $\mathbb{R}^{n+1}$ connecting (x_{i-1}, t_{i-1}) to (x_i, t_i) $(i = 1, \dots, m)$ lie in U_T. According to Step 1, $u \equiv M$ on each such segment and so $u(x,t) = M$. □

Remark. The strong maximum principle implies that if U is connected and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

where $g \ge 0$, then u is positive *everywhere* within U_T if g is positive *somewhere* on U. This is another illustration of infinite propagation speed for disturbances. □

An important application of the maximum principle is the following uniqueness assertion.

THEOREM 5 (Uniqueness on bounded domains). *Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of the initial/boundary-value problem*

$$\text{(22)} \qquad \begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \qquad\; u = g & \text{on } \Gamma_T. \end{cases}$$

Proof. If u and $\tilde{u}$ are two solutions of (22), apply Theorem 4 to $w := \pm(u - \tilde{u})$. □

We next extend our uniqueness assertion to the *Cauchy problem*, that is, the initial value problem for $U = \mathbb{R}^n$. As we are no longer on a bounded region, we must introduce some control on the behavior of solutions for large $|x|$.

THEOREM 6 (Maximum principle for the Cauchy problem). *Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves*

$$\text{(23)} \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ \qquad\; u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

and satisfies the growth estimate

$$\text{(24)} \qquad u(x,t) \leq Ae^{a|x|^2} \quad (x \in \mathbb{R}^n, 0 \leq t \leq T)$$

for constants $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.$$

Proof. 1. First assume

$$\text{(25)} \qquad 4aT < 1;$$

in which case

$$\text{(26)} \qquad 4a(T + \varepsilon) < 1$$

for some $\varepsilon > 0$. Fix $y \in \mathbb{R}^n$, $\mu > 0$, and define

$$v(x,t) := u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \quad (x \in \mathbb{R}^n,\ t > 0).$$

A direct calculation (cf. §2.3.1) shows

$$v_t - \Delta v = 0 \quad \text{in } \mathbb{R}^n \times (0,T].$$

Fix $r > 0$ and set $U := B^0(y,r)$, $U_T = B^0(y,r) \times (0,T]$. Then according to Theorem 4,

$$\max_{\bar{U}_T} v = \max_{\Gamma_T} v. \tag{27}$$

2. Now if $x \in \mathbb{R}^n$,

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \\ &\leq u(x,0) = g(x); \end{aligned} \tag{28}$$

and if $|x-y| = r$, $0 \leq t \leq T$, then

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\ &\leq Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \qquad \text{by (24)} \\ &\leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}}. \end{aligned}$$

Now according to (26), $\frac{1}{4(T+\varepsilon)} = a+\gamma$ for some $\gamma > 0$. Thus we may continue the calculation above to find

$$v(x,t) \leq Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2} \leq \sup_{\mathbb{R}^n} g, \tag{29}$$

for r selected sufficiently large. Thus (27)–(29) imply

$$v(y,t) \leq \sup_{\mathbb{R}^n} g$$

for all $y \in \mathbb{R}^n$, $0 \leq t \leq T$, provided (25) is valid. Let $\mu \to 0$.

3. In the general case that (25) fails, we repeatedly apply the result above on the time intervals $[0,T_1]$, $[T_1, 2T_1,]$, etc., for $T_1 = \frac{1}{8a}$. □

THEOREM 7 (Uniqueness for Cauchy problem). *Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of the initial-value problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{30}$$

satisfying the growth estimate

$$|u(x,t)| \leq Ae^{a|x|^2} \quad (x \in \mathbb{R}^n,\ 0 \leq t \leq T) \tag{31}$$

for constants A, $a > 0$.

Proof. If u and $\tilde{u}$ both satisfy (30), (31), we apply Theorem 6 to $w := \pm(u - \tilde{u})$. $\square$

Remark. There are in fact infinitely many solutions of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}; \end{cases} \tag{32}$$

see for instance John [**J**, Chapter 7]. Each of the solutions besides $u \equiv 0$ grows very rapidly as $|x| \to \infty$.

There is an interesting point here: although $u \equiv 0$ is certainly the "physically correct" solution of (32), this initial-value problem in fact admits other, "nonphysical" solutions. Theorem 7 provides a criterion which excludes the "wrong" solutions. We will encounter somewhat analogous situations in our study of Hamilton–Jacobi equations and conservation laws, in Chapters 3, 10 and 11. $\square$

b. Regularity.

We next demonstrate that solutions of the heat equation are automatically smooth.

THEOREM 8 (Smoothness). *Suppose $u \in C^2_1(U_T)$ solves the heat equation in U_T. Then*

$$u \in C^\infty(U_T).$$

This regularity assertion is valid even if u attains nonsmooth boundary values on Γ_T.

Proof. 1. Recall from §A.2 that we write

$$C(x,t;r) = \{(y,s) \mid |x-y| \leq r,\ t - r^2 \leq s \leq t\}$$

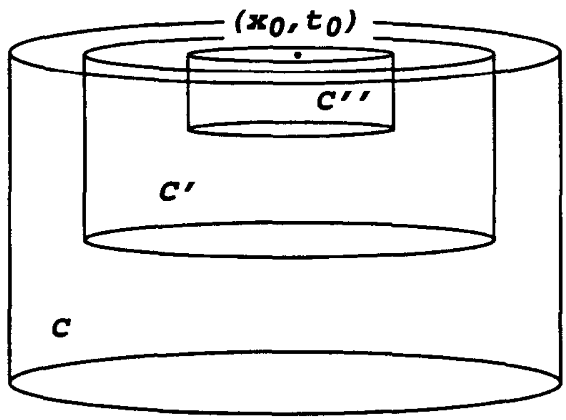
to denote the closed circular cylinder of radius r, height r^2, and top center point (x,t).

Fix $(x_0,t_0) \in U_T$ and choose $r > 0$ so small that $C := C(x_0,t_0;r) \subset U_T$. Define also the smaller cylinders $C' := C(x_0,t_0;\frac{3}{4}r)$, $C'' := C(x_0,t_0;\frac{1}{2}r)$, which have the same top center point (x_0,t_0).

Choose a smooth cutoff function $\zeta = \zeta(x,t)$ such that

$$\begin{cases} 0 \leq \zeta \leq 1,\ \zeta \equiv 1 \text{ on } C', \\ \zeta \equiv 0 \text{ near the parabolic boundary of } C. \end{cases}$$

Extend $\zeta \equiv 0$ in $(\mathbb{R}^n \times [0,t_0]) - C$.



2. Assume temporarily that $u \in C^\infty(U_T)$ and set

$$v(x,t) := \zeta(x,t)u(x,t) \quad (x \in \mathbb{R}^n,\, 0 \le t \le t_0).$$

Then

$$v_t = \zeta u_t + \zeta_t u, \;\; \Delta v = \zeta \Delta u + 2D\zeta \cdot Du + u\Delta\zeta.$$

Consequently

$$v = 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{33}$$

and

$$v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta =: \tilde{f} \tag{34}$$

in $\mathbb{R}^n \times (0, t_0)$. Now set

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds.$$

According to Theorem 2

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} & \text{in } \mathbb{R}^n \times (0, t_0) \\ \tilde{v} = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{35}$$

Since $|v|, |\tilde{v}| \le A$ for some constant A, Theorem 7 implies $v \equiv \tilde{v}$; that is,

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds. \tag{36}$$

Now suppose $(x,t) \in C''$. As $\zeta \equiv 0$ off the cylinder C, (34) and (36) imply

$$\begin{aligned} u(x,t) = \iint_C \Phi(x-y, t-s)[(\zeta_s(y,s) - \Delta\zeta(y,s))u(y,s) \\ - 2D\zeta(y,s) \cdot Du(y,s)]\, dyds. \end{aligned}$$

Note in this expression that the expression in the square brackets vanishes in some region *near* the singularity of Φ. Integrate the last term by parts:

$$
\begin{aligned}
(37)\qquad u(x,t) = \iint_C [\Phi(x-y,t-s)(\zeta_s(y,s) + \Delta\zeta(y,s)) \\
+ 2D_y\Phi(x-y,t-s)\cdot D\zeta(y,s)]u(y,s)\,dyds.
\end{aligned}
$$

We have proved this formula assuming $u \in C^\infty$. If u satisfies only the hypotheses of the theorem, we derive (37) with $u^\varepsilon = \eta_\varepsilon * u$ replacing u, η_ε being the standard mollifier in the variables x and t, and let $\varepsilon \to 0$.

3. Formula (37) has the form

$$
(38)\qquad u(x,t) = \iint_C K(x,t,y,s)u(y,s)\,dyds \quad ((x,t) \in C''),
$$

where

$$
K(x,t,y,s) = 0 \quad \text{for all points } (y,s) \in C',
$$

since $\zeta \equiv 1$ on C'. Note also K is smooth on $C - C'$. In view of expression (38), we see u is C^∞ within $C'' = C(x_0,t_0;\frac{1}{2}r)$. □

c. Local estimates for solutions of the heat equation.

Next we record some estimates on the derivatives of solutions to the heat equation, paying attention to the differences between derivatives with respect to x_i $(i = 1,\dots,n)$ and with respect to t.

THEOREM 9 (Estimates on derivatives). *There exists for each pair of integers $k,l = 0,1,\dots,$ a constant $C_{k,l}$ such that*

$$
\max_{C(x,t;\,r/2)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t;r))}
$$

for all cylinders $C(x,t;r/2) \subset C(x,t;r) \subset U_T$, and all solutions u of the heat equation in U_T.

Proof. 1. Fix some point in U_T. Upon shifting the coordinates, we may as well assume the point is $(0,0)$. Suppose first that the cylinder $C(1) := C(0,0;1)$ lies in U_T. Let $C\left(\frac{1}{2}\right) := C\left(0,0;\frac{1}{2}\right)$. Then, as in the proof of Theorem 8,

$$
u(x,t) = \iint_{C(1)} K(x,t,y,s)u(y,s)\,dyds \quad ((x,t) \in C(\tfrac{1}{2}))
$$

for some smooth function K. Consequently

$$
\begin{aligned}
|D_x^k D_t^l u(x,t)| &\leq \iint_{C(1)} |D_t^l D_x^k K(x,t,y,s)||u(y,s)|\,dyds \\
&\leq C_{kl}\|u\|_{L^1(C(1))}
\end{aligned} \tag{39}
$$

for some constant C_{kl}.

2. Now suppose the cylinder $C(r) := C(0,0;r)$ lies in U_T. Let $C(r/2) = C(0,0;r/2)$. We rescale by defining

$$v(x,t) := u(rx, r^2t).$$

Then $v_t - \Delta v = 0$ in the cylinder $C(1)$. According to (39),

$$|D_x^k D_t^l v(x,t)| \leq C_{kl}\|v\|_{L^1(C(1))} \quad ((x,t) \in C(\tfrac{1}{2})).$$

But $D_x^k D_t^l v(x,t) = r^{2l+k} D_x^k D_t^l u(rx, r^2t)$ and $\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}}\|u\|_{L^1(C(r))}$. Therefore

$$\max_{C(r/2)} |D_x^k D_t^l u| \leq \frac{C_{kl}}{r^{2l+k+n+2}}\|u\|_{L^1(C(r))}.$$

□

Remark. If u solves the heat equation within U_T, then for each fixed time $0 < t \leq T$, the mapping $x \mapsto u(x,t)$ is analytic. (See Mikhailov **[M]**.) However the mapping $t \mapsto u(x,t)$ is not in general analytic. □

2.3.4. Energy methods.

a. Uniqueness.

Let us investigate again the initial/boundary-value problem

$$
\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T. \end{cases} \tag{40}
$$

We earlier invoked the maximum principle to show uniqueness, and now—by analogy with §2.2.5—provide an alternative argument based upon integration by parts. We assume as usual that $U \subset \mathbb{R}^n$ is open, bounded and that ∂U is C^1. The terminal time $T > 0$ is given.

THEOREM 10 (Uniqueness). *There exists at most one solution $u \in C_1^2(\bar{U}_T)$ of (40).*

Proof. 1. If $\tilde{u}$ is another solution, $w := u - \tilde{u}$ solves

$$\begin{cases} w_t - \Delta w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T. \end{cases} \tag{41}$$

2. Set

$$e(t) := \int_U w^2(x,t)\,dx \quad (0 \leq t \leq T).$$

Then

$$\begin{aligned} \dot{e}(t) &= 2\int_U w w_t\,dx \quad \left(\cdot = \frac{d}{dt}\right) \\ &= 2\int_U w \Delta w\,dx \\ &= -2\int_U |Dw|^2 dx \leq 0, \end{aligned}$$

and so

$$e(t) \leq e(0) = 0 \quad (0 \leq t \leq T).$$

Consequently $w = u - \tilde{u} \equiv 0$ in U_T. □

Observe that the foregoing is a time-dependent variant of the proof of Theorem 16 in §2.2.5.

b. Backwards uniqueness.

A rather more subtle question concerns uniqueness *backwards in time* for the heat equation. For this, suppose u and $\tilde{u}$ are both smooth solutions of the heat equation in U_T, with the same boundary conditions on ∂U:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = g & \text{on } \partial U \times [0,T], \end{cases} \tag{42}$$

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 & \text{in } U_T \\ \tilde{u} = g & \text{on } \partial U \times [0,T], \end{cases} \tag{43}$$

for some function g. Note carefully that we are *not* supposing $u = \tilde{u}$ at time $t = 0$.

THEOREM 11 (Backwards uniqueness). *Suppose $u, \tilde{u} \in C^2(\bar{U}_T)$ solve* (42), (43). *If*

$$u(x,T) = \tilde{u}(x,T) \quad (x \in U),$$

then

$$u \equiv \tilde{u} \quad \textit{within } U_T.$$

In other words, if two temperature distributions on U agree at some time $T > 0$, and have had the same boundary values for times $0 \leq t \leq T$, then these temperatures must have been identically equal within U at all earlier times. This is not at all obvious.

Proof. 1. Write $w := u - \tilde{u}$ and, as in the proof of Theorem 10, set

$$e(t) := \int_U w^2(x,t)\,dx \quad (0 \leq t \leq T).$$

As before

$$\dot{e}(t) = -2 \int_U |Dw|^2 dx \quad \left(\dot{} = \frac{d}{dt} \right). \tag{44}$$

Furthermore

$$\begin{aligned} \ddot{e}(t) &= -4 \int_U Dw \cdot Dw_t\,dx \\ &= 4 \int_U \Delta w w_t\,dx \\ &= 4 \int_U (\Delta w)^2 dx \quad \text{by (41).} \end{aligned} \tag{45}$$

Now since $w = 0$ on ∂U,

$$\begin{aligned} \int_U |Dw|^2 dx &= -\int_U w \Delta w\,dx \\ &\leq \left(\int_U w^2 dx \right)^{1/2} \left(\int_U (\Delta w)^2 dx \right)^{1/2}. \end{aligned}$$

Thus (44) and (45) imply

$$\begin{aligned} (\dot{e}(t))^2 &= 4 \left(\int_U |Dw|^2 dx \right)^2 \\ &\leq \left(\int_U w^2 dx \right) \left(4 \int_U (\Delta w)^2 dx \right) \\ &= e(t)\ddot{e}(t). \end{aligned}$$

Hence

$$\ddot{e}(t)e(t) \geq (\dot{e}(t))^2 \quad (0 \leq t \leq T). \tag{46}$$

2. Now if $e(t) = 0$ for all $0 \leq t \leq T$, we are done. Otherwise there exists an interval $[t_1, t_2] \subset [0, T]$, with

$$e(t) > 0 \quad \text{for } t_1 \leq t < t_2, \quad e(t_2) = 0. \tag{47}$$

3. Now write

$$f(t) := \log e(t) \quad (t_1 \leq t < t_2). \tag{48}$$

Then

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geq 0 \quad \text{by (46);}$$

and so f is convex on the interval (t_1, t_2). Consequently if $0 < \tau < 1$, $t_1 < t < t_2$, we have

$$f((1-\tau)t_1 + \tau t) \leq (1-\tau) f(t_1) + \tau f(t).$$

Recalling (48), we deduce

$$e((1-\tau)t_1 + \tau t) \leq e(t_1)^{1-\tau} e(t)^\tau,$$

and so

$$0 \leq e((1-\tau)t_1 + \tau t_2) \leq e(t_1)^{1-\tau} e(t_2)^\tau \quad (0 < \tau < 1).$$

But in view of (47) this inequality implies $e(t) = 0$ for all times $t_1 \leq t \leq t_2$, a contradiction. □

2.4. WAVE EQUATION

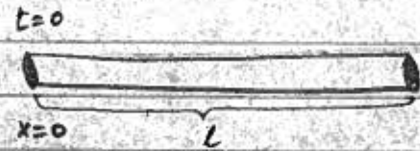
In this section we investigate the *wave equation*

$$u_{tt} - \Delta u = 0 \tag{1}$$

and the *nonhomogeneous wave equation*

$$u_{tt} - \Delta = f, \tag{2}$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables



Ex-1:

Consider a rod of length l with insulated sides is given an initial temperature distribution of $f(x)$ degree C, for $0<x<l$. Find $u(x,t)$ at subsequent times $t>0$ if end of the rod are kept $0°c$.

$$\begin{cases} u_t = u_{xx} & 0<x<l \\ u(x,0) = f(x) & 0<x<l \\ u(0,t) = u(l,t) = 0 & t>0 \end{cases}$$

Conservation of energy:

Consider a uniform rod of length l with non-uniform temperature lying on the x-axis from $x=0$ and $x=l$.
By uniform rod, we mean the density ρ, specific heat c, thermal conductivity k, cross-sectional area A are all constant.

Assume the sides of the rod are insulated and only the ends may be exposed. Also assume there is no heat source within the rod. Consider arbitrary thin slice of the rod of width Δx between x and $x+\Delta x$. The slice is so thin that the temperature throughout the slice is $u(x,t)$.

Thus, heat energy of segment $= c\rho A \Delta x u = c\rho A \Delta x u(x,t)$

By conservation of energy,

$$\left\{ \begin{array}{l} \text{Change of heat energy of} \\ \text{segment in time } \Delta t \end{array} \right\} = \begin{array}{l} \text{heat in from left} \\ \text{boundary} \end{array} - \begin{array}{l} \text{heat out from right} \\ \text{boundary.} \end{array}$$

$$\frac{\text{rate of heat transfer}}{\text{area}} = -k\frac{\partial u}{\partial x}$$

From Fourier's Law, rate of heat transfer $\propto$ − temperature gradient.

$$c\rho A\Delta x\, u(x, t+\Delta t) - c\rho A\Delta x\, u(x,t) = \Delta t A\left(-k\frac{\partial u}{\partial x}\right)_x - \Delta t A\left(-k\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

where ρ, c, A, k are constant

$$\Rightarrow \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \frac{k}{c\rho}\,\frac{\left(\frac{\partial u}{\partial x}\right)_x - \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}}{\Delta x}$$

Letting $\Delta t, \Delta x \to 0$, so that

$$\frac{\partial u}{\partial t} = \varepsilon\frac{\partial u}{\partial x^2} \qquad \text{where, } \varepsilon = \frac{k}{c\rho}$$

If Ω is a bounded domain in $\mathbb{R}^n$ that we view as a physical body with constant heat conductivity k, then the heat or diffusion equation

$$u_t = k \Delta u \quad \text{for } x \in \Omega \text{ and } t > 0$$

where $u(x,t)$ represents the temperature of the body at point x and time t. We generally take $k=1$, otherwise $t \to kt$.

The appropriate side conditions are the initial temperature.

$$u(x,0) = g(x)$$

Dirichlet: If the temperature controlled on the boundary $\partial\Omega$.
Set $\Omega = (0,l)$, then $u(0,t) = 0 = u(l,t)$

Neumann: If the heat flow across $\partial\Omega$ is controlled.

$$u_x(0,t) = 0 = u_x(l,t)$$

Robin: If heat flow obeys Newton's law of cooling:

$$u_x(0,t) - a_0 u(0,t) = 0$$

$$\left.\begin{array}{l} u_t = \Delta u \quad \text{for } x \in \Omega \text{ and } t>0. \\ u(x,0)\big|_{\partial\Omega} = g(x). \\ u(0,t) = 0 = u(L,t) \qquad \Omega = (0,L). \end{array}\right\} \text{ --- } (*)$$

Let $u(x,t) = X(x)T(t)$. Then, $u_t = X(x)T'(t)$, $u_{xx} = X''(x)\,T(t)$.

So, $XT' = X''T$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{T} = \cancel{p \text{ where } p \text{ is a constant}}$$

Then $\frac{X''}{X} = P \Rightarrow X'' - PX = 0$

<u>Case I:</u> $P=0$

$$X'' = 0 \Rightarrow X = c_1 x + c_2 \Rightarrow X(0) = c_2$$

Since $X(0)=0$, then $c_2 = 0$.

$$X(L) = c_1 L + c_2$$

Since $X(L)=0$, then $Lc_1 = 0 \Rightarrow c_1 = 0$ (trivial)

<u>Case II:</u> $P>0$. Let $P = \mu^2$ ($\mu \neq 0$)

Then $X'' - \mu^2 X = 0$

Characteristic eqn: $r^2 - \mu^2 = 0 \Rightarrow r = \pm\mu$.

General solution $X_1(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

Then $X_1(0) = c_1 + c_2 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$X_1(L) = c_1 e^{\mu L} + c_2 e^{-\mu L} \Rightarrow 0 = c_1 e^{\mu L} - c_1 e^{-\mu L}$

~~If~~ If $c_1 \neq 0$, then $e^{\mu L} = -e^{-\mu L} \Rightarrow \mu = 0$ (not possible)

If $c_1 = 0$, then $c_2 = 0$ (trivial)

Case III: $P<0$. Let $P=-\mu^2$ $(\mu\neq 0)$.

$$X''+\mu^2 X=0$$
$$\Rightarrow r^2+\mu^2=0 \Rightarrow r=\pm i\mu$$

General solution: $X(x)=C_1\cos\mu x+C_2\sin\mu x$

Since $X(0)=0$, then $0=C_1$. So, $X=C_2\sin\mu x$

Since $X(L)=0$, then $0=C_2\sin(\mu L)$

$$\Rightarrow C_2\sin(\mu L)=0$$

Take $C_2=1$ or $\sin\mu L=0$

$$\mu L=n\pi,\ \mu=\mu_n=\frac{n\pi}{L},\ n\in\mathbb{Z}$$

Thus. $X_n(x)=\sin\left(\frac{n\pi}{L}x\right)$, $n=1,2,\cdots$

Now, $\frac{T'}{T}=P \Rightarrow T'-PT=0$

$$\Rightarrow T'+\left(\frac{n\pi}{L}\right)^2 T=0 \quad \text{since } P=-\mu^2=-\left(\frac{n\pi}{L}\right)^2$$

$$\Rightarrow \frac{dT}{T}=-\left(\frac{n\pi}{L}\right)^2 dt$$

$$\Rightarrow \ln T=-\left(\frac{n\pi}{L}\right)^2 t+c$$

$$\Rightarrow T=a_1 e^{-\left(\frac{n\pi}{L}\right)^2 t}.$$

Hence $T_n=a_n e^{-\left(\frac{n\pi}{L}\right)^2 t}=a_n e^{-\lambda_n t}$ set $\lambda_n=\left(\frac{n\pi}{L}\right)^2$

Then, $u_n(x,t)=X_n(x)\,T_n(t)$

For each eigenfunction X_n with corresponding eigenvalue λ_n, we have a solution T_n such that

$$u_n(x,t) = X_n(x) \cdot T_n(t)$$ is a solution of heat equation.

Note that we have not accounted for our initial condition $u(x,0) = g(x)$.

If $\{u_n\}$ is a sequence of solutions of the heat equation on Ω which satisfies our boundary conditions, then any finite linear combination of these solutions will also give us a solution. That is

$$u(x,t) = \sum_{n=1}^{N} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \sin\left(\frac{n\pi}{L} x\right)$$

$$g(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L} x\right)$$

$$a_n = \frac{2}{L} \int_{\Omega} g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

set $\varphi_n(x) = \sin\left(\frac{n\pi}{L} x\right)$, $g(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$

where $a_n = \frac{2}{L} \int_{\Omega} g(x) \varphi_n(x)\, dx$.

Since $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \varphi_n(x)$, $\varphi_n(x) = \sin\left(\frac{n\pi}{L} x\right)$

and $g(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$ where $a_n = \int_{\Omega} g(x)\,\varphi_n(x)\,dx$, then

$$u(x,t) = \int_{\Omega} \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x)\varphi_n(y)\, g(y)\, dy.$$

If $k(x,y,t) := \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x)\varphi_n(y)$, then $u(x,t) = \int_{\Omega} k(x,y,t)\, g(y)\, dt$

The integral kernel $k(x,y,t)$ is called the heat kernel.

The method of eigenfunction expansion may also be applied to cover more general problems than (*), namely

$$(**)\begin{cases} u_t = \Delta u + f(x,t) & \text{for } x \in \Omega \text{ and } t > 0 \\ u(x,0) = g(x) & \text{for } x \in \bar{\Omega} \\ u(x,t) = h(x,t) & \text{for } x \in \partial\Omega \text{ and } t > 0. \end{cases}$$

in which $f(x,t)$ represents a forching term, such as a heat source, and $h(x,t)$ represents a nonhomogeneous temperature controle on $\partial\Omega$.

We look for solutions u in the form $u(x,t) = T(t)\,X(x)$.
As before we look at the eigenvalue problem.

$$X'' + \lambda X = 0$$
$$X(0) = X(L) = 0$$

The eigenvalues and the corresponding eigenfunctions are

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

Now, we set,

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right)$$

Then
$$u_x(x,t) = \sum_{n=1}^{\infty} T_n(t)\, \frac{n\pi}{L} \cos\left(\frac{n\pi}{L}x\right)$$

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} T_n(t) \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} T_n'(t) \sin\left(\frac{n\pi}{L}x\right)$$

So, $f(x,t) = u_t - \Delta u$

$$f(x,t) = \sum_{n=1}^{\infty} \left(T_n'(t) + T_n(t)\left(\frac{n\pi}{L}\right)^2\right) \sin\left(\frac{n\pi}{L}x\right)$$

Hence expanding f and $h(x,t)$ into the Fourier series

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\, \sin\left(\frac{n\pi}{L}x\right) \qquad \text{set } f_n(t) = T_n'(t) + T_n(t)\left(\frac{n\pi}{L}\right)^2$$

$$g(x,t) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right)$$

where, $f_n(t) = \frac{2}{L}\int_0^L f(x,t)\sin\left(\frac{n\pi}{L}x\right)dx$ & $g_n = \frac{2}{L}\int_0^L g(x,t)\sin\left(\frac{n\pi}{L}x\right)dx$

The uniqueness of the Fourier expansion leads to the family of ODE's.

$$T_n'(t) + \left(\frac{n\pi}{L}\right)^2 T_n = f_n(t) \quad \text{—— (i)}$$

In addition, $u(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right) = g(x)$

$$\Rightarrow T_n(0) = g_n \qquad n \geq 1 \quad \text{—— (ii)}$$

Solving (i) & (ii) $\Rightarrow$ $\mu = e^{\left(\frac{n\pi}{L}\right)^2 t}$

$$\text{So, } T_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t}\left[\int_0^t e^{\left(\frac{n\pi}{L}\right)^2 t} f_n(t)\,dt + c\right]$$

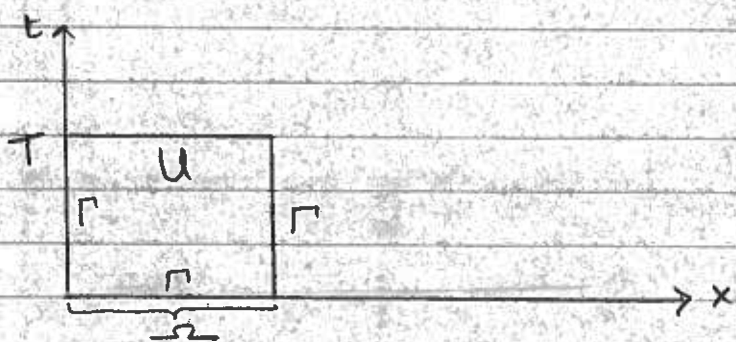
$$T_n(0) = c \Rightarrow c = g_n$$

$$\text{Then } T_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t}\left[\int_0^t e^{\left(\frac{n\pi}{L}\right)^2 t} f_n(t)\,dt + g_n\right]$$

Therefore, $u(x,t) = \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)\int_0^t e^{\left(\frac{n\pi}{L}\right)^2 t} f_n(t)\,dt + \sum_{n=1}^{\infty} g_n e^{-\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)$

The maximum principle:

To formulate this principle, let us introduce the cylinder $U = U_T = \Omega \times (0,T)$. From previous chapter (4), the harmonic (and subharmonic) functions achieve their maximum on the boundary of the domain. For heat equation the result is improved in that the maximum is achieved on a certain part of its boundary, which we call the parabolic boundary:
$\Gamma = \{(x,t) \in \bar{U} : x \in \partial\Omega \text{ or } t = 0\}$.

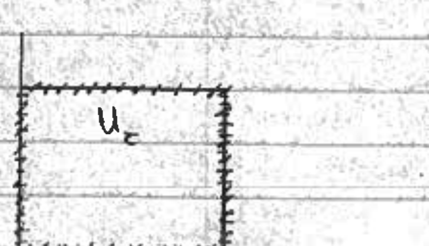


Weak Maximum Principle: Let $u \in C^{2;1}(U) \cap C(\bar{U})$ satisfy $\Delta u \geq u_t$ in U. Then u achieves its maximum on the parabolic boundary of U:
$$\max_{(x,t)\in\bar{U}} u(x,t) = \max_{(x,t)\in\Gamma} u(x,t). \quad (*)$$

Proof:

First step: Assume $\Delta u > u_t$ in U.
Consider $U_\tau = \Omega \times (0,\tau)$, $\Gamma_\tau = \{(x,t) \in \bar{U}_\tau : x \in \partial\Omega \text{ or } t = 0\}$

If the maximum of u on $\bar{U}_\tau$ occurs at $x \in \Omega$ and $t = \tau$, then $u_t(x,\tau) \geq 0$ and $\Delta u(x,\tau) \leq 0$ since $u_t - \Delta u = 0$, which contradicts our assumption.
Similarly u cannot attain an interior maximum U_τ. Thus $\max_{\bar{U}_\tau} u = \max_{\Gamma_\tau} u$ for U_τ



$\bar{U}_\tau = \bar{\Omega} \times (\overline{0,\tau})$

However $\max_{\Gamma_\tau} u \le \max_{\Gamma} u$.

By continuity of u $\lim_{\tau \to T} \max_{\bar{U}_\tau} u = \max_{\bar{U}} u$

Hence $\max_{\bar{U}} u = \max_{\Gamma} u$

Second step: Assume $\Delta u \geq u_t$ in U.

Introduce $v = u - kt$, $k > 0$. Then $v \le u$ on $\bar{U}$.

Since $\Delta v = \Delta u$ and $v_t = u_t - k$, then $\Delta v - v_t = \Delta u - u_t + k > 0$

because $\Delta v - u_t \geq 0$ in U.

Then we may apply (*) to v:

$$\max_{\bar{U}} u = \max_{\bar{U}} (v + kt) \le \max_{\bar{U}} v + kT = \max_{\Gamma} v + kT \le \max_{\Gamma} u + kT$$

letting $k \to 0$ then $\max_{\bar{U}} u = \max_{\Gamma} u$.

Uniqueness:

If $u \in C^{2;1}(U) \cap C(\bar{U})$, then

$$u_t = \Delta u + f(x,t), \quad x \in \Omega, \ t>0$$
$$u(x,0) = g(x) \quad x \in \bar{\Omega}$$
$$u(x,t) = h(x,t) \quad \text{for } x \in \partial\Omega, \ t>0.$$

Let u and v are two solutions. Then $w = u - v$.

Now,
$$V_t = \Delta V + f(x,t) \quad x \in \Omega, \ t>0$$
$$V(x,0) = g(x) \quad x \in \bar{\Omega}$$
$$V(x,t) = h(x,t) \quad \text{for } x \in \partial\Omega, \ t>0$$

Then $W_t = u_t - V_t$ and $\Delta W = \Delta u - \Delta V$

$$\Rightarrow W_t = \Delta u - \Delta V$$
$$\Rightarrow W_t = \Delta W, \quad x \in \Omega, \ t>0$$
$$W(x,0) = g(x) - g(x) = 0, \quad x \in \bar{\Omega}$$
$$W(x,t) = 0 \quad x \in \partial\Omega, \ t>0.$$

Now define the following energy.

$$E(t) = \frac{1}{2}\int_0^t w^2\,dx \geq 0 \quad 0 \leq t \leq T$$

$$E_t = \int_0^t w\,w_t\,dx$$

$$E_t = \int_0^t w\,w_{xx}\,dx$$

$$E_t = w\,w_x\Big|_{x=0}^{t} - \int_{x=0}^{t} w_x\,w_x\,dx$$

$$E_t = 0 - \int_0^t w_x^2\,dx \leq 0$$

and $E(0) = \frac{1}{2}\int_0^{l} w^2(x,0)\,dx = 0$ but $E(t) \geq 0$

i.e $E(t)$ is non-negative, non-increasing function of time whose initial value is zero.

Thus $E(t) = 0$ for all t and $w(x,t) = 0$, $x,t \in \Omega$.

Therefore, $u = v$ and the solution of heat equation is unique

PART 2

Diffusion-Type Problems

LESSON 2

Diffusion-Type Problems (Parabolic Equations)

PURPOSE OF LESSON: To show how parabolic PDEs are used to model heat-flow and diffusion-type problems. The physical meaning of different terms (such as u_t, u_x, u_{xx}, and u) are explained and a few examples of parabolic equations presented.

The idea of an *initial-boundary-value problem* is introduced along with an example. One of the major goals of this lesson is to give the reader an intuitive feeling for parabolic-type problems.

We begin this lesson by introducing a simple physical problem and showing how it can be described by means of a mathematical model (which will involve a PDE). We then complicate the problem and show how new partial differential equations can describe the new situations. The partial differential equations in this lesson are not derived or solved now, but will be in later lessons.

A Simple Heat-Flow Experiment

Suppose we have the following simple experiment that we break into steps:

STEP 1 We start with a reasonably long (say $L = 2$ m) rod (say copper) 2 cm in diameter whose lateral sides (but not the ends) we wrap with insulation. We could even use copper tubing provided we pour some sort of insulation down the inside. In other words, heat can flow in and out of the rod *at the ends*, but not across the lateral boundary.

STEP 2 Next, we place this rod in an environment whose temperature is fixed at some temperature T_0 (degrees °C) for a sufficiently long time, so that the temperature of the entire rod comes to a steady-state temperature similar to the environment. For simplicity, we let the temperature of the environment $T_0 = 10$°C.

STEP 3 We take the rod out of the environment at a time that we call $t = 0$ and attach two *temperature elements* to the ends of the rod. The purpose of these elements is to keep the ends at specific temperatures T_1 and T_2 (say $T_1 = 0°C$ and $T_2 = 50°C$). In other words, two thermostats constantly monitor the temperature at the ends of the rod, and if the temperatures differ from their prescribed values T_1 and T_2, strong heating (or cooling) elements come into operation to adjust the temperature accordingly. Our experiment is illustrated in Figure 2.1.

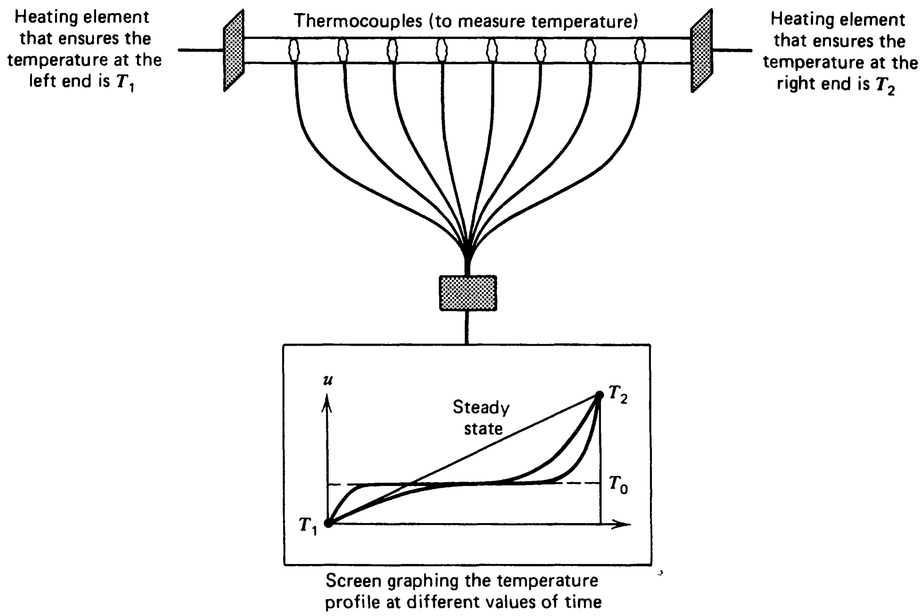


FIGURE 2.1 Schematic diagram of the experiment.

STEP 4 We now monitor the temperature profile of the rod on some type of display. (Why we want to perform an experiment of this kind is another question; we will talk about that later.) This completes our discussion of the experiment. The main purpose of this lesson is to show how this physical problem (and variations of it) can be explained (modeled) by parabolic PDEs.

The Mathematical Model of the Heat-Flow Experiment

The description of our physical problem requires three types of equations

1. The *PDE* describing the physical phenomenon of heat flow.
2. The *boundary conditions* describing the physical nature of our problem on the boundaries.
3. The *initial conditions* describing the physical phenomenon at the start of the experiment.

The Heat Equation

The basic equation of *one-dimensional* heat flow is the relationship

$$\boxed{\text{PDE} \quad u_t = \alpha^2 u_{xx} \qquad 0 < x < L \qquad 0 < t < \infty} \tag{2.1}$$

which relates the quantities

u_t = the *rate of change* in temperature with respect to time (measured in deg/sec)

and

u_{xx} = the *concavity* of the temperature profile $u(x,t)$ (which essentially compares the temperature at one point to the temperature at neighboring points).

This equation will be derived from the basic *conservation of heat equation* in later lessons, but for the time being, we examine it by itself. This equation simply says that the temperature $u(x,t)$ (at some point along the rod x and at some point in time t) is increasing ($u_t > 0$) or decreasing ($u_t < 0$) according to whether u_{xx} is positive or negative. Figure 2.2 illustrates the change in temperature at different points along the rod.

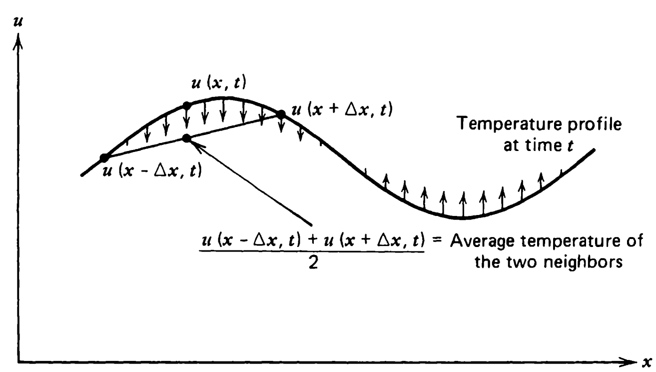


FIGURE 2.2 Arrows indicating change in temperature according to $u_t = \alpha^2 u_{xx}$.

To see how u_{xx} can be interpreted to measure heat flow, suppose we approximate u_{xx} by the difference quotient

$$u_{xx}(x,t) \cong \frac{1}{\Delta x^2}[u(x + \Delta x,t) - 2u(x,t) + u(x - \Delta x,t)]$$

Since this can be rewritten

$$u_{xx}(x,t) \cong \frac{2}{\Delta x^2}\left[\frac{u(x+\Delta x,t)+u(x-\Delta x,t)}{2} - u(x,t)\right]$$

we have the following interpretation of u_{xx}:

1. If the temperature $u(x,t) <$ average of the two neighboring temperatures, then $u_{xx} > 0$ (here, the net flow of heat into x is positive).
2. If the temperature $u(x,t) =$ average of the two neighboring temperatures, then $u_{xx} = 0$ (here the net flow of heat into x is zero).
3. If the temperature $u(x,t) >$ average of the two neighboring temperatures, then $u_{xx} < 0$ (here the net flow of heat into x is negative).

This is illustrated in Figure 2.2. In other words, if the temperature at a point x is greater than the average of the temperature at two nearby points $x - \Delta x$ and $x + \Delta x$, then the temperature at x will be decreasing. Furthermore, the exact rate of decrease u_t is proportional to this difference. The proportionality constant α^2 is a property of the material, and we will discuss this constant more in the next few lessons.

Boundary Conditions

All physical problems have boundaries of some kind, so we must describe mathematically what goes on there in order to adequately describe the problem. In our experiment, the boundary conditions (BCs) are quite easy. Since the temperature u was fixed for all time $t > 0$ at T_1 and T_2 at the two ends $x = 0$ and $x = L$, we would simply say

$$\text{(2.2)} \qquad \text{BCs} \begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \qquad 0 < t < \infty$$

Initial Conditions

All physical problems must start from some value of time (generally called $t = 0$), so we must specify the physical apparatus at this time. Since we started monitoring the rod temperature in our example from the time the rod had achieved a constant temperature of T_0, we have

$$\text{(2.3)} \qquad \boxed{\text{IC} \quad u(x,0) = T_0 \quad 0 \leq x \leq L}$$

We have now mathematically described the experiment. By writing equations (2.1), (2.2), and (2.3) together, we have what is called an *initial-boundary-value problem* (IBVP)

$$
(2.4) \qquad \begin{array}{lll}
\text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < L \quad 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = T_0 & 0 \leqslant x \leqslant L
\end{array}
$$

The interesting thing here, which is not at all obvious, is that there is *only one function* $u(x,t)$ that satisfies the problem (2.4), and that function will describe the temperature of the rod. Hence, our goal in the near future will be to find that unique solution $u(x,t)$ to (2.4).

Before finishing this lesson, we will discuss some variations of this basic problem. We start with a few modifications of the heat equation $u_t = \alpha^2 u_{xx}$.

More Diffusion-Type Equations

Lateral Heat Loss Proportional to the Temperature Difference

The equation

$$u_t = \alpha^2 u_{xx} - \beta\,(u - u_0) \qquad \beta > 0$$

describes heat flow in the rod with both diffusion $\alpha^2 u_{xx}$ along the rod and heat loss (or gain) across the *lateral* sides of the rod. Heat loss ($u > u_0$) or gain ($u < u_0$) is proportional to the difference between the temperature $u(x,t)$ of the rod and the surrounding medium u_0 (with β the proportionality constant). If β is very large in contrast to α^2, then the flow of heat *back and forth* along the rod will be small in contrast to the flow *in and out the sides,* and, hence, the heat will drain out the sides (at each point) according to the approximate equation $u_t = -\beta\,(u - u_0)$.

In chemistry where u may stand for concentration, the equation

$$u_t = \alpha^2 u_{xx} - \beta(u - u_0)$$

says that the rate of change (u_t) of the substance is due both to the diffusion $\alpha^2 u_{xx}$ (in the x-direction) and to the fact that the substance is being created ($u < u_0$) or destroyed ($u > u_0$) by a chemical reaction proportional to the difference between two concentrations u and u_0.

Internal Heat Source

The *nonhomogeneous* equation

$$u_t = \alpha^2 u_{xx} + f(x, t)$$

corresponds to the situation where the rod is being supplied with an internal heat source (everywhere along the rod and for all time t). It may be that a wire carrying electrical current passes through the rod and the resistance generates a constant heat source $f(x, t) = K$.

Diffusion-convection Equation

Suppose a pollutant is being carried along in a stream moving with velocity v. It is obvious that the concentration $u(x,t)$ of the substance changes as a function of both x (positive x measures the distance downstream) and time t. The rate of change u_t is measured by the *diffusion-convection equation*

$$u_t = \alpha^2 u_{xx} - v u_x$$

The term $\alpha^2 u_{xx}$ is the diffusion contribution and $-vu_x$ is the convection component. Whether the pollutant primarily diffuses or convects depends on the relative size of the two coefficients α^2 and v. You have probably seen smoke rising from a smoke stack. Here, the smoke particles are *convected* upward with the hot air and, at the same time, *diffuse* within the air currents.
In addition to these modifications in the heat equation, the *boundary conditions* of the rod can also be changed to correspond to other physical situations. We will discuss some of these modifications in Lesson 3.

NOTES

The heat equation $u_t = \alpha^2(x)u_{xx}$ with a variable coefficient $\alpha(x)$ would correspond to a problem where the diffusion within the rod depends on x (the material is *nonhomogeneous*). For example, if copper and steel slabs were placed next to each other (see Figure 2.3) and if the left side of the copper slabs were fixed at

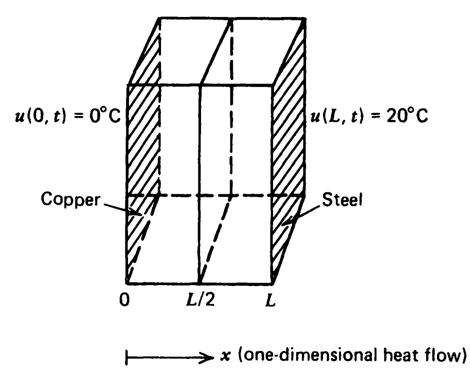


FIGURE 2.3

$u(0,t) = 0°\text{C}$ and the right side of the steel sheet were fixed at $u(L,t) = 20°\text{C}$, then the PDE that describes the heat flow would be

$$u_t = \alpha^2(x)u_{xx} \qquad 0 < x < L$$

$$\text{where } \alpha(x) = \begin{cases} \alpha_1 \text{ (diffusion coefficient of copper)} & 0 < x < L/2 \\ \alpha_2 \text{ (diffusion coefficient of steel)} & L/2 < x < L \end{cases}$$

PROBLEMS

1. If the initial temperature of the rod were

$$u(x,0) = \sin \pi x \qquad 0 \leq x \leq 1$$

and if the BCs were

$$u(0,t) = 0$$
$$u(1,t) = 0$$

what would be the behavior of the rod temperature $u(x,t)$ for later values of time?
HINT Use the physical interpretation of the heat equation $u_t = \alpha^2 u_{xx}$.

2. Suppose the rod has a constant internal heat source, so that the basic equation describing the heat flow within the rod is

$$u_t = \alpha^2 u_{xx} + 1 \qquad 0 < x < 1$$

Suppose we fix the boundaries' temperatures by $u(0,t) = 0$ and $u(1,t) = 1$. What is the steady-state temperature of the rod? In other words, does the temperature $u(x,t)$ converge to a constant temperature $U(x)$ independent of time?
HINT Set $u_t = 0$. It would be useful to graph this temperature. Also start with an initial temperature of zero and draw some temperature profiles.

3. Suppose a metal rod loses heat across the lateral boundary according to the equation

$$u_t = \alpha^2 u_{xx} - \beta u \qquad 0 < x < 1$$

and suppose we keep the ends of the rod at $u(0,t) = 1$ and $u(1,t) = 1$. Find the steady-state temperature of the rod (graph it). Where is heat flowing in this problem?

4. Suppose a laterally insulated metal rod of length $L = 1$ has an initial temperature of $\sin(3\pi x)$ and has its left and right ends fixed at temperatures zero and 10°C. What would be the IBVP that describes this problem?

*Note that the boundary and initial data do not match up in this problem.

OTHER READING

1. *Equations of Mathematical Physics* by A. N. Tikhonov and A. A. Samarskii. Macmillan, 1963; Dover, 1990. An encyclopedia of information; contains many good examples and problems.

LESSON 3

Boundary Conditions for Diffusion-Type Problems

PURPOSE OF LESSON: To show how heat-flow and diffusion-type problems can give rise to a variety of boundary conditions and to introduce the important concept of *flux*.

Three important types of BCs discussed are

1. $u = g(t)$ (temperature specified on the boundary).
2. $\dfrac{\partial u}{\partial n} + \lambda u = g(t)$ (temperature of the *surrounding medium* is specified; n is the *outward normal* direction to the boundary).
3. $\dfrac{\partial u}{\partial n} = g(t)$ (heat flow across the boundary specified).

When describing the various types of boundary conditions that can occur for heat-flow problems, three basic types generally come to mind. Lesson 3 discusses these three kinds of BCs and gives an example of how they occur in experiments.

Type 1 BC (Temperature specified on the boundary)

Consider the heat flow in the one-dimensional rod illustrated in Figure 3.1 and suppose we make the ends of the rod follow the temperature curves $g_1(t)$ and $g_2(t)$.

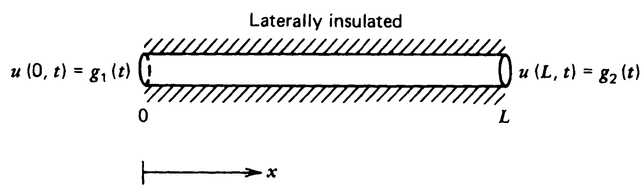


FIGURE 3.1 Temperature specified on the boundary.

As we mentioned in the previous lesson, an apparatus that keeps the ends at specified temperatures requires a thermostat at each end and heating elements to adjust the temperature accordingly. Problems with BCs of this kind are fairly common. It may even be that the goal of the problem is to find the *boundary temperatures* (boundary control) $g_1(t)$ and $g_2(t)$ that will force the temperature to behave in a suitable manner. In the steel industry, it is often necessary to determine the *boundary controls* so that the temperature of the metal inside the furnace changes over time but the temperature *gradient* from one point to another is small.

Similar types of BCs also apply to higher dimensional domains, for example, in two dimensions, we could imagine the interesting problem of finding the temperature inside the circular disc (of radius R) when the boundary temperature is specified in polar coordinates to be

$$u(R,\theta,t) = \cos t \sin \theta$$

See Figure 3.2.

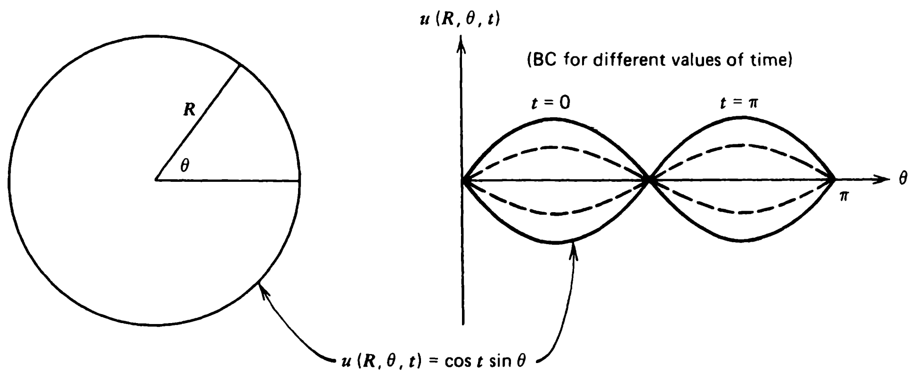


FIGURE 3.2 Oscillating boundary temperature.

Of course, we'd have to have an initial temperature to get this experiment started, but in this case, the effects of our IC would vanish after a short period of time, and the resulting temperature inside the circle would depend on the boundary temperature.

Type 2 BC (Temperature of the surrounding medium specified)

Suppose we consider again our laterally insulated copper rod, but now instead of requiring the two boundaries to be specified at temperatures $g_1(t)$ and $g_2(t)$, we only bring them in contact with surrounding mediums that have those temperatures. In other words, suppose the left side of the rod is enclosed in a container of liquid that has a changing temperature $g_1(t)$, while the right end is enclosed in another liquid with temperature $g_2(t)$ (Figure 3.3).

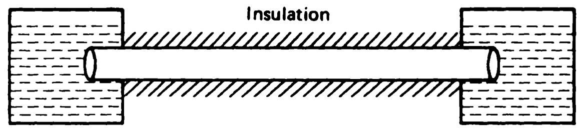


FIGURE 3.3 Convection cooling at the boundaries.

By specifying these types of BCs, we cannot say the boundary temperatures of the rod will be the same as the liquid temperatures $g_1(t)$ and $g_2(t)$, but we do know (Newton's law of cooling) that whenever the rod temperature at one of the boundaries is *less* than the respective liquid temperatures, then heat will flow into the rod at a rate proportional to this difference. In other words, for the one-dimensional rod with boundaries at $x = 0$ and L, Newton's law of cooling states

$$(3.1) \qquad \begin{cases} \text{Outward flux of heat (at } x = 0) = h[u(0,t) - g_1(t)] \\ \text{Outward flux of heat (at } x = L) = h[u(L,t) - g_2(t)] \end{cases}$$

where h is a **heat-exchange coefficient,** which is a measure of how many calories flow across the boundary per unit of temperature difference per second per cm and the **outward flux of heat** is the number of calories crossing the ends of the rod per second. Note that the outward flux of heat will be positive at either end provided the temperature of the rod is greater than the surrounding medium. Equations (3.1) can now be used in conjunction with what is known as Fourier's Law of Cooling to arrive at our BCs. Fourier's law gives us another representation (the first one is 3.1) for the outward flux of heat and by setting these two representations equal to each other, we get our BCs. First, we state Fourier's law (proven experimentally):

(3.2) | Outward flux of heat across a boundary is proportional to the inward normal derivative across the boundary. |

This law says that if the temperature is increasing rapidly in the direction *outward* from the boundary of D (Figure 3.4), then heat will flow *from* the surrounding medium *into* the domain D.

In our one-dimensional problem, Fourier's law takes the form:

$$(3.3) \qquad \begin{cases} \text{Outward flux of heat (at } x = 0) = k\dfrac{\partial u(0,t)}{\partial x} \\ \text{Outward flux of heat (at } x = L) = -k\dfrac{\partial u(L,t)}{\partial x} \end{cases}$$

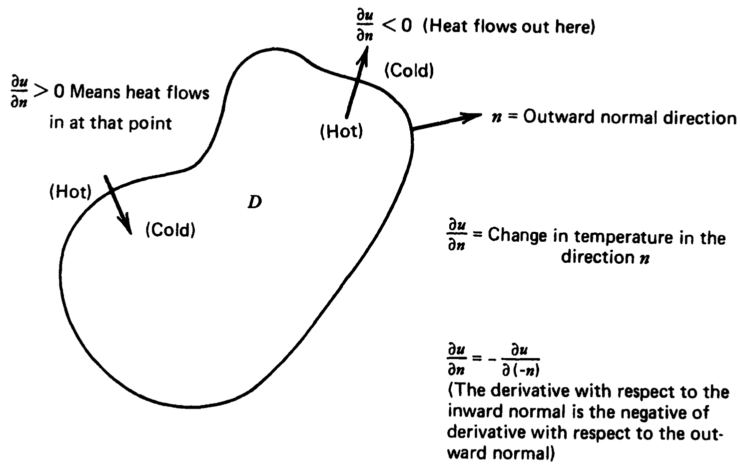


FIGURE 3.4 Illustration of Fourier's law.

where k is the **thermal conductivity** of the metal, which is a measure of how well the material conducts heat. (Poorly conducting materials have values near zero in cgs. units, while copper and aluminum have values close to one.)

Fourier's law (3.3) actually holds anywhere inside the rod and not just at the boundary; for example,

$$\boxed{\text{Flux of heat crossing } x_0 \text{ (from left to right)} = -kA\frac{\partial u}{\partial x}(x_0,t)} \tag{3.4}$$

See Figure 3.5.

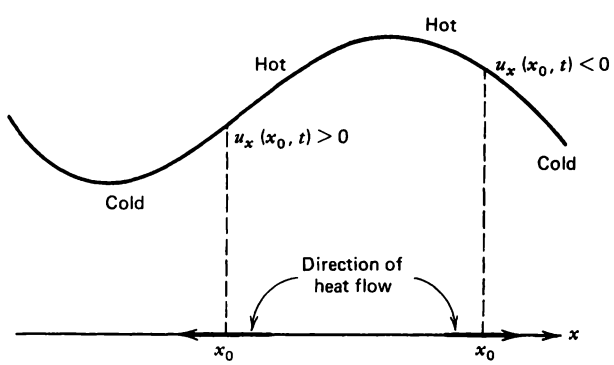


FIGURE 3.5 Another illustration of Fourier's law.

Fourier's law (3.4) says that if $u_x(x_0,t) < 0$, then heat will flow from *left to right*; if $u_x(x_0,t) > 0$, then the flow of heat through point x_0 will be from *right to left* (heat always flows from high to low temperatures).

Finally, if we use the two expressions (3.1) and (3.3) for heat flux, we have our desired BCs for Figure 3.3 in purely mathematical terms; namely,

$$\text{BCs}\quad \begin{cases} \dfrac{\partial u(0,t)}{\partial x} = \dfrac{h}{k}\,[u(0,t) - g_1(t)] \\[2ex] \dfrac{\partial u(L,t)}{\partial x} = -\dfrac{h}{k}\,[u(L,t) - g_2(t)] \end{cases} \qquad 0 < t < \infty$$

Quite often, the constant h/k is simply written as λ, and so we have the BCs for heat flow across the boundary

$$\begin{aligned} u_x(0,t) &= \lambda\,[u(0,t) - g_1(t)] \\ u_x(L,t) &= -\lambda\,[u(L,t) - g_2(t)] \end{aligned} \tag{3.5}$$

In higher dimensions, we have similar BCs; for example, if the boundary of a circular disc is interfaced with a moving liquid that has a temperature $g(\theta,t)$, our BC would be

$$\frac{\partial u}{\partial r}(R,\theta,t) = -\frac{h}{k}\,[u(R,\theta,t) - g(\theta,t)]$$

Here, $\dfrac{\partial u}{\partial r}(R,\theta,t)$ represents the outward normal derivative (in the positive r-direction) of u evaluated at a point (R,θ) on the boundary. This type of BC would be called a *linear* BC (since it is linear in u and u_r) but nonhomogeneous due to the right-hand side $g(\theta,t)$.

Type 3 BC (Flux specified—including the special case of insulated boundaries)

Insulated boundaries are those that do not allow any flow of heat to pass, and, hence, the normal derivative (inward or outward) must be zero on the boundary (since the normal derivative is proportional to the flux). In the case of the one-dimensional rod with insulated ends at $x = 0$ and $x = L$, the BCs are

$$\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0 \end{aligned} \qquad 0 < t < \infty$$

In two-dimensional domains, an insulated boundary would mean that the *normal derivative* of the temperature across the boundary is zero. For example, if the circular disc were insulated on the boundary, then the BC would be $u_r(R,\theta,t) = 0$ for all $0 \leq \theta < 2\pi$ and all $0 < t < \infty$.

On the other hand, if we specify the amount of heat entering across the boundary of our disc, the BC is

$$u_r(R,\theta,t) = f(\theta,t)$$

where $f(\theta,t)$ would represent the amount of heat crossing *into* the circular disc from an outside heating source.

We now illustrate different types of BCs.

Typical BCs for One-Dimensional Heat Flow

Suppose we have a copper rod 200 cm long that is laterally insulated and has an initial temperature of 0°C. Suppose the top of the rod ($x = 0$) is insulated, while the bottom ($x = 200$) is immersed in moving water that has a constant temperature of $g_2(t) = 20°C$ (Figure 3.6).

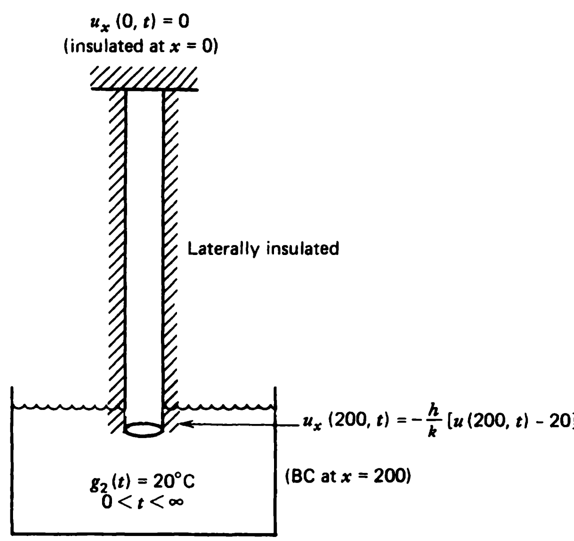


FIGURE 3.6 Initial-boundary-value problem.

The mathematical model for this problem would be the following four equations:

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 200 \qquad 0 < t < \infty$$

$$\text{(3.6)} \qquad \text{BCs} \qquad \begin{cases} u_x(0,t) = 0 \\ u_x(200,t) = -\dfrac{h}{k}[u(200,t) - 20] \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = 0°C \qquad 0 \le x \le 200$$

where

α^2 = 1.16 cm²/sec (diffusivity constant for copper)
k = 0.93 cal/cm-sec°C (thermal conductivity of copper)
h = heat exchange coefficient. To find h is a hard problem in itself. It measures the rate that heat is being exchanged between the bottom of the rod and the surrounding water. It is a function of how fast the water is being circulated, the nature of the interface, and so forth. The reader would have to carry out an experiment to determine its value.

NOTES

1. A typical heat-flow problem inside a square is shown in Figure 3.7.

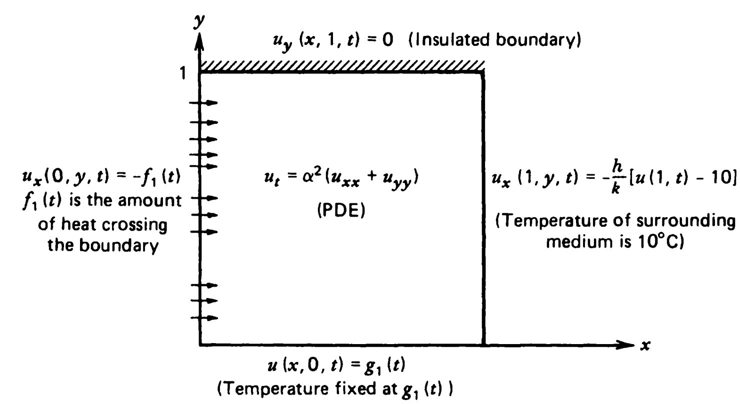


FIGURE 3.7 Typical BCs for diffusion problems inside a square.

In this problem, after we specify the initial temperature $u(x,y,0)$ at $t = 0$ inside the square, the PDE and BCs in the diagram will take over for $0 < t < \infty$ and determine the subsequent temperature values $u(x,y,t)$. Whatever the temperature is, however, it must satisfy the BCs in Figure 3.7.

2. Note that the BC

$$u_r(R,\theta,t) = -\frac{h}{k}[u(R,\theta,t) - g(\theta,t)]$$

on the circle will not require the boundary temperature to be $g(\theta,t)$, but when the heat-exchange coefficient h is large, then the BC essentially says that the boundary temperature $u(R,\theta,t)$ is almost equal to $g(\theta,t)$.

PROBLEMS

1. Draw rough sketches of the solution to the IBVP (3.6) for different values of time. Do your sketches satisfy the BCs? What is the steady-state temperature of the rod? Is this obvious based on your intuition?
2. What is your interpretation of the initial-boundary-value problem?

$$\begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < 1 \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 1 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = \sin(\pi x) & 0 \leq x \leq 1 \end{array}$$

   Can you draw rough sketches of the solution for different values of time? Will the solution come to a steady state; is this obvious?
3. What is your physical interpretation of the problem?

$$\begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < 1 \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = \sin(\pi x) & 0 \leq x \leq 1 \end{array}$$

   Can you draw rough sketches of this solution for various values of time? What about the steady-state temperature?
4. Suppose a metal rod laterally insulated has an initial temperature of 20°C but immediately thereafter has one end fixed at 50°C. The rest of the rod is immersed in a liquid solution of temperature 30°C. What would be the IBVP that describes this problem?

OTHER READING

1. *Conduction of Heat in Solids* by H. S. Carslaw and J. C. Jaeger. Oxford University Press, 1959. An excellent reference that discusses BCs of many physical problems.

2. *Partial Differential Equations in Biology* by C. S. Peskin. Courant Institute of Mathematical Sciences, 1976. Several biological phenomena such as nerve cells, the inner ear, and the cardiovascular system are modeled by PDEs.

LESSON 4

Derivation of the Heat Equation

PURPOSE OF LESSON: To show how the one-dimensional heat equation

$$u_t = \alpha^2 u_{xx} + f(x,t)$$

is derived from the basic principle of *conservation of heat.* Physical concepts such as *thermal conductivity, thermal capacity,* and *density* are discussed, and it is shown how the rate of heat transfer depends on these three basic physical parameters. A few variations of the basic heat equation are also discussed.

In all areas of science, we begin with a given set of assumptions that are taken to be self-evident and from which all other ideas are derived. Of course, what is self-evident to one person may hold doubts for others. The history of science consists of pushing back the basic axioms further and further so that there is a universally agreed upon starting point.

For example, one person may think that all relevant facts will spring from a basic assumption, say assumption B. From assumption B, he or she may prove theorem C, which in turn proves theorem D, which in turn proves others (Figure 4.1).

? $\longrightarrow$ Assumption A $\longrightarrow$ Assumption B $\longrightarrow$ Theorem C $\longrightarrow$ Theorem D $\longrightarrow$?

FIGURE 4.1 The axiomatic method.

This of course is progress—the more new results a person can prove, the better. Physicists, chemists, and biologists all proceed in this basic manner.

On the other hand, instead of proving new theorems we may ask if it is possible to find a new assumption, say assumption A, more basic than assumption B, so that assumption B can be proven from A. In this way, we are pushing back the frontiers of knowledge. In the general area of heat flow, the concept of *conservation of energy* (heat energy) is the basis from which other principles are derived (Figure 4.2).

Conservation of energy (Assumption) $\longrightarrow$ $u_t = \alpha^2 u_{xx} + f(x,t)$ $\longrightarrow$ Other properties of heat flow

FIGURE 4.2 Conservation of energy: the cornerstone of heat-flow problems.

We could, of course, forget this lesson and use the heat equation as the starting point (some people may think it is self-evident in itself), but this would be shortchanging serious students, since conservation of energy assumptions are basic to science. Scientists often begin modeling specific problems by writing conservation of energy relationships and then rewriting them as partial differential equations.

We now turn to the goal of the lesson—to derive the heat equation from the conservation of heat equation.

Derivation of the Heat Equation

Suppose we have a one-dimensional rod of length L for which we make the following assumptions:

1. The rod is made of a single homogeneous conducting material.
2. The rod is laterally insulated (heat flows only in the x-direction).
3. The rod is thin (the temperature at all points of a cross section is constant).

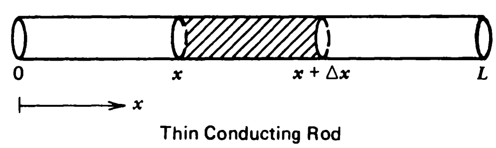


FIGURE 4.3 Thin conducting rod.

If we apply the principle of conservation of heat to the segment $[x,x + \Delta x]$, we can claim

$$
\begin{aligned}
&\text{Net change of heat inside } [x,x + \Delta x] \\
&\quad = \text{Net flux of heat across the boundaries} \\
&\quad + \text{Total heat generated inside } [x,x + \Delta x]
\end{aligned} \tag{4.1}
$$

Now, inasmuch as the total amount of heat (in calories) inside $[x,x + \Delta x]$ at any time t is measured by (see reference 1 in this lesson)

$$\text{Total heat inside } [x,x + \Delta x] = \int_x^{x+\Delta x} c\rho A u(s,t)\, ds$$

where

c = thermal capacity of the rod (measures the ability of the rod to store heat).
ρ = density of the rod.
A = cross-section area of the rod

we can write the conservation of energy equation (4.1) via calculus as

$$\frac{d}{dt}\int_x^{x+\Delta x} c\rho A u(s,t)\, ds = c\rho A \int_x^{x+\Delta x} u_t(s,t)\, ds \tag{4.2}$$
$$= kA\,[u_x(x+\Delta x,t) - u_x(x,t)] + A\int_x^{x+\Delta x} f(s,t)\, ds$$

where

k = thermal conductivity of the rod (measures the ability to conduct heat).
$f(x,t)$ = external heat source (calories per cm per sec).

The problem now is to replace equation (4.2) by one that does not contain integrals. The reader may recall the Mean Value Theorem from calculus.

Mean Value Theorem

If $f(x)$ is a continuous function on $[a,b]$, then there exists at least one number ξ, $a < \xi < b$ that satisfies

$$\int_a^b f(x)dx = f(\xi)\,(b-a)$$

Applying this result to equation (4.2), we arrive at the following equation:

$$c\rho A u_t(\xi_1,t)\Delta x = kA[u_x(x+\Delta x,t) - u_x(x,t)] + Af(\xi_2,t)\,\Delta x \quad x < \xi < x + \Delta x$$

or

$$u_t(\xi,t) = \frac{k}{c\rho}\left\{\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right\} + \frac{1}{c\rho}f(\xi,t)$$

Finally letting $\Delta x \to 0$, we have the desired result

$$\boxed{u_t(x,t) = \alpha^2 u_{xx}(x,t) + F(x,t)} \tag{4.3}$$

where

$$\alpha^2 = \frac{k}{c\rho} \qquad \text{(called the diffusivity of the rod)}$$

$$F(x,t) = \frac{1}{c\rho} f(x,t) \qquad \text{(heat source density)}$$

This completes our discussion. Before we close, however, suppose the rod were not laterally insulated and that heat can flow in and out across the lateral boundary at a rate proportional to the difference between the temperature $u(x,t)$ and the surrounding medium that we keep at zero. In this case, the conservation of heat principle will give.

$$u_t = \alpha^2 u_{xx} - \beta u + F(x;t) \tag{4.4}$$

where β = rate constant for the lateral heat flow ($\beta > 0$).

NOTES

1. The constant k is the **thermal conductivity** of the rod and a measure of the heat flow (in calories) that is transmitted per second through a plate 1 cm thick across an area of 1 cm^2 when the temperature difference is 1°C; values for k can be found in *The Handbook of Chemistry and Physics.* Typical values of k are close to 1 for copper and near zero for insulating-type materials.

   If the material of the rod is uniform, then k will not depend on x. For some materials, the value of k depends on the temperature u and hence the heat equation

   $$u_t = \frac{1}{c\rho}\frac{\partial}{\partial x}\{k(u)u_x\}$$

   is *nonlinear.* Most of the time, however, k changes very slowly with u and this nonlinearity is neglected.
2. The constant c is known as the **thermal capacity** (or specific heat) of the substance and measures the amount of energy the substance can store. For example, a baked potato would have a large thermal capacity, since it can store a large amount of heat per unit mass of potato (that's why it takes a long time to heat). Technically, the thermal capacity is the amount of heat (in calories) necessary to produce a 1°C change in temperature of 1 g of the substance. For most of our problems, c is taken as a constant independent of x and u; typical values can be found in *The Handbook of Chemistry and Physics*.
3. The units of some of the basic quantities of heat flow (in the cgs. measurement system) are

   u = temperature (degrees centigrade).

   u_t = rate of change in temperature (°C/sec).

u_x = slope of temperature curve (°C/cm).
u_{xx} = concavity of temperature curve (°C/cm²).
c = thermal capacity (cal/g-°C).
k = thermal conductivity (cal/cm-sec-°C).
ρ = density (g/cm³).
α^2 = diffusivity (cm²/sec).

4. Note that the diffusivity $\alpha^2 = \dfrac{k}{c\rho}$ of a material is proportional to the conductivity k of the material and inversely proportional to the density ρ and thermal capacity c; this should have some intuitive appeal to the reader.

PROBLEMS

1. Substitute the units of each quantity $u, u_t, \ldots$ into the equation

$$u_t = \alpha^2 u_{xx} - \beta u$$

to see that every term has the same units of °C/sec.

2. Substitute the units of each quantity into the equation

$$u_t = \alpha^2 u_{xx} - v u_x$$

where v has units of velocity to see that every term has the same units.

3. Derive the heat equation

$$u_t = \frac{1}{c\rho}\frac{\partial}{\partial x}[k(x)u_x] + f(x,t)$$

for the situation where the thermal conductivity $k(x)$ depends on x.

4. Suppose $u(x,t)$ measures the concentration of a substance in a moving stream (moving with velocity v). Suppose the concentration $u(x,t)$ changes both by diffusion and convection; derive the equation

$$u_t = \alpha^2 u_{xx} - v u_x$$

from the fact that at any instant of time, the total mass of the material is not created or destroyed in the region $[x, x + \Delta x]$.
HINT Write the conservation equation

Change of mass inside $[x, x + \Delta x]$
= Change due to *diffusion* across the boundaries
+ Change due to the material being *carried* across the boundaries

OTHER READING

1. *Applied Mathematics for the Engineer and Physicist* by L. A. Pipes. McGraw-Hill, 1958. An older reference, but still a good one for the practicing scientist.

2. *Equations of Mathematical Physics* by A. N. Tikhonov and A. A. Samarskii. Macmillan, 1963; Dover, 1990. A good text for derivations of equations. A companion volume, *A Collection of Problems in* [*on*] *Mathematical Physics*, by B. M. Budak, A. A. Samarskii, and A. N. Tikhonov (Pergamon, 1964; Dover, 1988) is available, which will give the student additional experience with solving problems in partial differential equations.

LESSON 5

Separation of Variables

PURPOSE OF LESSON: To introduce the powerful method of separation of variables and to show how this method can be used to solve a well-known diffusion problem. Inasmuch as the method is not well understood by some students due to its complicated algebraic nature, several intuitive explanations are given along the way.

The basic idea is to break down the *initial conditions* of the problem into simple components, find the response to each component, and then add up these individual responses. This gives the response to the *arbitrary initial condition.*

The actual step-by-step methodology of separation of variables somewhat hides this basic interpretation, but that's what's going on nevertheless.

Separation of variables is one of the oldest techniques for solving initial-boundary-value problems (IBVPs) and applies to problems where

1. The PDE is linear and homogeneous (not necessarily constant coefficients).
2. The boundary conditions are of the form

$$\begin{aligned} \alpha u_x(0,t) + \beta u(0,t) &= 0 \\ \gamma u_x(1,t) + \delta u(1,t) &= 0 \end{aligned}$$

   where α, β, γ, and δ are constants (boundary conditions of this form are called **linear homogeneous BCs**).

It dates back to the time of Joseph Fourier (in fact, it's occasionally called *Fourier's method*) and is probably the most widely used method of solution (when applicable).

Instead of showing how the method works in general, let's apply it to a specific problem (later we will discuss it in more generality). Consider the IBVP (diffusion problem)

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

IC $\qquad u(x,0) = \phi(x) \qquad 0 \leq x \leq 1$

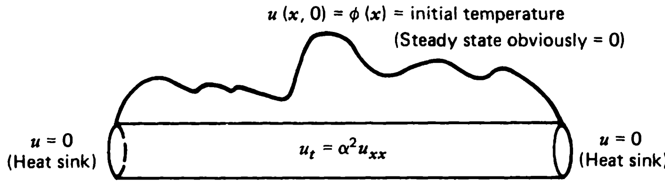


FIGURE 5.1 Diagram of the diffusion problem.

Before getting to separation of variables, let's first think about our problem. Here we have a finite rod where temperature at the ends is fixed at zero (suppose it's a temperature problem where zero means so many degrees). We are also given data for the problem in the form of an initial condition; our goal is to find the temperature $u(x,t)$ at later points in time.

Now for the method itself—but first an overview.

Overview of Separation of Variables

Separation of variables looks for simple-type solutions to the PDE of the form

$$u(x,t) = X(x)T(t)$$

where $X(x)$ is some function of x and $T(t)$ is some function of t. The solutions are simple because any temperature $u(x,t)$ of this form will retain its basic "shape" for different values of time t (Figure 5.2).

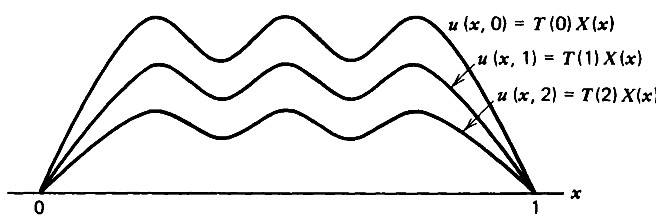


FIGURE 5.2 Graph of $X(x)T(t)$ for different values of t.

The general idea is that it is possible to find an infinite number of these solutions to the PDE (which, at the same time, also satisfy the BCs). These simple functions $u_n(x,t) = X_n(x)T_n(t)$ (called **fundamental solutions**) are the building blocks of our problem, and the solution $u(x,t)$ we are looking for is found by adding the simple fundamental solutions $X_n(x)T_n(t)$ in such a way that the resulting sum

$$\sum_{n=1}^{\infty} A_n X_n(x) T_n(t)$$

satisfies the initial conditions. Inasmuch as this sum still satisfies the PDE and the BCs, we now have the solution to our problem. Let's now carry this out in detail.

Separation of Variables

STEP 1 (Finding elementary solutions to the PDE)

We wish to find the function $u(x,t)$ that satisfies the following four conditions:

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leq x \leq 1$$

To begin, we look for solutions of the form $u(x,t) = X(x)T(t)$ by substituting $X(x)T(t)$ into the PDE and solving for $X(x)T(t)$. Making this substitution gives

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Now, here is the part that makes all this work: If we *divide* each side of this equation by $\alpha^2 X(x)T(t)$, we have

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

and obtain what is called **separated variables**, that is, the left side of the equation depends only on t and the right side, only on x. Inasmuch as x and t are *independent of each other*, each side must be a fixed constant (say k); hence, we can write

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = k$$

or

$$\begin{aligned} T' - k\alpha^2 T &= 0 \\ X'' - kX &= 0 \end{aligned}$$

So now we can solve each of these two ODEs, multiply them together to get a solution to the PDE (note that we have essentially changed a second-order PDE to two ODEs). However, we now make an important observation, namely, that we want the separation constant k to be *negative* (or else the $T(t)$ factor doesn't go to zero as $t \to \infty$). With this in mind, it is general practice to rename $k = -\lambda^2$, where λ is nonzero ($-\lambda^2$ is guaranteed to be negative). Calling our separation constant by its new name, we can now write the two ODEs as

$$T' + \lambda^2\alpha^2 T = 0$$
$$X'' + \lambda^2 X = 0$$

We will now solve these equations. Both equations are standard-type ODEs and have solutions

$$T(t) = Ae^{-\lambda^2\alpha^2 t} \qquad (A \text{ an arbitrary constant})$$
$$X(x) = A\sin(\lambda x) + B\cos(\lambda x) \qquad (A,\ B \text{ arbitrary})$$

and hence all functions

$$u(x,t) = e^{-\lambda^2\alpha^2 t}[A\sin(\lambda x) + B\cos(\lambda x)]$$

(with A, B, and λ arbitrary) will satisfy the PDE $u_t = \alpha^2 u_{xx}$; this verification is problem 1 in the problem set. At this point, we have an infinite number of functions that satisfy the PDE.

STEP 2 (Finding solutions to the PDE and the BCs)

We are now to the point where we have many solutions to the PDE but not all of them satisfy the BCs or the IC. The next step is to choose a certain *subset* of our current crop of solutions

$$e^{-\lambda^2\alpha^2 t}[A\sin(\lambda x) + B\cos(\lambda x)] \tag{5.1}$$

that satisfy the boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0$$

To do this, we substitute our solutions (5.1) into these BCs, getting

$$u(0,t) = Be^{-\lambda^2\alpha^2 t} = 0 \Rightarrow B = 0$$
$$u(1,t) = Ae^{-\lambda^2\alpha^2 t}\sin\lambda = 0 \Rightarrow \sin\lambda = 0$$

This last BC restricts the separation constant λ from being any nonzero number, it must be a root of the equation $\sin\lambda = 0$. In other words, in order that $u(1,t) = 0$, it is necessary to *pick*

$$\lambda = \pm\pi,\ \pm2\pi,\ \pm3\pi,\ \ldots$$

or

$$\lambda_n = \pm n\pi \qquad n = 1, 2, 3, \ldots$$

Note that the last BC could also imply $A = 0$, but if we choose this, we would get the zero solution in (5.1).

We have now finished the second step; we have found an infinite number of functions

$$\boxed{u_n(x,t) = A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \qquad n = 1, 2, \ldots} \tag{5.2}$$

each one satisfying the PDE and the BCs.* These are the building blocks of the problem, and our desired solution will be a certain sum of these simple functions; the specific sum will depend on the initial conditions. See Figure 5.3 for the graphs of these fundamental solutions $u_n(x,t)$:

STEP 3 (Finding the solution to the PDE, BCs, and the IC)

The last step (and probably the most interesting from a mathematical point of view) is to add the fundamental solutions

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)$$

in such a way (pick the coefficients A_n) that the initial condition

$$u(x,0) = \phi(x)$$

is satisfied. Substituting the sum into the IC gives

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \tag{5.3}$$

This equation leads us to the interesting question asked by the French mathematician Joseph Fourier, is it possible to expand the initial temperature $\phi(x)$ as the sum of the elementary functions as follows:

$$A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + \ldots$$

The answer to this question is yes provided $\phi(x)$ is a reasonably nice function—continuous. Hence, the question now becomes how to find the coefficients A_n.

* Notice that the functions u_n and u_{-n} are essentially the same except for a minus sign.

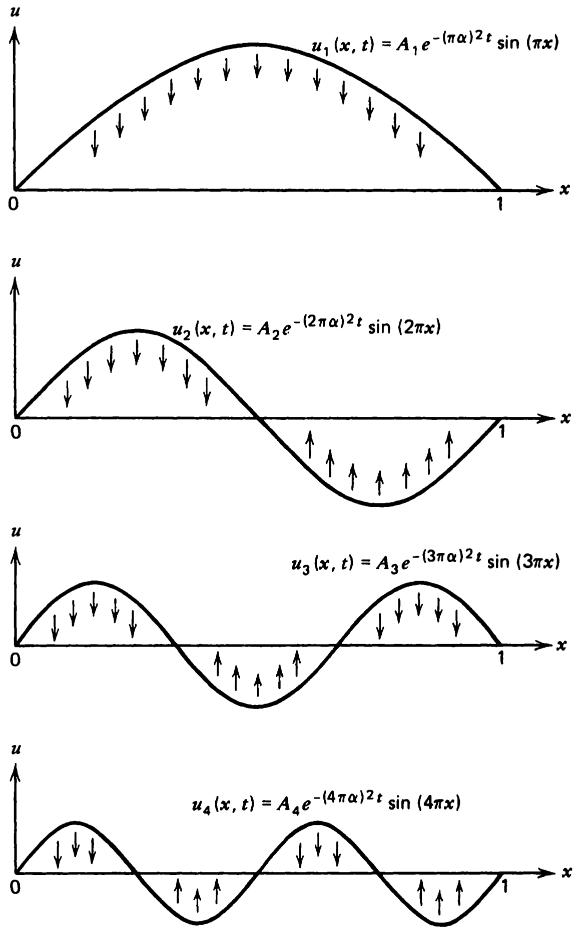


FIGURE 5.3 Fundamental solutions $u_n(x,t) = A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)$.

This is actually very easy: One uses a property of the functions

$$\{\sin(n\pi x); \quad n = 1, 2, \ldots\}$$

known as **orthogonality**. It turns out (see problem 2) that these functions are orthogonal to each other in the sense

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$$

This property can be illustrated by looking at the graphs of these functions (Figure 5.4).

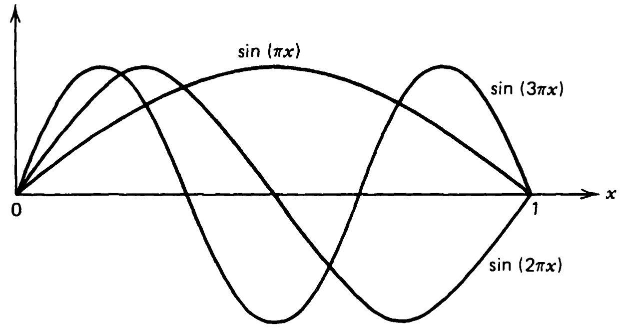


FIGURE 5.4 Orthogonal sequence of functions.

So, we are now in position to solve for the coefficients in the expression

$$\phi(x) = A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + A_4 \sin(4\pi x) + \ldots$$

We *multiply* each side of this equation by $\sin(m\pi x)$ (m, an arbitrary integer) and *integrate* from zero to one; doing this, we get

$$\int_0^1 \phi(x) \sin(m\pi x)\, dx = A_m \int_0^1 \sin^2(m\pi x)\, dx = \frac{1}{2} A_m$$

(all other terms drop out due to orthogonality). Solving for A_m gives

$$A_m = 2 \int_0^1 \phi(x) \sin(m\pi x)\, dx$$

We're done; the solution is

(5.4) $$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)}$$

where the coefficients A_n are given by

(5.5) $$\boxed{A_n = 2\int_0^1 \phi(x) \sin(n\pi x)\, dx}$$

We can check this answer to see that it satisfies all four of our original conditions in the problem. This ends step 3.

Many students are disappointed when they finally discover that the solution is this complicated, and many hardly give the solution a second look (that's too bad). The solution is not all that difficult if one takes the time to analyze it; in fact, the more complicated it is, the more information it contains. Here are a few notes that will help you interpret this solution.

NOTES

1. Observe that the only difference between the *Fourier sine expansion* of $\phi(x)$ in (5.3) and the solution (5.4) is the insertion of the time factor

$$e^{-(n\pi\alpha)^2 t}$$

in each term. Hence, if our IC were a very simple expression like

$$\phi(x) = \sin(\pi x) + \frac{1}{2}\sin(3\pi x)$$

then the solution would simply be

$$u(x,t) = e^{-(\pi\alpha)^2 t}\sin(\pi x) + \frac{1}{2}e^{-(3\pi\alpha)^2 t}\sin(3\pi x)$$

In this case, it's obvious that if we expanded $\phi(x)$ as a Fourier sine series, we would get

$$\begin{aligned} A_1 &= 1 \\ A_2 &= 0 \\ A_3 &= ½ \\ A_4 &= A_5 = \ldots = 0 \end{aligned}$$

2. We can interpret the solution (5.4) in the following manner: We expand the initial temperature $\phi(x)$ as a sum of simple functions, $A_n \sin(n\pi x)$ and then find the response to each of these (which is $A_n e^{-(n\pi\alpha)^2 t}\sin(n\pi\alpha)$; and then add these individual responses to get the solution corresponding to the IC $u(x,0) = \phi(x)$.
3. The terms in the solution

$$u(x,t) = A_1 e^{-(\pi\alpha)^2 t}\sin(\pi x) + A_2 e^{-(2\pi\alpha)^2 t}\sin(2\pi x) + \ldots$$

are functions of x and t. Note that the terms further out in the series get small very fast due to the factor

$$e^{-(n\pi\alpha)^2 t}$$

Hence, for long time periods, the solution is approximately equal to the first term

$$u(x,t) \cong A_1 e^{-(\pi\alpha)^2 t} \sin(\pi x)$$

which is the shape of a damped sine curve (Figure 5.5).

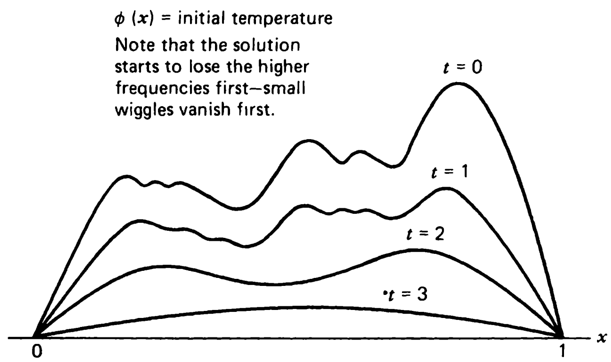


FIGURE 5.5 Higher-order terms damp faster in diffusion problems.

PROBLEMS

1. Show that $u(x,t) = e^{-\lambda^2\alpha^2 t}[A \sin(\lambda x) + B \cos(\lambda x)]$ satisfies the PDE $u_t = \alpha^2 u_{xx}$ for arbitrary A, B, and λ.

2. Show $\int_0^1 \sin(\pi m x) \sin(\pi n x)\, dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$

   HINT Use the identity

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m - n)x - \cos(m + n)x]$$

3. Find the Fourier sine expansion of $\phi(x) = 1 \quad 0 \leq x \leq 1$. Draw the first three or four terms.

4. Using the results of problem 3, what is the solution to the IBVP

$$\text{PDE} \quad u_t = u_{xx} \quad 0 < x < 1$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 1 \quad 0 \leq x \leq 1$$

(Note that this problem is physically impossible, since we are pulling the temperature from one to zero instantaneously. In most problems, if the BCs are zero, then the initial temperature $\phi(x)$ should also be zero at $x = 0$ and $x = 1$.)

5. What is the solution to problem 4 if the IC is changed to

$$u(x,0) = \sin(2\pi x) + \frac{1}{3}\sin(4\pi x) + \frac{1}{5}\sin(6\pi x)$$

6. What would be the solution to problem 4 if the IC were

$$u(x,0) = x - x^2 \qquad 0 < x < 1$$

OTHER READING

Partial Different Equations of Mathematical Physics by Tyn Myint-U. Elsevier, 1973. A well-written text slightly more advanced than the current one; Chapter 6. A large chapter on separation of variables with several good problems.

LESSON 6

Transforming Nonhomogeneous BCs into Homogeneous Ones

PURPOSE OF LESSON: To show how the initial-boundary-value problem

$$\text{PDE} \quad u_t - \alpha^2 u_{xx} = f(x,t)$$

$$\text{BCs} \quad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = g_1(t) \\ \alpha_2 u_x(L,t) + \beta_2 u(L,t) = g_2(t) \end{cases}$$

$$\text{IC} \quad u(x,0) = \phi(x)$$

can be transformed into a new one (with *zero* BCs) like

$$\begin{aligned} &U_t - \alpha^2 U_{xx} = F(x,t) \\ &\alpha_1 U_x(0,t) + \beta_1 U(0,t) = 0 \\ &\alpha_2 U_x(L,t) + \beta_2 U(L,t) = 0 \\ &U(x,0) = \phi(x) \end{aligned}$$

This new problem can then be solved by

1. Separation of variables if the new PDE just happens to be homogeneous $[F(x,t) = 0]$.
2. Integral transforms and eigenfunction expansions if $F(x,t) \neq 0$.

Although the method of separation of variables that we discussed in the last lesson is very powerful and gives us a nice series solution, the reader should realize it doesn't apply to all problems. In order for separation of variables to apply, the BCs must be of the following form (*linear homogeneous* BCs):

$$\begin{aligned} &\alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ &\alpha_2 u_x(L,t) + \beta_2 u(L,t) = 0 \end{aligned} \tag{6.1}$$

The purpose of this lesson is to show how problems with *nonhomogeneous* BC like

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx}$$

$$(6.2) \qquad \text{BCs} \qquad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = g_1(t) \\ \alpha_2 u_x(L,t) + \beta_2 u(L,t) = g_2(t) \end{cases} \qquad \text{(nonhomogeneous BCs)}$$

$$\text{IC} \qquad u(x,0) = \phi(x)$$

can be solved by transforming them into others with zero BCs. The new problem can then be solved by other methods (like eigenfunction expansions). We start our discussion by transforming an extremely simple problem with nonhomogeneous BCs into one with zero BCs.

Transforming Nonhomogeneous BCs to Homogeneous Ones

Consider heat flow in an insulated rod where the two ends are kept at constant temperatures k_1 and k_2; that is,

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < L \qquad 0 < t < \infty$$

$$(6.3) \qquad \text{BCs} \qquad \begin{cases} u(0,t) = k_1 \\ u(L,t) = k_2 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \le x \le L$$

The difficulty here is that since the BCs are not homogeneous, we cannot solve this problem by separation of variables. However, it is obvious that the solution will have a steady-state solution (solution when $t = \infty$) that varies *linearly* (in x) between the boundary temperatures k_1 and k_2 (Figure 6.1).

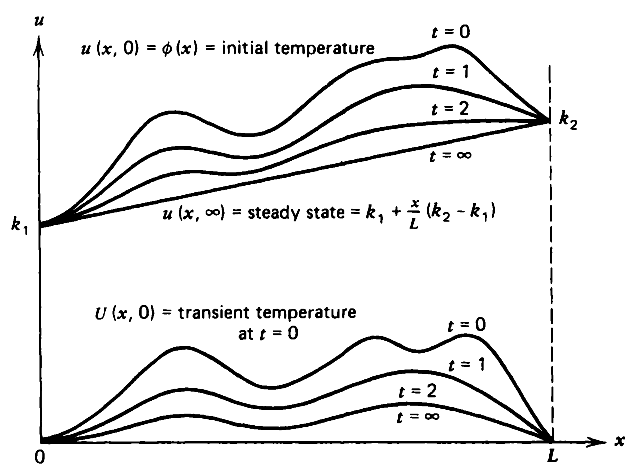


FIGURE 6.1 Solution of (6.3) for various values of time.

In other words, it seems reasonable to think of our temperature $u(x,t)$ as the sum of two parts

$$u(x,t) = \underbrace{\text{steady state}}_{\substack{\text{Eventual solution} \\ \text{for large time}}} + \underbrace{\text{transient}}_{\substack{\text{Part of the solution that} \\ \text{depends on the IC (and} \\ \text{will go to zero)}}}$$

$$= \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x,t)$$

This being the case, our goal is to find the *transient* $U(x,t)$. By substituting

$$u(x,t) = \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x,t)$$

in the original problem (6.3), we will arrive at a new problem in $U(x,t)$. We can then solve this new one for $U(x,t)$ and add it to the steady state to get $u(x,t)$. Carrying out this simple substitution in (6.3) gives us

$$\text{(6.4)} \qquad \begin{array}{lll} \text{PDE} & U_t = \alpha^2 U_{xx} & 0 < x < L \\[1em] \text{BCs} & \begin{cases} U(0,t) = 0 \\ U(L,t) = 0 \end{cases} & 0 < t < \infty \\[1em] \text{IC} & U(x,0) = \underbrace{\phi(x) - \left[k_1 + \frac{x}{L}(k_2 - k_1)\right]}_{\bar{\phi}(x) \,=\, \text{new IC—but known}} & \end{array}$$

This problem (fortunately) has a homogeneous PDE as well as homogeneous BCs, and so we can solve it by separation of variables; in fact, the reader probably remembers the solution:

$$\text{(6.5)} \qquad U(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x/L)$$

where

$$a_n = \frac{2}{L}\int_0^L \bar{\phi}(\xi) \sin(n\pi\xi/L)\, d\xi$$

So much for rods with *fixed* temperatures at the boundaries. What about more realistic-type derivative BCs with *time-varying* right-hand sides? The ideas are similar to the previous problem but a little more complicated.

Transforming Time Varying BCs to Zero BCs

Consider the typical problem

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < L \qquad 0 < t < \infty$$

$$(6.6) \qquad \text{BCs} \qquad \begin{cases} u(0,t) = g_1(t) & 0 < t < \infty \\ u_x(L,t) + hu(L,t) = g_2(t) \end{cases}$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leqslant x \leqslant L$$

To change these nonzero BCs to homogeneous ones, we (after some trial and error) seek a solution of the form

$$u(x,t) = A(t)[1 - x/L] + B(t)[x/L] + U(x,t) \tag{6.7}$$

where $A(t)$ and $B(t)$ are chosen so that the steady-state part

$$S(x,t) = A(t)[1 - x/L] + B(t)[x/L] \tag{6.8}$$

satisfies the BCs of the problem. In this way, the transformed problem in $U(x,t)$ will have homogeneous BCs. Substituting $S(\mathrm{x},t)$ into the BCs

$$\begin{aligned} &S(0,t) = g_1(t) \\ &S_x(L,t) + hS(L,t) = g_2(t) \end{aligned}$$

gives us two equations in which we can solve for $A(t)$ and $B(t)$. Doing this, we get

$$\begin{aligned} A(t) &= g_1(t) \\ B(t) &= \frac{g_1(t) + Lg_2(t)}{1 + Lh} \end{aligned} \tag{6.9}$$

Hence, we have

$$u(x,t) = g_1(t)[1 - x/L] + \frac{g_1(t) + Lg_2(t)}{1 + Lh}[x/L] + U(x,t)$$

and so if we substitute this into the original problem (6.6), we get our transformed problem in $U(\mathrm{x},t)$ (the reader should do this)

$$
\begin{array}{llll}
 & \text{PDE} & U_t = \alpha^2 U_{xx} - S_t & \text{(nonhomogeneous PDE)} \\
(6.10) & \text{BCs} & \begin{cases} U_x(L,t) + hU(L,t) = 0 \\ U(0,t) = 0 \end{cases} & \text{(homogeneous BCs)} \\
 & \text{IC} & U(x,0) = \phi(x) - S(x,0) & \text{(new IC—but known)}
\end{array}
$$

We now have our new problem with zero BCs (unfortunately, the PDE is nonhomogeneous). We can't solve this problem by separation of variables, but if the reader can wait for a few lessons, we will solve it by integral transforms and eigenfunction expansions.

NOTES

1. Our goal in this lesson was to transform problems with nonhomogeneous BCs into those with zero BCs. In so doing, if the new PDE just happens to be homogeneous, we are fortunate (like the first example) because we can then solve the problem by separation of variables.

   If, on the other hand, the new transformed PDE is nonhomogeneous, then we must solve the new problem by some other method.
2. The most general nonhomogeneous linear BCs

$$
\begin{aligned}
\alpha_1 u_x\,(0,t) + \beta_1 u(0,t) &= g_1(t) \\
\alpha_2 u_x\,(L,t) + \beta_2 u(L,t) &= g_2(t)
\end{aligned}
$$

   can also be transformed into zero BCs in a manner similar to the technique in the second example. Of course, the new PDE would most likely be nonhomogeneous.
3. Some methods of solution do not require the BCs to be homogeneous at all, and, hence, it isn't necessary to make any preliminary transformation. Later, when we study the *Laplace transform*, we will see it isn't necessary to have zero BCs (it's just that it's sometimes easier).
4. For BCs of the form

$$
\begin{aligned}
u(0,t) &= g_1(t) \\
u(L,t) &= g_2(t)
\end{aligned}
$$

   the method discussed in the second example will give us the transformation

$$
u(x,t) = \{g_1(t) + \frac{x}{L}[g_2(t) - g_1(t)]\} + U(x,t)
$$

PROBLEMS

1. Solve the initial-boundary-value problem

$$
\begin{array}{lll}
\text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < 1 \\
\text{BCs} & \begin{cases} u(0,t) = 1 \\ u_x(1,t) + hu(1,t) = 1 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = \sin(\pi x) + x & 0 \leqslant x \leqslant 1
\end{array}
$$

by transforming it into homogeneous BCs and then solving the transformed problem. Does the solution agree with your intuition of the problem?

2. Transform

$$
\begin{array}{lll}
\text{PDE} & u_t = u_{xx} & 0 < x < 1 \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 1 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = x^2 & 0 \leqslant x \leqslant 1
\end{array}
$$

to zero BCs and solve the new problem. What will the solution to this problem look like for different values of time? Does the solution agree with your intuition? What is the steady-state solution? What does the transient solution look like?

3. Transform

$$
\begin{array}{lll}
\text{PDE} & u_t = u_{xx} & 0 < x < 1 \\
\text{BCs} & \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) + hu(1,t) = 1 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = \sin(\pi x) & 0 \leqslant x \leqslant 1
\end{array}
$$

into a new problem with zero BCs; is the new PDE homogeneous?

OTHER READING

Analysis and Solution of Partial Differential Equations by R. L. Street. Brooks-Cole, 1973. This excellent text contains an extensive section on transforms of the type we discuss in this lesson and a good section on separation of variables.

LESSON 7

Solving More Complicated Problems by Separation of Variables

PURPOSE OF LESSON: To show how more complicated heat-flow problems can be solved by separation of variables. This lesson essentially consists of a worked problem that will give the reader more familiarity with the method. Hopefully, the reader will be able to extrapolate the ideas presented here to solve problems on his or her own.

Eigenvalue problems, known as *Sturm-Liouville problems*, are introduced, and some properties of these general problems are discussed.

The purpose of this lesson is to solve an *initial-boundary-value problem* by the separation of variables method that the reader might have trouble working on his or her own. Hopefully, the reader can extrapolate from this problem to other problems not specifically mentioned in this text.

We start with a one-dimensional heat-flow problem where one of the BCs contains derivatives.

Heat-Flow Problem with Derivative BC

Consider an apparatus

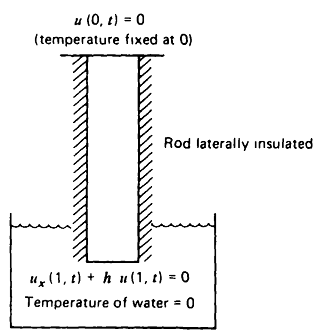


FIGURE 7.1 Diagram for the initial-boundary-value problem.

in which we fix the temperature at the top of the rod at $u(0,t) = 0$ and immerse the bottom of the rod in a solution of water fixed at the same temperature of zero (zero refers to some reference temperature). The natural flow of heat (Newton's law of cooling) says that the BC at $x = 1$ is

$$u_x(1,t) = -hu(1,t)$$

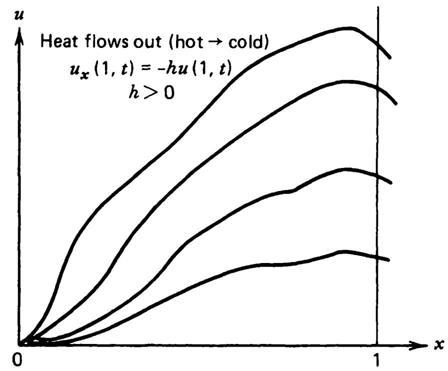


FIGURE 7.2 The nature of curves with BCs $\begin{cases} u(0,t) = 0 \\ u_x(1,t) = -hu(1,t) \end{cases}$

Suppose now the *initial temperature* of the rod is $u(x,0) = x$, but instantaneously thereafter ($t > 0$), we apply our BCs. To find the ensuing temperature, we must solve the IBVP

$$\begin{array}{llll} & \text{PDE} & u_t = \alpha^2 u_{xx} \quad 0 < x < 1 & 0 < t < \infty \\ (7.1) & \text{BCs} & \begin{cases} u(0,t) = 0 \\ u_x(1,t) + hu(1,t) = 0 \end{cases} & \text{(homogeneous BCs)} \\ & \text{IC} & u(x,0) = x \quad 0 \leq x \leq 1 & \end{array}$$

To apply the separation of variables method, we carry out the following steps:

STEP 1 (Separating the PDE into two ODEs)
Substituting $u(x, t) = X(x)T(t)$ into the PDE gives

$$XT' = \alpha^2 X''T$$

and dividing by $\alpha^2 XT$, we get

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X}$$

Since the left-hand side depends *only* on time and the right-hand side depends only on x (and since x and t are independent), both sides of this equation must be constants. Setting them both equal to μ gives the two ODEs

$$\begin{aligned} T' - \mu\alpha^2 T &= 0 \\ X'' - \mu X &= 0 \end{aligned} \tag{7.2}$$

We have now completed the *separation process*.

STEP 2 (Finding the separation constant)
First of all, μ must not be positive or else $T(t)$ will grow exponentially to infinity (which would make $u = XT$ go to infinity—which we can reject on physical grounds).

Secondly, suppose $\mu = 0$. This being the case, we have

$$X'' = 0$$

and thus

$$X(x) = A + Bx$$

But since the BCs of the problem are

$$\begin{aligned} u(0, t) &= X(0)T(t) = 0 \\ u_x(1,t) + h\,u(1,t) &= X'(1)T(t) + h\,X(1)\,T(t) = 0 \end{aligned}$$

we could conclude that

$$\begin{aligned} X(0) &= 0 \\ X'(1) + hX(1) &= 0 \end{aligned} \Rightarrow \begin{aligned} A &= 0 \\ B &= 0 \end{aligned}$$

which would mean $u(x,t) = 0$. In other words, $\mu = 0$ gives only $u = 0$; hence we throw it out (we are looking for nonzero solutions).

Finally, if $\mu < 0$, we call $\mu = -\lambda^2$ and write the two ODEs (7.2) as

$$\begin{aligned} T' + \lambda^2\alpha^2 T &= 0 \\ X'' + \lambda^2 X &= 0 \end{aligned}$$

which gives us solutions

$$\begin{aligned} T(t) &= Ae^{-(\lambda\alpha)^2 t} \\ X(x) &= B\sin(\lambda x) + C\cos(\lambda x) \end{aligned}$$

Hence, what we have is that *any function*

$$u(x, t) = e^{-(\lambda\alpha)^2 t} [A \sin (\lambda x) + B \cos (\lambda x)] \tag{7.3}$$

for *any* λ and *any* A and B will satisfy the PDE (the reader can verify this calculation on his or her own). What we'd like to do now is find out how many of these functions will satisfy the BCs

$$\begin{aligned} u(0,t) &= 0 \\ u_x(1,t) + hu(1,t) &= 0 \end{aligned} \tag{7.4}$$

Substituting the solution (7.3) into the BCs (7.4) gives us conditions on λ, A, and B that must be satisfied; namely,

$$\begin{aligned} &Be^{-(\lambda\alpha)^2 t} = 0 \Rightarrow B = 0 \\ &A\lambda e^{-(\lambda\alpha)^2 t} \cos \lambda + hAe^{-(\lambda\alpha)^2 t} \sin \lambda = 0 \end{aligned}$$

Performing a little algebra on this last equation gives us our desired condition on λ

$$\tan \lambda = -\lambda/h$$

In other words, to find λ, we must find the intersections of the curves $\tan \lambda$ and $-\lambda/h$ (Figure 7.3).

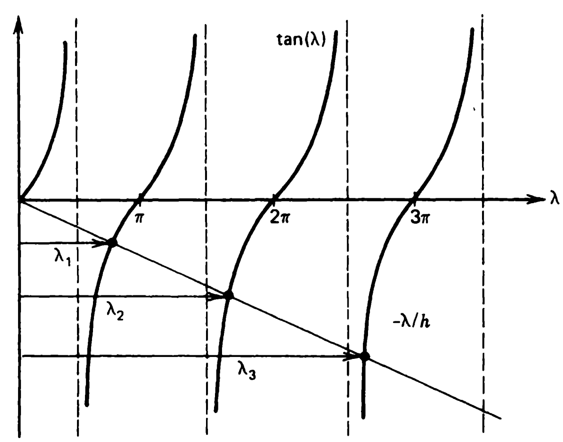


FIGURE 7.3 Graph showing intersections of tan (λ) and $-\lambda/h$.

These values $\lambda_1, \lambda_2, \ldots$ can be computed numerically for a given h on a computer and are called the **eigenvalues** of the boundary-value problem

$$\begin{aligned} &X'' + \lambda^2 X = 0 \\ &X(0) = 0 \\ &X'(1) + hX(1) = 0 \end{aligned} \tag{7.5}$$

In other words, they are the values of λ for which there exists a *nonzero solution*. The eigenvalues λ_n of (7.5), which, in this case, are the roots of $\tan \lambda = -\lambda/h$, have been computed (for $h = 1$) numerically, and the first five values are listed in Table 7.1.

TABLE 7.1 Roots of $\tan \lambda = -\lambda$

n	λ_n
1	2.02
2	4.91
3	7.98
4	11.08
5	14.20

The solutions of (7.5) corresponding to the eigenvalues λ_n are called the **eigenfunctions** $X_n(x)$, and for this problem, we have

$$X_n(X) = \sin(\lambda_n x)$$

See Figure 7.4.

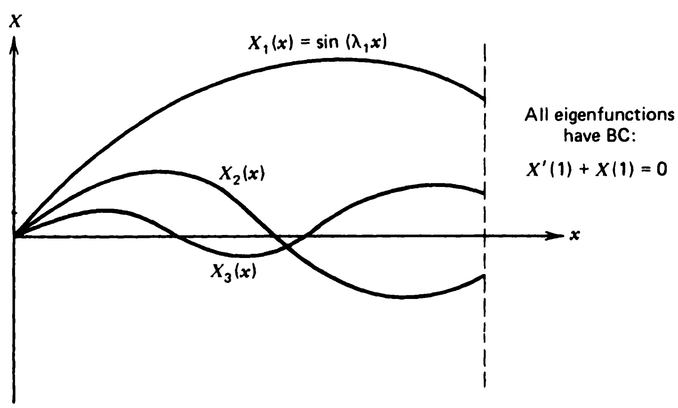


FIGURE 7.4 Eigenfunctions $X_n(x)$ of (7.5) for $h = 1$.

STEP 3 (Finding the fundamental solutions)
We now have an infinite number of functions (fundamental solutions),

$$u_n(x, t) = X_n(x)\, T_n(t) = e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x)$$

each one satisfying the PDE and the BCs. The final step is to add these functions together (the sum will still satisfy the PDE and BCs, since both the PDE and BCs are *linear* and *homogeneous*) in such a way that they agree with the IC when $t = 0$; that is, we sum

$$u(x, t) = \sum_{n=1}^{\infty} a_n X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x)$$

so that the IC $u(x,0) = x$ is satisfied. In other words,

$$u(x, 0) = x = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) \tag{7.6}$$

This brings us to our final step.

STEP 4 (Expansion of the IC as a sum of eigenfunctions)
To find the constants a_n in the eigenfunction expansion (7.6), we must multiply each side of the equation by $\sin(\lambda_m x)$ and integrate x from 0 to 1; that is;

$$\int_0^1 \xi \, sin\,(\lambda_m \xi)\, d\xi = \sum_{n=1}^{\infty} a_n \int_0^1 \sin(\lambda_n \xi) \sin(\lambda_m \xi)\, d\xi$$

$$= a_m \int_0^1 \sin^2(\lambda_m \xi)\, d\xi$$

$$= a_m \left(\frac{\lambda_m - \sin \lambda_m \cos \lambda_m}{2\lambda_m} \right)$$

Solving for a_m (we'll change the notation to a_n), we get our desired result

$$a_n = \frac{2\lambda_n}{(\lambda_n - \sin \lambda_n \cos \lambda_n)} \int_0^1 \xi \sin(\lambda_n \xi)\, d\xi \tag{7.7}$$

In other words, our solution to (7.1) is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x) \tag{7.8}$$

where the constants a_n are given by (7.7). In this problem, the first five constants a_n have been computed and are listed in Table 7.2.

TABLE 7.2 Coefficients a_n in (7.8)

n	a_n
1	0.24
2	0.22
3	−0.03
4	−0.11
5	−0.09

Hence, the first three terms of the IBVP

$$
(7.9)\qquad
\begin{array}{lll}
\text{PDE} & u_t = u_{xx} \quad 0 < x < 1 & 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u_x(1,t) + u(1,t) = 0 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = x \quad 0 \leqslant x \leqslant 1 &
\end{array}
$$

are

$$u(x, t) = 0.24\, e^{-4t} \sin(2x) + 0.22\, e^{-24t} \sin(4.9x) + 0.03e^{-63.3t} \sin(7.98x) + \ldots$$

The graph of this solution is drawn for various values of time in Figure 7.5. The reader can ask himself or herself if this solution agrees with his or her intuition and whether or not it satisfies the BCs of the problem.

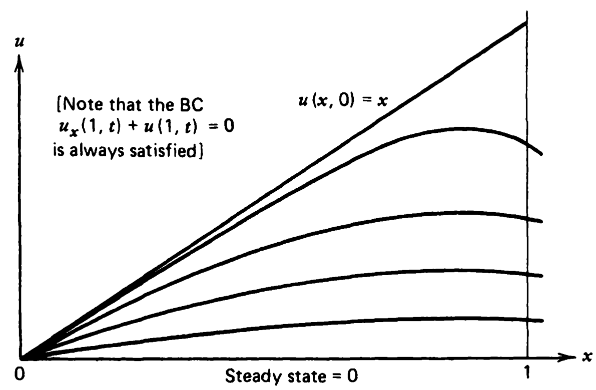


FIGURE 7.5 Solution to (7.8).

NOTES

The eigenvalue problem (7.5) is a special case of the general problem

$$
(7.10) \qquad \begin{array}{ll} \text{ODE} & [p(x)\,y']' - q(x)y + \lambda\, r(x)y = 0 \qquad 0 < x < 1 \\ \text{BCs} & \begin{cases} \alpha_1 y\,(0) + \beta_1 y'(0) = 0 \\ \alpha_2 y\,(1) + \beta_2 y'(1) = 0 \end{cases} \end{array}
$$

known as the **Sturm-Liouville problem**. When we solve PDEs by separation of variables with linear homogeneous BCs, the ODE in $X(x)$ along with its BCs will always be some particular Sturm-Liouville problem. We observe that the eigenvalue problem (7.5) is a special case of (7.10).

What Sturm and Liouville proved is that under suitable conditions on the functions $p(x)$, $q(x)$, and $r(x)$, the problem (7.10) has

1. An infinite sequence of eigenvalues

$$
\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots \rightarrow \infty
$$

2. Corresponding to *each* eigenvalue λ_n, there is *one* nonzero solution $y_n\,(x)$ [not including other constant multiples of $y_n(x)$].

3. If $y_n(x)$ and $y_m(x)$ are two *different* eigenfunctions (corresponding to $\lambda_n \neq \lambda_m$), then they are *orthogonal* with respect to the *weight function* $r(x)$ on the interval of [0,1]; that is, they satisfy

$$
\int_0^1 r(x)\, y_n(x)\, y_m(x)\, dx = 0
$$

More details of Sturm-Liouville-type problems can be found in references 1 and 2.

PROBLEMS

1. Solve the following heat-flow problem:

$$
\begin{array}{lll} \text{PDE} & u_t = u_{xx} & 0 < x < 1 \qquad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = x & 0 \leqslant x \leqslant 1 \end{array}
$$

by separation of variables. Does your solution agree with your intuition? What is the steady-state solution?

2. What are the eigenvalues and eigenfunctions of the Sturm-Louiville problem

$$\text{ODE} \quad X'' + \lambda X = 0 \quad 0 < x < 1$$

$$\text{BCs} \quad \begin{cases} X(0) = 0 \\ X'(1) = 0 \end{cases}$$

What are the functions $p(x)$, $q(x)$, and $r(x)$ in the general Sturm-Liouville problem for this equation?

3. Solve the following problem with insulated boundaries:

$$\text{PDE} \quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = x \quad 0 \leqslant x \leqslant 1$$

Does your solution agree with your interpretation of the problem? What is the steady-state solution?; does this make sense?

4. What are the eigenvalues and eigenfunctions of

$$\text{ODE} \quad X'' + \lambda X = 0 \quad 0 < x < 1$$

$$\text{BCs} \quad \begin{cases} X'(0) = 0 \\ X'(1) = 0 \end{cases}$$

OTHER READING

1. *Elementary Differential Equations and Boundary-Value Problems* by W. E. Boyce and R. C. DiPrima. John Wiley & Sons, 1965. Chapter 11. This is an ordinary-differential-equations text that contains an excellent section on the Sturm-Liouville problem, one of the better undergraduate texts in ODEs.

2. *Advanced Engineering Mathematics* by E. Kreyszig. John Wiley & Sons, 1967. This text contains many worked examples of typical problems; very readable.

LESSON 8

Transforming Hard Equations into Easier Ones

PURPOSE OF LESSON: To show how one can transform a PDE in $u(x,t)$ into a new (easier) one in a new variable $w(x,t)$. The transformation is generally based on intuition, and in this lesson, the PDEs

$$u_t = \alpha^2 u_{xx} - \beta u$$
$$u_t = \alpha^2 u_{xx} - v u_x$$

are transformed into the simple heat equation

$$w_t = \alpha^2 w_{xx}$$

by means of the transformations

$$u(x,t) = e^{-\beta t} w(x,t)$$
$$u(x,t) = e^{v[x - vt/2]/2\alpha^2} w(x,t)$$

After the transformations are made, the heat equation (the easy one) can be solved for $w(x,t)$, hence,

$$u = e^{-\beta t} w(x,t)$$
$$u = e^{v[x - vt/2]/2\alpha^2} w(x,t)$$

are the solutions of the original equations (of course, the BCs and the IC must be transformed too).

The reader may get the impression from the last two lessons that the only type of PDE that can be solved by separation of variables is

$$u_t = \alpha^2 u_{xx}$$

It is true the heat equation is the easiest parabolic PDE to solve by separation of variables, but it is in no way the *only* equation we can solve by this technique.

As mentioned earlier, as long as the equation is *linear* and *homogeneous*, we can separate variables. For example, two-dimensional heat flow inside a circle would be described by the equation

$$u_t = \alpha^2\left[u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right]$$

and although it has *variable coefficients,* it can still be separated into three ODEs.

This lesson will show the reader that sometimes a PDE doesn't have to be attacked directly but that the original PDE can be *transformed* into an easier one. In this way, the easier problem can be solved (by separation of variables or some other technique). We now present an example that illustrates this technique.

Transforming a Heat-Flow Problem with Lateral Heat Loss into an Insulated Problem

Consider the following problem:

$$\begin{array}{lll} & \text{PDE} & u_t = \alpha^2 u_{xx} - \beta u \qquad 0 < x < 1 \qquad 0 < t < \infty \\ (8.1) & \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty \\ & \text{IC} & u(x,0) = \phi(x) \qquad 0 \leqslant x \leqslant 1 \end{array}$$

where the term $-\beta u$ represents heat flow across the lateral boundary (Figure 8.1).

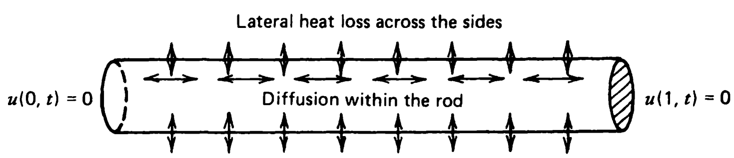


FIGURE 8.1 Heat flow described by $u_t = \alpha^2 u_{xx} - \beta u$.

The goal of this lesson is to introduce a *new temperature* $w(x,t)$ in place of $u(x,t)$, so that the PDE in w is simpler than the original one

$$u_t = \alpha^2 u_{xx} - \beta u$$

This is a common technique in PDEs, and the transformation is generally based on an intuitive feeling of how the solution of the original PDE behaves. For example, in our problem (8.1), the temperature $u(x,t)$ at any point x_0 is changing as a result of two phenomena

1. *diffusion* of heat within the rod (due to $\alpha^2 u_{xx}$).
2. *heat flow* across the lateral boundary (due to $-\beta u$).

The important point is that if there were *no* diffusion *within* the rod ($\alpha = 0$), then the temperature at each point x_0 would "damp" exponentially to zero according to

$$u(x_0,t) = u(x_0,0)e^{-\beta t}$$

By means of this observation, we wonder if we can essentially decompose the temperature $u(x,t)$ of problem (8.1) into two factors

$$u(x,t) = e^{-\beta t}w(x,t) \tag{8.2}$$

or

$$\text{Noninsulated temperature} = e^{-\beta t} \qquad \text{(insulated temperature)}$$

where $w(x,t)$ would represent the temperature due to diffusion only. Let's see what happens if we substitute this expression into problem (8.1); this is a routine calculation (the reader can do it on his or her own), and we arrive at

$$\begin{array}{lll} \text{PDE} & w_t = \alpha^2 w_{xx} & 0 < x < 1 \qquad 0 < t < \infty \\ \text{BCs} & \begin{cases} w(0,t) = 0 \\ w(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & w(x,0) = \phi(x) & 0 \leqslant x \leqslant 1 \end{array} \tag{8.3}$$

This is exactly the same problem we started with except that now the PDE doesn't contain $-\beta u$; so all we have to do to solve (8.1) is solve the transformed problem (8.3) and then multiply the solution $w(x,t)$ by $e^{-\beta t}$. In this case, we have already solved (8.3) previously by the separation of variables method and found

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \\ a_n &= 2\int_0^1 \phi(\xi)\sin(n\pi\xi)\,d\xi \end{aligned} \tag{8.4}$$

and, hence, the solution of the original problem (8.1) is

$$u(x,t) = e^{-\beta t}w(x,t)$$

The following example in the notes is solved by this technique.

NOTES

1. To solve the problem

$$
\begin{aligned}
u_t &= u_{xx} - u \qquad 0 < x < 1 \qquad 0 < t < \infty \\
u(0,t) &= 0 \\
u(1,t) &= 0 \\
u(x,0) &= \sin(\pi x) + 0.5 \sin(3\pi x)
\end{aligned}
$$

by the preceding strategy, we
   (a) neglect the convection term $-u$ for the time being.
   (b) solve the initial-boundary-value problem without the term $-u$ to get

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x) + 0.5\, e^{-(3\pi)^2 t} \sin(3\pi x)$$

   (c) multiply this solution by the convection factor $e^{-\beta t} = e^{-t}$ to get the solution

$$u(x,t) = e^{-t}\,[e^{-\pi^2 t} \sin(\pi x) + 0.5\, e^{-(3\pi)^2 t} \sin(3\pi x)]$$

2. The *diffusion-convection* equation

$$u_t = \alpha^2 u_{xx} - v u_x$$

(v is a constant) can also be transformed to

$$w_t = \alpha^2 w_{xx}$$

In this case, the transformation is

$$u(x,t) = e^{v(x - vt/2)/2\alpha^2}\, w(x,t)$$

This transformation essentially factors out the part of the solution (exponential factor) that is due to the moving medium. Note that the exponential factor consists of a moving exponential (moving to the right with velocity $v/2$). The reader will get a chance to use this transformation in the problem set.

PROBLEMS

1. Solve the diffusion problem

$$\begin{array}{llll} \text{PDE} & u_t = u_{xx} - u_x & 0 < x < 1 & 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty & \\ \text{IC} & u(x,0) = e^{x/2} & 0 \leq x \leq 1 & \end{array}$$

by transforming it into an easier problem. What does the solution look like? We could interpret this problem as describing the concentration $u(x,t)$ in a moving medium (moving from left to right with velocity $v = 1$) where the concentration at the *ends* of the medium are kept at zero (by some filtering device) and the *initial concentration* is $e^{x/2}$. Does your solution agree with this interpretation?

2. Solve the problem

$$\begin{array}{llll} \text{PDE} & u_t = u_{xx} - u + x & 0 < x < 1 & 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 1 \end{cases} & 0 < t < \infty & \\ \text{IC} & u(x,0) = 0 & 0 \leq x \leq 1 & \end{array}$$

by

(a) changing the nonhomogeneous BCs to homogeneous ones.
(b) transforming into a new equation without the term $-u$.
(c) solving the resulting problem.

3. Solve

$$\begin{array}{llll} \text{PDE} & u_t = u_{xx} - u & 0 < x < 1 & 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty & \\ \text{IC} & u(x,0) = \sin(\pi x) & 0 \leq x \leq 1 & \end{array}$$

directly by separation of variables without making any preliminary transformation. Does your solution agree with the solution you would obtain if the transformation

$$u(x,t) = e^{-t}w(x,t)$$

were made in advance?

OTHER READING

Nonlinear Partial Differential Equations in Engineering by W. F. Ames. Academic Press, 1965. This text discusses many types of transformations for changing old problems into new ones, so that sometimes even nonlinear problems can be transformed into linear ones.

LESSON 9

Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

PURPOSE OF LESSON: To show how to solve the initial-boundary-value problem

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} + f(x,t) \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leq x \leq 1$$

A nonhomogeneous PDE can be solved by finding a series solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

where the $X_n(x)$ are the eigenfunctions we find when solving the associated homogeneous problem

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \phi(x) \qquad 0 \leq x \leq 1$$

and $T_n(t)$ are functions that can be found by solving a sequence of ODEs.

In Lesson 6, we discussed how transformations could be made to transform nonhomogeneous BCs into *homogeneous* ones. Unfortunately, the PDE was left nonhomogeneous by this process, and we were left with the problem

$$
(9.1) \qquad \begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} + f(x,t) & 0 < x < 1 \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = \phi(x) & 0 \leq x \leq 1 \end{array}
$$

The purpose of this lesson is to solve this problem by a method that is analogous to the method of *variation of parameters* in ODEs and is known as the **eigenfunction expansion method**.

The idea is quite simple. Inasmuch as the solution of (9.1) with $f(x,t) = 0$ (so-called corresponding homogeneous problem) is given by

$$
u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} X_n(x)
$$

where λ_n and $X_n(x)$ are the eigenvalues and eigenfunctions of the Sturm-Liouville problem,

$$
(9.2) \qquad \begin{aligned} &X'' + \lambda^2 X = 0 \\ &\alpha_1 X'(0) + \beta_1 X(0) = 0 \\ &\alpha_2 X'(1) + \beta_2 X(1) = 0 \end{aligned}
$$

we ask whether the solution of the nonhomogeneous problem (9.1) can be written in the slightly more general form

$$
u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)
$$

The reason for this speculation is physically appealing, inasmuch as a source of heat $f(x,t)$ *within* the rod will give rise to a new time component and not the damping factor

$$
e^{-(\lambda_n \alpha)^2 t}
$$

as was the case when the only input into the problem was the IC.

To show how this method works, we apply it to a simple problem so the details aren't as complicated.

Solution by the Eigenfunction Expansion Method

Consider the nonhomogeneous problem

$$
\begin{array}{lll}
 & \text{PDE} & u_t = \alpha^2 u_{xx} + f(x,t) \qquad 0 < x < 1 \qquad 0 < t < \infty \\
(9.3) & \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty \\
 & \text{IC} & u(x,0) = \phi(x) \qquad 0 \leqslant x \leqslant 1
\end{array}
$$

To solve this problem, we divide the procedure into the following steps:

STEP 1 The basic idea in this method is to decompose the heat source $f(x,t)$ into simple components

$$f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + \ldots + f_n(t)X_n(x) + \ldots$$

and find the response $u_n(x,t)$ to *each* of these individual components $f_n(t)X_n(x)$. The solution to our problem is then

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

To determine how to decompose $f(x,t)$ into its component parts $f_n(t)X_n(x)$ is one of the major problems. It turns out that the $X_n(x)$ factors in this problem are the *eigenvectors* of the Sturm-Liouville system we get when solving the *associated homogeneous problem* to (9.3) by separation of variables; that is,

$$
(9.4) \qquad
\begin{array}{l}
u_t = \alpha^2 u_{xx} \qquad \text{(note that } f(x,t) = 0) \\
u(0,t) = 0 \\
u(1,t) = 0 \\
u(x,0) = \phi(x)
\end{array}
$$

in this case, the Sturm-Liouville problem we find when separating variables is

$$
\begin{array}{l}
X'' + \lambda^2 X = 0 \\
X(0) = 0 \\
X(1) = 0
\end{array}
$$

and, hence, the $X_n(x)$ are

$$X_n(x) = \sin(n\pi x) \qquad n = 1, 2, \ldots$$

Hence, our decomposition of the heat source has the form

$$(9.5) \quad f(x,t) = f_1(t)\sin(\pi x) + f_2(t)\sin(2\pi x) + \ldots + f_n(t)\sin(n\pi x) + \ldots$$

Finally, to find the functions $f_n(t)$, we merely multiply each side of this equation by $\sin(m\pi x)$ and integrate from zero to one (with respect to x); hence, we have

$$\int_0^1 f(x,t) \sin(m\pi x)\, dx = \sum_{n=1}^{\infty} f_n(t) \int_0^1 \sin(m\pi x) \sin(n\pi x)\, dx$$
$$= \frac{1}{2} f_m(t)$$

or (changing m to n)

$$f_n(t) = 2 \int_0^1 f(x,t) \sin(n\pi x)\, dx \tag{9.6}$$

This will give us an equation for the coefficients $f_n(t)$ in terms of the heat source $f(x,t)$.

STEP 2 (Find the response $u_n(x,t) = T_n(t)X_n(x)$ to input $f_n(t)X_n(x)$)
We now replace the heat source $f(x,t)$ by its decomposition

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x)$$

and try to find the individual responses

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$$

in other words, we seek the functions $T_n(t)$. Knowing these, the answer to our problem is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$$

Substituting $u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$ into the system

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x) \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= \phi(x) \end{aligned}$$

gives us

$$\sum_{n=1}^{\infty} T_n'(t)\sin(n\pi x) = -\alpha^2\sum_{n=1}^{\infty}(n\pi)^2 T_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)$$

(9.7)

$$\sum_{n=1}^{\infty} T_n'(t)\sin 0 = 0 \qquad \text{(says nothing; zero = zero)}$$

$$\sum_{n=1}^{\infty} T_n(t)\sin(n\pi) = 0 \qquad \text{(says nothing; zero = zero)}$$

$$\sum_{n=1}^{\infty} T_n(0)\sin(n\pi x) = \phi(x)$$

Rewriting the PDE and the IC as

$$\text{PDE} \qquad \sum_{n=1}^{\infty}[T_n' + (n\pi\alpha)^2 T_n - f_n(t)]\sin(n\pi x) = 0$$

$$\text{IC} \qquad \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x) = \phi(x)$$

we can see fairly easily that $T_n(t)$ will satisfy the simple initial value problem

(9.8)

$$T_n' + (n\pi\alpha)^2 T_n = f_n(t)$$

$$T_n(0) = 2\int_0^1 \phi(\xi)\sin(n\pi\xi)\,d\xi = a_n$$

This ODE is one of the easier ones to solve (use an integrating factor) and has the solution

(9.9)

$$T_n(t) = a_n e^{-(n\pi\alpha)^2 t} + \int_0^t e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau)\,d\tau$$

Hence, the solution of problem (9.3) is

(9.10)

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$$

$$= \sum_{n=1}^{\infty}[a_n e^{-(n\pi\alpha)^2 t}\sin(n\pi x)] + \sum_{n=1}^{\infty}\left[\sin(n\pi x)\int_0^t e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau)\,d\tau\right]$$

Transient part (because of the initial condition)

Steady state (because of the right-hand side $f(x,t)$)

We can see from this solution that the *temperature response* is due to *two parts*: the first part that is due to the IC and the second part that is due to the heat source $f(x,t)$. The phrase steady state is not the best phrase to describe the second part, since it doesn't necessarily come to rest (it may approach a periodic steady state, *if* $f(x,t)$ is periodic in t).

This completes the problem. Before stopping, however, we will show how this method can be applied to a specific example.

Solution of a Problem by the Eigenfunction-Expansion Method

Consider the simple problem

$$(9.11)\qquad \begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} + \sin(3\pi x) & 0 < x < 1 \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = \sin(\pi x) & 0 \leqslant x \leqslant 1 \end{array}$$

Our goal is to compute the coefficients $T_n(t)$ in the expansion

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$$

(the eigenfunctions $X_n(x)$ are still the same for this problem). If we substitute this expansion in the problem, we will get an ODE for the functions $T_n(t)$ In fact, we will get

$$T_n' + (n\pi\alpha)^2 T_n = f_n(t) = 2\int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx = \begin{cases} 1 & n = 3 \\ 0 & n \neq 3 \end{cases}$$
$$T_n(0) = 2\int_0^1 \sin(\pi\xi)\sin(n\pi\xi)\,d\xi = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Writing out these equations for $n = 1, 2, \ldots$, we see

$$(n = 1)\qquad \left.\begin{array}{l} T_1' + (\pi\alpha)^2 T_1 = 0 \\ T_1(0) = 1 \end{array}\right\} \Rightarrow T_1(t) = e^{-(\pi\alpha)^2 t}$$

$$(n = 2)\qquad \left.\begin{array}{l} T_2' + (2\pi\alpha)^2 T_2 = 0 \\ T_2(0) = 0 \end{array}\right\} \Rightarrow T_2(t) = 0$$

$$(n = 3)\qquad \left.\begin{array}{l} T_3' + (3\pi\alpha)^2 T_3 = 1 \\ T_3(0) = 0 \end{array}\right\} \Rightarrow T_3(t) = \frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi\alpha)^2 t}]$$

$$(n \geqslant 4)\qquad \left.\begin{array}{l} T_n' + (n\pi\alpha)^2 T_n = 0 \\ T_n(0) = 0 \end{array}\right\} \Rightarrow T_n(t) = 0$$

Hence the solution of our problem is

$$u(x,t) = e^{-(\pi\alpha)^2 t} \sin(\pi x) + \frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi\alpha)^2 t}] \sin(3\pi x) \tag{9.12}$$

Transient (because of initial conditions) — Steady state (because of the right-hand side of the PDE)

NOTES

1. The method of eigenfunction expansion is one of the most powerful for solving nonhomogeneous PDEs. Later, when we study integral transforms, we will see that there are other methods for solving these types of problems.
2. The eigenfunctions $X_n(x)$ in the expansion *change* from problem to problem and depend on the PDE and BCs. The reader should look at problem 4 in the problem set to make sure he or she knows how to find the eigenfunctions $X_n(x)$.
3. If the reader remembers ODE theory, he or she will remember that solutions of equations corresponding to nonhomogeneous terms like

$$P_n(x)\, e^{\alpha x}\begin{cases} \sin(\beta x) \\ \cos(\beta x) \end{cases}$$

could be found by the method of *undetermined coefficients*. The same is true here. Problem (9.11) could be solved by this method. Any reader interested in this method should consult the reference.

PROBLEMS

1. The solution of the problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx} + \sin(3\pi x) & 0 < x < 1 \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = \sin(\pi x) & 0 \leq x \leq 1 \end{array}$$

is given by (9.12). Does this solution agree with your intuition? What does the solution look like?

2. Solve the problem

$$\text{PDE} \quad u_t = u_{xx} + \sin(\pi x) + \sin(2\pi x) \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 0 \quad 0 \le x \le 1$$

3. Solve the problem

$$\text{PDE} \quad u_t = u_{xx} + \sin(\pi x) \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 1 \quad 0 \le x \le 1$$

by the method of eigenfunction expansion.

4. Find the solution of

$$\text{PDE} \quad u_t = u_{xx} + \sin(\lambda_1 x) \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u_x(1,t) + u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 0 \quad 0 \le x \le 1$$

by the method of eigenfunction expansion where λ_1 is the first root of the equation $\tan \lambda = -\lambda$. What are the eigenfunctions $X_n(x)$ in this problem?

5. Solve the problem

$$\text{PDE} \quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = \cos t \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = x \quad 0 \le x \le 1$$

by

(a) transforming it to one with zero BCs.

(b) solving the resulting problem by expanding it in terms of eigenfunctions.

OTHER READING

Elementary Partial Differential Equations by P. W. Berg and J. L. McGregor. Holden-Day, 1966. One of the more popular texts on PDEs; slightly more advanced than this text; clearly written. An extensive section on nonhomogeneous problems (Chapter 5).

LESSON 10

Integral Transforms (Sine and Cosine Transforms)

PURPOSE OF LESSON: To introduce the idea of integral transforms and show how they transform PDEs in n variables into differential equations in $n-1$ variables.

It is also shown that these transforms can be interpreted as resolving the input of the problem into simple parts (frequency resolution), finding the solution for each subpart, and adding the results.

In summary, integral transforms change differentiation to multiplication, and, hence, certain partial derivatives are changed into algebraic expressions.

The sine and cosine transforms are introduced and are used to solve an infinite-diffusion problem. The solution is interesting in that it involves the complementary-error function.

An integral transformation is merely a transformation that assigns to one function $f(t)$ a new function $F(s)$ by means of a formula like

$$F(s) = \int_A^B K(s,t)f(t)\,dt$$

Note that we *start* with a function of t and *end* with a function of s. The function $K(s,t)$ is called the **kernel of the transformation** and is the major ingredient that distinguishes one transform from another; it is chosen so that the transform has certain desirable properties. The limits of integration A and B also change from transformation to transformation.

The general philosophy behind integral transformations is that they eliminate *partial derivatives* with respect to one of the variables; hence, the new equation has one less variable. For example, if we apply a transform to the PDE

$$u_t = u_{xx}$$

for the purpose of eliminating the time derivative, then we would arrive at an ODE in x. On the other hand, if we had the PDE

$$u_{xx} + u_{yy} + u_{zz} = 0$$

and applied the Fourier transform to the x-variable, then we would eliminate the derivative u_{xx} and would have a new PDE in y and z. We could, of course, apply the Fourier transform again to eliminate one of the other variables (and arrive at an ODE in the last remaining variable). In other words, integral transforms change problems into easier ones. The transformed problem is then solved, and its *inverse* is obtained to find the solution to the original problem; this general strategy is illustrated in Figure 10.1.

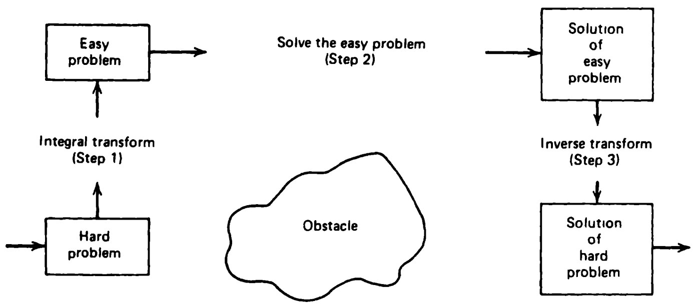


FIGURE 10.1 General philosophy of transforms.

In Figure 10.1, we see that along with every integral transform, there is an *inverse transform* that will reproduce that original function from its transform. The transform and its inverse together form what is called a **transform pair**. Table 10.1 lists several common transform pairs that we will use to solve PDEs.

TABLE 10.1 Some Common Transform Pairs

Transform pairs

1. $$\begin{cases} \mathcal{F}_s[f] = F(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(t)\sin(\omega t)\,dt & \text{(Fourier sine transform)} \\ \mathcal{F}_s^{-1}[F] = f(t) = \displaystyle\int_0^\infty F(\omega)\sin(\omega t)\,d\omega & \text{(inverse sine transform)} \end{cases}$$

2. $$\begin{cases} \mathcal{F}_c[f] = F(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(t)\cos(\omega t)\,dt & \text{(Fourier cosine transform)} \\ \mathcal{F}_c^{-1}[F] = f(t) = \displaystyle\int_0^\infty F(\omega)\cos(\omega t)\,d\omega & \text{(inverse cosine transform)} \end{cases}$$

3. $$\begin{cases} \mathcal{F}[f] = F(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^\infty f(x)\,e^{-i\omega x}dx & \text{(Fourier transform)} \\ \mathcal{F}^{-1}[F] = f(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^\infty F(\omega)\,e^{i\omega x}d\omega & \text{(inverse Fourier transform)} \end{cases}$$

4. $$\begin{cases} \mathcal{F}_s[f] = S_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin(n\pi x/L)\,dx & \text{(finite-sine transform)} \\ \mathcal{F}_s^{-1}[F_n] = f(x) = \displaystyle\sum_{n=1}^{\infty} S_n \sin(n\pi x/L) & \text{(inverse finite-sine transform)} \end{cases}$$

5. $$\begin{cases} \mathcal{F}_c[f] = C_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos(n\pi x/L)\,dx & \text{(finite-cosine transform)} \\ \mathcal{F}_c^{-1}[F_n] = f(x) = \dfrac{C_0}{2} + \displaystyle\sum_{n=1}^{\infty} C_n \cos(n\pi x/L) & \text{(inverse finite-cosine transform)} \end{cases}$$

6. $$\begin{cases} \mathcal{L}[f] = F(s) = \displaystyle\int_0^\infty f(t)\,e^{-st}\,dt & \text{(Laplace transform)} \\ \mathcal{L}^{-1}[F] = f(t) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds & \text{(inverse Laplace transform)} \end{cases}$$

7. $$\begin{cases} H[f] = F_n(\xi) = \displaystyle\int_0^\infty rJ_n(\xi r)f(r)\,dr & \text{(Hankel transform)} \\ H^{-1}[F_n] = f(r) = \displaystyle\int_0^\infty J_n(\xi r)F_n(\xi)\,d\xi & \text{(inverse Hankel transform)} \end{cases}$$

Note that in these transforms we have alternative notations. For instance, in the case of the Laplace transform, the notation $\mathcal{L}[f]$ indicates that we are taking the transform of f, whereas the alternative notation $F(s)$ indicates a function of s.

The current lesson does not attempt to study all of these transform pairs—only the sine and cosine transforms (1. and 2.); later, we will study several of the others. Questions about the relationship between the transforms, when to apply them, advantages and disadvantages of each, will be answered as we go along. However, before we begin the study of integral transforms, it will be useful to study what is called the *spectrum of a function* (or the *spectral resolution* of a function).

The Spectrum of a Function

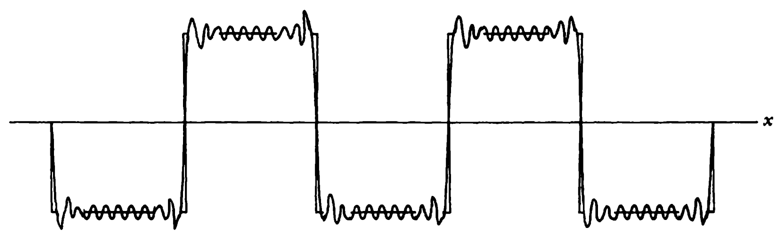
Integral transforms and the spectrum of a function are closely related; in fact, an **integral transformation** can be thought of as a resolution of a function into a certain spectrum of components. How the transform actually resolves the function changes from transform to transform, but the function is being resolved into something nevertheless.

For instance, let's consider the resolution of a periodic funciton $f(x)$ into sines and cosines (Fourier series)*

* Fourier series will be discussed in detail in Chapter 11.

$$f(x) = \sum_{n=0}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]$$

(Figure 10.2).



A square wave approximated by sines and cosines

FIGURE 10.2 Expansion of a periodic function into sines and cosines.

Here, the coefficients A_n and B_n represent the amount of the function $f(x)$ made up by cos (nx) and sin (nx), respectively, while the square root

$$\sqrt{A_n^2 + B_n^2}$$

(called the **spectrum of the function**) measures the amount of $f(x)$ with frequency n.

For example, if the function $f(x)$ were a simple sum of sines and cosines

$$f(x) = 1 + \sin x + \frac{1}{5}\sin(3x) + \cos x + \frac{1}{2}\cos(2x) + \frac{1}{4}\cos(4x)$$

then its spectrum (discrete) would be as given in Figure 10.3.

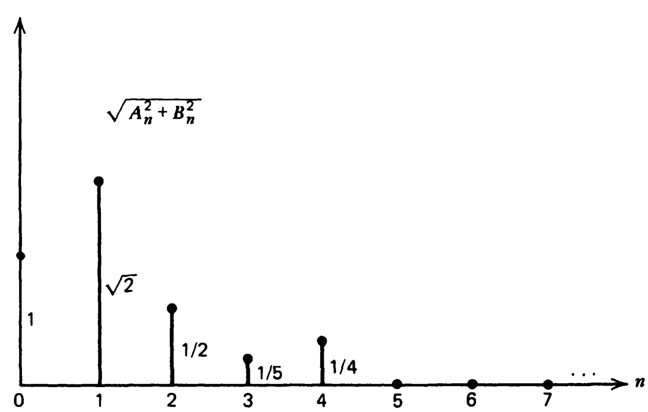


FIGURE 10.3 Discrete spectrum of *f(x)*.

By reading off the values of $\sqrt{A^2{}_n + B^2{}_n}$, we can tell the magnitude of the component in $f(x)$ with frequency n.

Functions that are *periodic* can be resolved into *infinite series* (they have discrete spectrums), whereas functions that are *not periodic* must be resolved into a *continuous spectrum* of values (of course, if a function is defined only on a *finite interval*, we could extend the function outside the interval in a periodic way, so that a Fourier series representation could be obtained for the function inside the interval).

For example, although a nonperiodic function $f(x)$ cannot be represented by an infinite series of sines and cosines, we might be tempted to write it as a *continuous analog* of the Fourier series; that is,

$$f(x) = \int_{-\infty}^{\infty} [C(\omega) \cos(\omega x) + S(\omega) \sin(\omega x)]\, d\omega$$

where the functions $S(\omega)$ and $C(\omega)$ measure the sine and cosine component of $f(x)$ and

$$\sqrt{S^2(\omega) + C^2(\omega)}$$

measures the ω frequency component of $f(x)$ and is called the spectrum (continuous spectrum) of $f(x)$.

With this intuitive explanation of the spectrum of a function, we now get to the nuts and bolts of integral transforms. The first step would be to list a few properties of these transforms that make them work.

Sine and Cosine Transforms of Derivatives

(10.1)

1. $\mathcal{F}_s[f'] = -\omega\, \mathcal{F}_c[f]$ (proved by integration by parts)
2. $\mathcal{F}_s[f''] = \dfrac{2}{\pi}\omega f(0) - \omega^2 \mathcal{F}_s[f]$
3. $\mathcal{F}_c[f'] = \dfrac{-2}{\pi} f(0) + \omega \mathcal{F}_s[f]$
4. $\mathcal{F}_c[f''] = -\dfrac{2}{\pi} f'(0) - \omega^2 \mathcal{F}_c[f]$

Several other sine and cosine transforms and their inverses can be found in the tables at the end of the lessons. We now show how the sine transform can solve an important initial-boundary-value problem.

Solution of an Infinite-Diffusion Problem via the Sine Transform

The problem we are interested in is the *infinite diffusion problem*

$$\begin{array}{llll} \text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < \infty & 0 < t < \infty \\ \text{BC} & u(0, t) = A & 0 < t < \infty & \\ \text{IC} & u(x, 0) = 0 & 0 \leq x < \infty & \end{array}$$

(Figure 10.4).

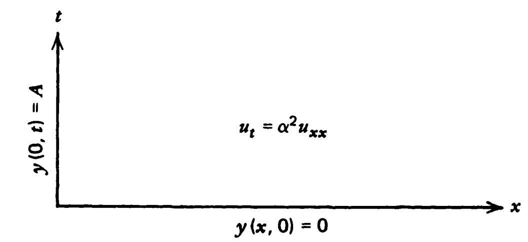


FIGURE 10.4 Diffusion problem in a semi-infinite medium.

STEP 1 To solve this, we break it into three simple steps. First our strategy is to transform the *x-variable* via the Fourier sine transform so that we get an ODE in time. We start by transforming each side of the PDE; in other words

$$\mathcal{F}_s[u_t] = \alpha^2 \mathcal{F}_s[u_{xx}]$$

Let's consider each term individually:

$\mathcal{F}_s[u_t]$: The partial derivative u_t in this problem is what we could call the *off derivative*, since our transform is *with respect to x*. In this case, we can write

$$\begin{aligned} \mathcal{F}_s[u_t] &= \frac{2}{\pi} \int_0^\infty u_t(x,t) \sin(\omega x)\, dx \\ &= \frac{\partial}{\partial t}\left[\frac{2}{\pi}\int_0^\infty u(x,t) \sin(\omega x)\, dx\right] \\ &= \frac{d}{dt}\mathcal{F}_s[u] \\ &= \frac{d}{dt} U(t) \end{aligned}$$

The fact that we took the derivative *outside* the integral is a property from calculus. Note that u is a function of x and t, whereas its transform

$$\mathcal{F}_s[u] = U(\omega, t)$$

is a function of ω and t. The new variable ω will be treated like a parameter in the new problem, and, hence, we call the sine transform a function of t alone; that is,

$$\mathcal{F}_s[u] = U(t)$$

$\mathcal{F}_s[u_{xx}]$: For this one, we have the identity

$$\begin{aligned}\mathcal{F}_s[u_{xx}] &= \frac{2}{\pi}\omega u(0,t) - \omega^2\mathcal{F}_s[u] \\ &= \frac{2}{\pi}\omega u(0,t) - \omega^2 U(t) \\ &= \frac{2A\omega}{\pi} - \omega^2 U(t)\end{aligned}$$

Note here that when you proved these identities (10.1), you did it for functions of *one* variable $f(x)$. We now have a slight modification, since $u(x,t)$ depends on x and t. You should use the formulas according to which variable is being transformed and treat the others as constants. In this case, the transform is with respect to x, and, hence, t is just carried along as a constant. Also note that the BC $u(0,t) = A$ is used at this point in our operation.

Substituting these expressions into our PDE, we arrive at the ODE

$$\frac{dU}{dt} = \alpha^2\left[-\omega^2 U(t) + \frac{2A\omega}{\pi}\right]$$

The only thing missing is an IC for $U(t)$; we get this by transforming the IC $u(x,0) = 0$ to get

$$\mathcal{F}_s[u(x,0)] = U(0) = 0$$

This completes the first step in the transform process—we have changed the original problem into an initial-value problem

$$\text{(10.2)}\qquad \begin{aligned}&\text{ODE} && \frac{dU}{dt} + \omega^2\alpha^2 U = \frac{2A\omega\alpha^2}{\pi} \\ &\text{IC} && U(0) = 0\end{aligned}$$

STEP 2 To solve this IVP, we could use a variety of elementary techniques from ordinary differential equations (integrating factor, homogeneous and particular solution); in any case, the solution is

$$U(t) = \frac{2A}{\pi\omega}(1 - e^{-\omega^2\alpha^2 t})$$

We have now found the sine transform for the answer $u(x,t)$. The last step is to find the inverse transform of $U(t)$; that is,

$$u(x,t) = \mathcal{F}_s^{-1}[U]$$

STEP 3 To find the solution, we can either evaluate the inverse transform directly from the integral or else resort to the tables. Using the tables, we see that

$$u(x,t) = A\ erfc\ (x/2\alpha\sqrt{t})$$

where $erfc(x)$, $0 < x < \infty$, is called the **complementary-error function** and is given by

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

See Figure 10.5 for its graph.

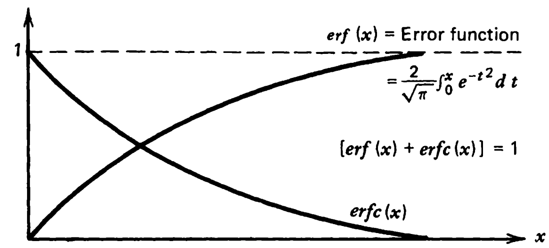


FIGURE 10.5 Graphs of *erf(x)* and *erfc(x)*.

The exact values of these well-known functions can be found in most tables for physics and chemistry. It should be noted that these integrals cannot be integrated by the usual elementary tricks of calculus.

Interpretation of the Solution

The solution

$$u(x,t) = A\ erfc\ [x/2\alpha\sqrt{t}]$$

makes a lot of sense. For different values of time, we have the graph of a complementary-error function with different scale factors on the x-axis. As time increases, the error function gets pulled out; for a graph of the solution at different values of time, see Figure 10.6.

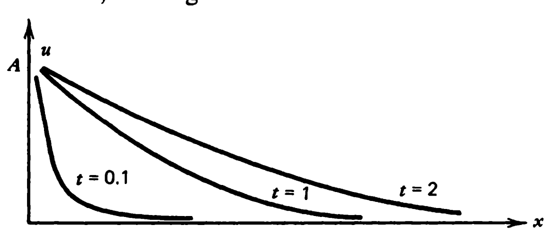


FIGURE 10.6 Solution to semi-infinite rod with fixed temperature *A* at the end.

PROBLEMS

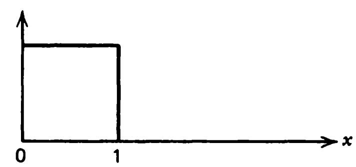
1. Prove the identities (10.1). What assumptions do you need to assume about the function f?
2. Solve the ordinary-differential equation problem (10.2).
3. Solve by means of the sine *or* cosine transform

$$
\begin{array}{rll}
\text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < \infty \\
\text{BC} & u_x(0, t) = 0 & 0 < t < \infty \\
\text{IC} & u(x,0) = H(1 - x) & 0 \leq x < \infty
\end{array}
$$

where $H(x)$ is the *Heaviside function*:

$$H(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

In other words, the IC looks like



What does the graph of the solution look like for various values of time?

OTHER READING

1. *Operational Mathematics* by R. V. Churchill. McGraw-Hill, 1958. An excellent text covering many of the integral transforms; good problems and many tables.

2. *Tables of Integral Transform* by A. Erdelyi. McGraw-Hill, 1954. One of the most comprehensive tables of integral transform.

3. *Integral Transforms in Mathematical Physics* by C. J. Tranter. Chapman and Hall (Science Paperbacks), 1971. A small, but concise paperback; easy to read with many examples.

LESSON 11

The Fourier Series and Transform

PURPOSE OF LESSON: To introduce the Fourier series and to show how it can represent certain periodic functions $f(x)$ by sums of sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$$

In the case of nonperiodic functions on $(-\infty, \infty)$, to show also how the Fourier series is replaced by the *Fourier transform* and how a function $f(x)$ can be represented by a continuous resolution of simple functions. This resolution (the Fourier integral) can be written in the complex form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx\right] e^{i\xi x}\, d\xi$$

which gives rise to the Fourier and inverse Fourier transforms

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx \qquad \text{(Fourier transform)}$$

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi)e^{i\xi x}\, d\xi \qquad \text{(inverse Fourier transform)}$$

The importance of the Fourier series in PDE theory is that periodic functions $f(x)$ defined on $(-\infty, \infty)$ *or* functions defined on *finite intervals* can be represented by infinite series of sines and cosines, and in this way, problems can be resolved into simple ones. For example, the so-called **sawtooth wave**

$$\begin{aligned} &f(x) = x \qquad -L < x < L \\ &f(x + 2L) = f(x) \qquad \text{(periodic condition)} \end{aligned}$$

shown in Figure 11.1

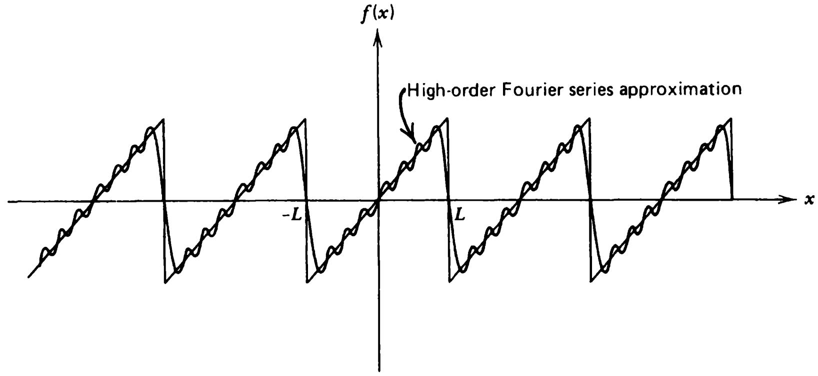


FIGURE 11.1 Sawtooth wave represented by a Fourier series.

can be represented by the Fourier series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]} \tag{11.1}$$

where the **Fourier coefficients** a_n and b_n are given by the **Euler formulas**

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx = 0 \qquad n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx = -(2L/n\pi)(-1)^n \qquad n = 1, 2, \ldots \end{aligned} \tag{11.2}$$

These integrations are routine calculus evaluations. To find Euler's formulas for a_n and b_n, respectively, we multiply each side of equation (11.1) by sin (nx) or cos (nx) and integrate the resulting equation from $-L$ to L. The *orthogonality* of the functions $\{\sin(n\pi x/L)\}$ and $\{\cos(n\pi x/L)\}$ allows us to solve for the coefficients a_n and b_n; see problem 6. For the sawtooth wave, the Fourier representation is given by

$$f(x) = \frac{2L}{\pi}\left[\sin(\pi x/L) - \frac{1}{2}\sin(2\pi x/L) + \frac{1}{3}\sin(3\pi x/L) - \ldots\right] \tag{11.3}$$

where each term (called **harmonic**) has a larger frequency than the previous term, and all frequencies are *multiples* of a fundamental frequency that has the same period as the function $f(x)$

One of the drawbacks of the Fourier series is that in order for a function to have a Fourier series representation, the function must be periodic. Of course, if we want to expand a function (say, $f(x) = x$ for $0 \leq x \leq 1$) defined on a *finite interval*, we could use expansion (11.1). The fact that the Fourier series is periodic *outside* the interval [0,1] doesn't concern us, since we're only interested in the function *inside* the interval. As a matter of fact, we can represent a function inside an interval with many different types of Fourier series by considering different types of extensions outside the interval (some converge faster than others).

The reader shouldn't get the idea that every periodic function can be represented by a Fourier series expansion. What we do know is that if a function $f(x)$ can be represented by a Fourier series (11.1), then the coefficients a_n and b_n are given by the *Euler formulas* (11.2). What's more, even if a function $f(x)$ can be represented by a Fourier series, it isn't always true that the *derivative* $f'(x)$ can be found by differentiating the series term by term. In fact, we can easily see that the derivative of $f(x) = x$ (the sawtooth function) cannot be found by differentiating each term of the Fourier series (11.3). Indeed, the differentiated series will not even converge for any x (the reader can verify this himself or herself).

The *exact conditions* that insure that a function $f(x)$ will have a Fourier series representation and that the representation can be differentiated term by term are found in the recommended reading for this lesson. For our purposes, we are content to know the important result of P. G. Dirichlet's (Deer-ish-lay) theorem, which states

> *Dirichlet's theorem* (sufficient conditions for a function to have a Fourier series representation):
>
> If $f(x)$ is a bounded periodic function that contains a finite number of maximum points, minimum points, and points of discontinuity in each period, then the Fourier series of $f(x)$ converges to $f(x)$ at each point x where $f(x)$ is continuous and to the *average* of the left- and right-hand limits of $f(x)$ at those points where $f(x)$ is discontinuous.

For example, in Figure 11.1, the Fourier series converges to the function $f(x)$ for all except $x = \pm L, \pm 3L, \ldots$ (points of discontinuity), in which case it converges to zero (the average of $+L$ and $-L$).

We are now almost ready to introduce the *Fourier transform*. Before we do, however, it will be useful to introduce the idea of the *frequency spectrum* of a periodic function.

Discrete Frequency Spectrum of a Periodic Function

For periodic functions, we can interpret the Fourier series as the replacement of a periodic function $f(x)$ by a sequence $\{c_n\}$ of numbers

$$c_n = \sqrt{a_n^2 + b_n^2} \qquad n = 0, 1, 2, \ldots .$$

where the numbers c_n can be taken as measuring the contributions of the various frequency components of the function $f(x)$. For example, the sawtooth wave $f(x)$ has the Fourier series representation

$$f(x) = \frac{2L}{\pi}\left[\sin(\pi x/L) - \frac{1}{2}\sin(2\pi x/L) + \frac{1}{3}\sin(3\pi x/\mathrm{L}) - \ldots\right]$$

and, hence, the frequency spectrum $\{c_n\}$ is $c_n = 2L/n\pi$ for $n = 1, 2, \ldots$ and $c_0 = \left|\frac{a_0}{2}\right| = 0$ (Figure 11.2).

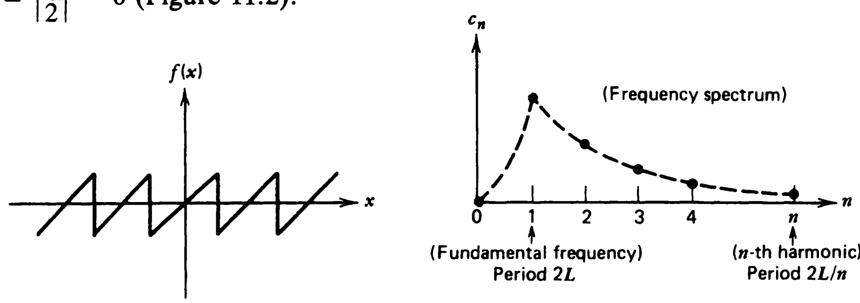


FIGURE 11.2 Discrete frequency spectrum of the sawtooth wave.

The sequence $\{c_n\}$ is somewhat similar to the decomposition of white light into the frequency spectrum of colors obtained with a spectroscope.
We now introduce the Fourier transform.

The Fourier Transform

The major difficulty with Fourier series representation is that nonperiodic functions defined on $(-\infty,\infty)$ cannot be represented. It is possible, however, to find an analogous representation for some of these functions. Without going through the details of the proof, we can show that the Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]$$

changes to the *Fourier integral* representation (continuous frequency resolution)

$$f(x) = \int_0^{\infty} a(\xi)\cos(\xi x)\,d\xi + \int_0^{\infty} b(\xi)\sin(\xi x)\,d\xi \tag{11.4}$$

where

$$\begin{aligned} a(\xi) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\xi x)\,dx \\ b(\xi) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\xi x)\,dx \end{aligned} \tag{11.5}$$

for nonperiodic functions defined on $(-\infty,\infty)$. Here, we see that the Fourier integral representation has resolved the function $f(x)$ into *all* frequencies $0 < \xi < \infty$ (and not just multiples of *one basic frequency*, as with periodic functions). As we did in the Fourier series, we define the **frequency spectrum**

$$C(\xi) = \sqrt{a^2(\xi) + b^2(\xi)}$$

which measures the composition of the function $f(x)$ in terms of its frequencies. Some examples of functions $f(x)$ and their frequency spectrums are given in Figure 11.3.

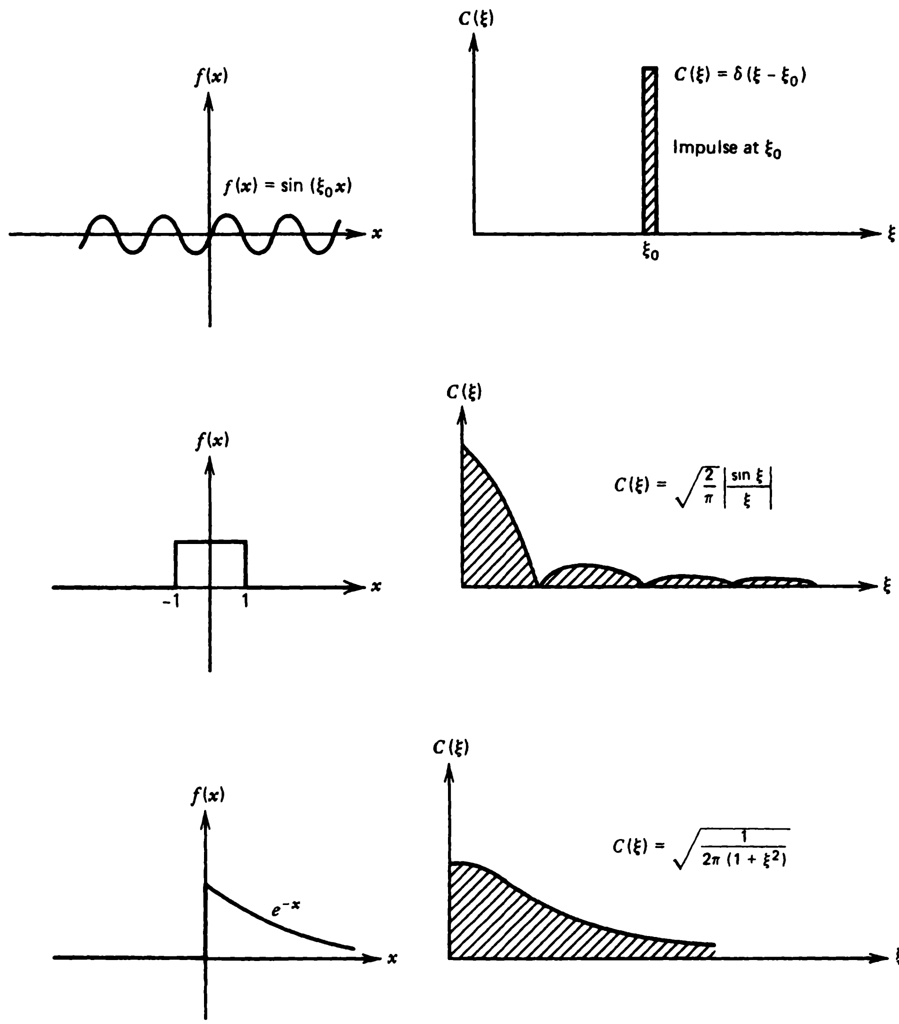


FIGURE 11.3 Frequency spectra for various functions.

Note that functions $f(x)$ that have sharp corners give rise to frequency spectra with large frequencies, since sharp corners require high-frequency components to represent them. On the other hand, the simple periodic function $f(x) = \sin(\xi_0 x)$ obviously has a frequency spectrum that is zero everywhere except at $\xi = \xi_0$.

We are now in a position to define what is generally known as the *exponential Fourier transform* (Equations 11.5 are known as the **Fourier sine and cosine transforms**). By use of Euler's (Oy-ler) equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we can rewrite equation (11.4) after a little effort as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \right] e^{i\xi x}\, d\xi \tag{11.6}$$

which is known as the **Fourier integral representation**. From this, we can write the two equations

$$\boxed{\begin{aligned} &\mathscr{F}[f] \equiv F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx && \text{(Fourier transform)} \\ &\mathscr{F}^{-1}[F] \equiv f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x}\, d\xi && \text{(inverse Fourier transform)} \end{aligned}} \tag{11.7}$$

which are the Fourier and inverse Fourier transforms. Properties of this transform pair will be discussed in the next lesson along with problems using these transforms.

NOTES

1. The Fourier transform $F(\xi)$ of $f(x)$ can be a *complex function*; for example, the Fourier transform of

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x} & x > 0 \end{cases}$$

   is $F(\xi) = \dfrac{1}{\sqrt{2\pi}} \dfrac{1 - i\xi}{1 + \xi^2}$

2. The absolute value of the Fourier transform $F(\xi)$ is the frequency spectrum of $f(x)$. For example, in note 1, the frequency spectrum of $f(x)$ is

$$|F(\xi)| = \sqrt{\frac{1}{2\pi\,(1 + \xi^2)}}$$

(the reader should be able to find the magnitude of a complex number).

3. Not all functions have Fourier transforms [the integral (11.7) may not exist]; in fact, $f(x) = c$, $\sin x$, e^x, x^2, do *not* have Fourier transforms. Only functions that go to zero sufficiently fast as $|x| \to \infty$ have transforms. In applications, we apply the Fourier transform to temperature functions, wave functions, and other physical phenomena that go to zero as $|x| \to \infty$.

PROBLEMS

1. What is the Fourier series expansion of the square sine wave

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 \leq x < 1 \end{cases}$$
$$f(x + 2) = f(x) \qquad \text{(periodic condition)}$$

   Graph the first 2, 3, 4 terms of the series to see how it is converging to $f(x)$. Also graph the frequency spectrum of $f(x)$.

2. Show that if we differentiate the Fourier series expansion (11.3) of the sawtooth wave term by term, we arrive at an infinite series that clearly does not represent the derivative of the sawtooth curve.
3. Graph the frequency spectrum of the following periodic functions:
   (*a*) $f(x) = \sin x$
   (*b*) $f(x) = \sin x + \cos 2x$
   (*c*) $f(x) = \sin x + \cos x + 0.5 \sin 3x$
4. What is the Fourier transform $F(\xi)$ and the frequency spectrum $C(\xi) = |F(\xi)|$ of the function

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

5. Show that the absolute value of the function $F(\xi) = 1/(1 + i\xi)$ is $|F(\xi)| = \sqrt{1/(1 + \xi^2)}$.

HINT First multiply the numerator and denominator by $1 - i\xi$ to get rid of the complex number i in the denominator.

6. Verify the *orthogonality* properties of sines and cosines on the interval $[-L, L]$

$$\int_{-L}^{L} \sin(m\pi x/L)\sin(n\pi x/L)\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \cos(m\pi x/L)\cos(n\pi x/L)\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \sin(m\pi x/L)\cos(n\pi x/L)\,dx = 0 \qquad \text{all } m, n = 1, 2, 3, \ldots$$

OTHER READING

Partial Differential Equations of Mathematical Physics by Tyn Myint-U. Elsevier, 1973. A well-written text with a fairly extensive section on the Fourier series and transform (Chapters 5, 11). Most of the important questions dealing with whether a function actually has a Fourier series or integral representation, whether the representation can be differentiated term by term or under the integral to get the derivative of the function, and so forth, are answered in these chapters.

LESSON 12

The Fourier Transform and Its Application to PDEs

PURPOSE OF LESSON: To illustrate several useful properties of the Fourier transform and to show how these properties can be used to solve PDEs. In particular, it is shown how the Fourier transform changes *differentiation* to *multiplication,* so differential equations change into algebraic equations. Also, the idea of the *infinite convolution* is introduced.

The Fourier transform of the function $f(x)$ for $-\infty < x < \infty$ is given by the formula

$$\boxed{\mathscr{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx} \tag{12.1}$$

That is, we start with a function $f(x)$ defined on the real x-axis, substitute it into equation (12.1), and arrive at the new function $F(\xi)$ for $-\infty < \xi < \infty$. For example, Table 12.1 lists some common Fourier transforms.

TABLE 12.1 Some Common Fourier Transforms

	Function $f(x)$	Fourier Transform $F(\xi)$
1.	$f(x) = \begin{cases} e^{-x} & x \geqslant 0 \\ -e^{x} & x < 0 \end{cases}$	$F(\xi) = -i\sqrt{\frac{2}{\pi}}\frac{\xi}{1+\xi^2}$ (complex function)
2.	$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$	$F(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin \xi}{\xi}$ (real function)
3.	$f(x) = e^{-x^2}$	$F(\xi) = \frac{1}{\sqrt{2}}e^{-(\xi/2)^2}$ (real function)

The reader can refer to the tables in the appendix for additional transforms. We can see from the examples that the transformed function $F(\xi)$ may or may not be a complex-valued function of ξ. In the first example, the transformed function $F(\xi)$ contains the complex number i, so we call it a **complex-valued function** of the *real variable* ξ (ξ ranges from $-\infty$ to ∞). In other words, the argument ξ is real, but the value of the function is complex.

The usefulness of the Fourier transform (as with most integral transforms) comes from the fact that it changes the operation of differentiation into multiplication; that is, differential equations are changed into algebraic equations. There are also a host of other properties that make the Fourier transform a useful operational tool; we list a few of the more important ones.

Useful Properties of the Fourier Transform

Property 1 (Fourier Transform Pair)

The Fourier transform of $f(x)$, $-\infty < x < \infty$, produces a new function $F(\xi)$ defined by the formula

$$\mathcal{F}[f] = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$$

and the *inverse* Fourier transform of $F(\xi)$, $-\infty < \xi < \infty$ will reproduce the original function $f(x)$ according to

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x}\, d\xi$$

For example,

$$e^{-|x|} \xrightarrow{\mathcal{F}} \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2} \xrightarrow{\mathcal{F}^{-1}} e^{-|x|}$$

See Figure 12.1.

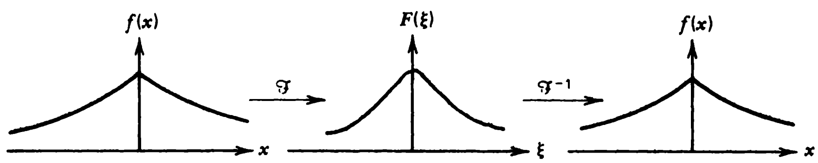


FIGURE 12.1 Graph of a function and its transform.

Property 2 (Linear Transformation)

The Fourier transform is a linear transformation; that is

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$$

This is easy to see. The reader can spend a few minutes to verify this property, which is used over and over again. For example, the Fourier transform of the expression

$$\frac{1}{x^2+1} + 3e^{-x^2}$$

would be

$$\mathcal{F}\left[\frac{1}{x^2+1}\right] + 3\mathcal{F}[e^{-x^2}]$$

Property 3 (Transformation of Partial Derivatives)

When we discuss how derivatives transform, we must distinguish partial derivatives with respect to various variables. For instance, if the Fourier transform transforms the x-variable (the variable of integration in the transform) and *if* the function being transformed is a partial derivative of a function $u(x,t)$ *with respect to* x, then the **rules of transformation are**

$$\mathcal{F}[u_x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_x(x,t)e^{-i\xi x}\,dx = i\xi\mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_{xx}(x,t)e^{-i\xi x}\,dx = -\xi^2\mathcal{F}[u]$$

On the other hand, if we transform the partial derivative $u_t(x,t)$ (and if the variable of integration in the transform is x), then the transform is given by

$$\mathcal{F}[u_t] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(x,t)\,e^{-i\xi x}dx = \frac{\partial}{\partial t}\mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_{tt}(x,t)e^{-i\xi x}\,dx = \frac{\partial^2}{\partial t^2}\mathcal{F}[u]$$

Property 4 (Convolution Property)

Every integral transform has what is called a *convolution* property. The general idea is that the transform of a product of two functions $f(x)g(x)$ is *not* the product of the individual transforms; that is,

$$\mathcal{F}[f(x)g(x)] \neq \mathcal{F}[f]\mathcal{F}[g]$$

However, in transform theory there is something called the convolution $f * g$ of two functions that more or less plays the role of the product. What is true about this convolution $f * g$ is that

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g] \tag{12.2}$$

So what is this mysterious convolution $f * g$? It's given by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi)g(\xi)\, d\xi \tag{12.3}$$

and it can be shown without too much trouble that (12.2) holds. We see from the definition of the convolution that given two functions $f(x)$ and $g(x)$, the convolution $(f * g)(x)$ is a new function.

Example of a Convolution of Two Functions

Given the two functions

$$f(x) = x$$
$$g(x) = e^{-x^2}$$

the convolution is given by

$$\begin{aligned}(f * g)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \xi)e^{-\xi^2}\, d\xi \\ &= x/\sqrt{2} \qquad \text{(a new function)}\end{aligned}$$

We have used the formula

$$\int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \sqrt{\pi}$$

to arrive at this value.

The importance of the convolution (12.3) in applications is due to the fact that quite often, the final step in solving a PDE boils down to finding the inverse transform of some expression that we can interpret as the product of two transforms $\mathcal{F}[f]\mathcal{F}[g]$; that is, we must find

$$\mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\} \tag{12.4}$$

By taking the inverse of each side of (12.2), we arrive at the result

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}[f]\mathcal{F}[g]\} \tag{12.5}$$

Hence, to find (12.4), all we have to do is find the inverse transform of *each* factor to get f and g and *then* compute their convolution. We are now in a position to work an important problem in PDE theory.

Solution of an Initial-Value Problem

Consider the heat flow in an *infinite* rod where the initial temperature is $u(x,0) = \phi(x)$. In other words, we look for the solution to the *initial-value problem* (IVP), sometimes called a *Cauchy problem*

$$\begin{aligned}
&\text{PDE} \quad u_t = \alpha^2 u_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty \\
&\text{IC} \quad u(x,0) = \phi(x) \qquad -\infty < x < \infty
\end{aligned} \tag{12.6}$$

There are three basic steps in solving this problem.

STEP 1 (Transforming the problem)
Since the space variable x ranges from $-\infty$ to ∞, we take the Fourier transform of the PDE and IC with respect to this variable x (the variable of integration in the transform is x). Doing this, we get

$$\mathcal{F}[u_t] = \alpha^2 \mathcal{F}[u_{xx}]$$
$$\mathcal{F}[u(x,0)] = \mathcal{F}[\phi(x)]$$

and using the properties of the Fourier transform, we have

$$\begin{aligned}
&\frac{dU(t)}{dt} = -\alpha^2\xi^2 U(t) \\
&U(0) = \Phi(\xi) \qquad (\Phi \text{ is the Fourier transform of } \phi)
\end{aligned} \tag{12.7}$$

where $U(t) = \mathcal{F}[u(x,t)]$. The reader should note here that the function U actually depends on *both* t and the new transformed variable ξ, but, for simplicity, since ξ is a constant insofar as the differential equation (12.7) is concerned, we will drop the notation and just call $U = U(t)$.

STEP 2 (Solving the transformed problem)

Remember the new variable ξ is nothing more than a constant in this differential equation, so the solution to this problem is

$$U(t) = \Phi(\xi)\, e^{-\alpha^2\xi^2 t} \tag{12.8}$$

STEP 3 (Finding the inverse transform)

To find the solution $u(x,t)$, we merely compute

$$\begin{aligned} u(x,t) &= \mathscr{F}^{-1}[U(\xi,t)] \\ &= \mathscr{F}^{-1}[\Phi(\xi)e^{-\alpha^2\xi^2 t}] \end{aligned}$$

Here is where the convolution theorem (12.5) comes to the rescue. Using this property, we can write

$$\begin{aligned} u(x,t) &= \mathscr{F}^{-1}[\Phi(\xi)e^{-\alpha^2\xi^2 t}] \\ &= \mathscr{F}^{-1}[\Phi(\xi)] * \mathscr{F}^{-1}[e^{-\alpha^2\xi^2 t}] \\ &= \phi(x) * \left[\frac{1}{\alpha\sqrt{2t}} e^{-(x^2/4\alpha^2 t)}\right] \qquad \text{(using tables)} \\ &= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(\xi)\, e^{-(x-\xi)^2/4\alpha^2 t}\, d\xi \end{aligned} \tag{12.9}$$

We're done; equation (12.9) is the solution to our problem.

Before stopping, however, let's analyze this result. Note that the integrand is made up of two terms

1. The initial temperature $\phi(x)$
2. The function $G(x,t) = \dfrac{1}{2\alpha\sqrt{\pi t}} e^{-(x-\xi)^2/4\alpha^2 t}$ (which is called **Green's function** or the **impulse-response function**)

It can be shown that this impulse-response function $G(x,t)$ is the *temperature response* to an initial temperature impulse at $x = \xi$. In other words, $G(x,t)$ is the temperature along the rod at time t due to an initial *unit* impulse of heat at $x = \xi$ (Figure 12.2).

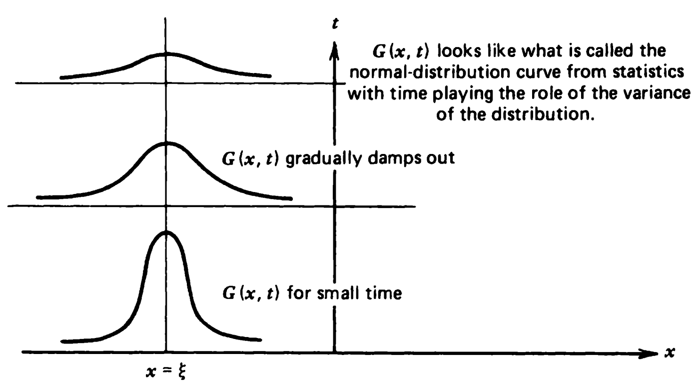


FIGURE 12.2 Impulse response $G(x,t)$ from a temperature impulse at $x = \xi$.

Hence, the interpretation of solution (12.9) is that the initial temperature $u(x,0) = \phi(x)$ is *decomposed* into a continuum of impulses of magnitude $\Phi(\xi)$ (at each point $x = \xi$) and the resulting temperature $\Phi(\xi)G(x,t)$ is found. These resulting temperatures are then added (integrated) to obtain solution (12.9). Later, we will see that this general idea is known as **superposition**.

From a practical point of view, solution (12.9) can often be integrated for some particular initial temperature $\phi(x)$. If this integration cannot be carried out *analytically*, the solution can be found at any point (x,t) by *numerically* integrating the integral.

NOTES

The major drawback of the Fourier transform is that all functions can not be transformed; for example, even simple functions like

$$\begin{aligned} f(x) &= \text{constant} \\ f(x) &= e^x \\ f(x) &= \sin x \end{aligned}$$

cannot be transformed, since the integral

$$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$$

does not exist. Only functions that damp to zero sufficiently fast as $|x| \to \infty$ have transforms. Also, the Fourier transform could not be used to transform the time variable in the previous initial value problem, since $0 < t < \infty$.

PROBLEMS

1. Find the Fourier transform of

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x} & 0 < x \end{cases}$$

   Check your answer by using the tables in the appendix.
2. Verify that the Fourier and inverse Fourier transforms are linear transforms.
3. Solve the initial-value problem

$$\begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} & -\infty < x < \infty \\ \text{IC} & u(x,0) = e^{-x^2} & -\infty < x < \infty \end{array}$$

   by using the Fourier transform.
4. Verify the properties

$$\mathcal{F}[u_x] = i\xi\,\mathcal{F}[u]$$
$$\mathcal{F}[u_{xx}] = -\xi^2\,\mathcal{F}[u]$$

HINT Use integration by parts.

5. Verify that the convolution of two functions f and g can be written as either

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi)\, d\xi$$

or

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x - \xi)\, d\xi$$

OTHER READING

Fourier Series and Orthogonal Functions by H. Davis. Allyn and Bacon, 1963; Dover, 1989. An excellent book gives the reader an intuitive as well as rigorous viewpoint of Fourier series and transforms.

LESSON 13

The Laplace Transform

PURPOSE OF LESSON: To introduce the important transform pair

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \text{(Laplace transform)}$$

$$\mathcal{L}^{-1}[F] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,e^{st}\,ds \qquad \text{(inverse Laplace transform)}$$

and illustrate useful properties. The Laplace transform has an advantage over the Fourier transform because it contains the damping factor e^{-st} that allow us to transform a wider class of functions. Inasmuch as the transform operates on functions defined on $[0,\infty)$, it is mostly applied to the time variable t.

After discussing some useful properties of the Laplace transform, we will solve an important problem in PDE theory.

Of all the integral transforms we will study in this book, the Laplace transform

$$\mathcal{L}[f] = \int_0^\infty f(t)e^{-st}\,dt \tag{13.1}$$

is probably the only one the reader has seen before, since it is a very powerful tool for transforming initial-value problems in ODE into algebraic equations. Not only is the Laplace transform useful in transforming ODEs into algebraic equations, but now we will use the Laplace transform to transform PDEs into ODEs. In particular, we will attempt to apply the Laplace transform to any variable $x, y, z, t, \ldots$ that ranges from 0 to ∞ (although it will generally be applied to time). The major difference in applying the Laplace transform to PDEs in contrast to ODEs is that now when the original PDE is transformed, the new resulting equation will be either a new PDE with one less independent variable or else an ODE in one variable. We must then decide how to solve this new problem (maybe by *another* transform, by separation of variables, and so on). Before actually solving a very interesting problem, we enumerate some useful properties of this transform.

Properties of the Laplace Transform

Property 1 (Transform Pair)

The Laplace transform and its inverse are given by

$$(13.2)\qquad \begin{aligned} \mathcal{L}[f] &= F(s) = \int_0^\infty f(t)e^{-st}\,dt &&\text{(Laplace transform)}\\ \mathcal{L}^{-1}[F] &= f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds &&\text{(inverse Laplace transform)}\end{aligned}$$

The Laplace transform has one major advantage over the Fourier transform in that the damping factor e^{-st} in the integrand allows us to transform a wider class of functions (the factor $e^{i\xi x}$ in the Fourier transform doesn't do any damping, since its absolute value is one). In fact, the exact conditions that insure that a function $f(t)$ has a Laplace transform are given by the following theorem:

Sufficient Conditions to Insure the Existence of a Laplace Transform

If

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A
2. we can find constants M and a such that $|f(t)| \leq Me^{at}$ for all values of t greater than some number T

then

$$\text{the Laplace transform } \mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \text{ exists for } s > a$$

We now list a few functions that have Laplace transforms and graph them on the s-axis.

1. $f(t) = 1 \quad 0 < t < \infty$ (pick $M = 1 \quad a = 0$) $\qquad F(s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$ (see Figure 13.1a)

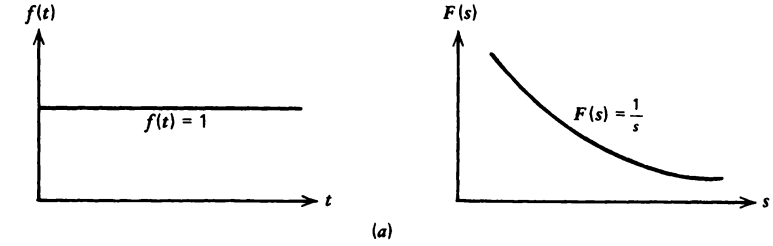
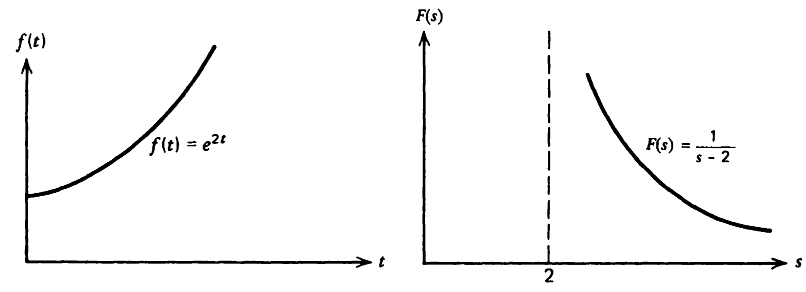


FIGURE 13.1a–13.1c Graphs of a few Laplace transforms.

2. $f(t) = e^{2t} \quad 0 < t < \infty$
(pick $M = 1 \quad a = 2$)

$$F(s) = \frac{1}{s-2} \qquad s > 2$$

(see Figure 13.1*b*)

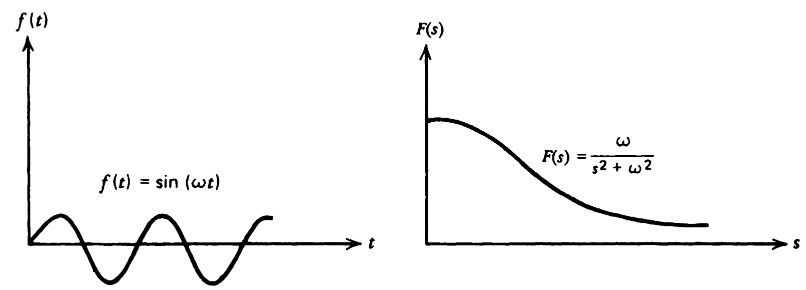


(*b*)

3. $f(t) = \sin(\omega t)$
(pick $M = 1 \quad a = 0$)

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

(see Figure 13.1*c*)



(*c*)

4. $f(t) = e^{t^2}$ (doesn't have a Laplace transform)

In the definition of the Laplace transform, the variable s is taken to be a real variable $0 < s < \infty$. It is possible, however (often desirable), to extend this definition to *complex* values of s and, in fact, to evaluate the *inverse Laplace transform*

$$\mathcal{L}^{-1}[F] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)e^{st}\, ds$$

We must often resort to *contour integration* in the complex plane and the theory of residues. We won't bother ourselves with this topic here but will use the tables in the appendix for finding inverse transforms.

Property 2 (Transforms of Partial Derivatives)

Suppose we have a function $u(x,t)$ of two variables and wish to transform various partial derivatives u_t, u_{tt}, u_x, u_{xx}, ... Since the Laplace transform transforms the *t-variable* (variable of integration), the rules of transformation for partial derivatives are

$$\mathscr{L}[u_t] = \int_0^\infty u_t(x,t)e^{-st}\,dt = sU(x,s) - u(x,0)$$

$$\mathscr{L}(u_{tt}) = \int_0^\infty u_{tt}(x,t)e^{-st}\,dt = s^2U(x,s) - su(x,0) - u_t(x,0)$$

$$\mathscr{L}[u_x] = \int_0^\infty u_x(x,t)e^{-st}\,dt = \frac{\partial U}{\partial x}(x,s)$$

$$\mathscr{L}[u_{xx}] = \int_0^\infty u_{xx}(x,s)e^{-st}\,dt = \frac{\partial^2 U}{\partial x^2}(x,s)$$

where $U(x,s) = \mathscr{L}[u(x,t)]$. The transform rules for u_x and u_{xx} are a result of a basic rule in calculus

$$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b \frac{\partial f}{\partial x}(x,y)\,dy$$

while the rules for u_t and u_{tt} can be proven by using the integration by parts formula.

Property 3 (Convolution Property)

Convolution plays the same role here as it did in the Fourier transform, but now the convolution is defined slightly differently.

Definition of the Finite Convolution

The **finite convolution** of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$
$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

(these two integrals are the same). In other words, in the *finite* convolution, we integrate from 0 to t instead of from $-\infty$ to ∞, as we did in the *infinite* convolution. An example of the finite convolution of two functions

$$f(t) = t$$
$$g(t) = t$$

would be

$$(f * g)(t) = \int_0^t \tau(t - \tau)\, d\tau = t^3/6$$

As in the case of the infinite convolution, the important property of this new convolution is that

$$\mathscr{L}[f * g] = \mathscr{L}[f]\, \mathscr{L}[g] \tag{13.4}$$

or the equivalent formula

$$\mathscr{L}^{-1}\{\mathscr{L}[f]\, \mathscr{L}[g]\} = f * g \tag{13.5}$$

This property will allow us to find the inverse Laplace transform of a product of two functions (which we interpret as $\mathscr{L}[f]\mathscr{L}[g]$) by finding the inverses of each factor $\mathscr{L}[f]$ and $\mathscr{L}[g]$ to get f and g and then finding their convolution. For example

$$\mathscr{L}^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2 + 1}\right] = \int_0^t \sin \tau \, d\tau = 1 - \cos t$$

$$F(s) = \frac{1}{s} \xrightarrow{\mathscr{L}^{-1}} f(t) = 1 \qquad G(s) \xrightarrow{\mathscr{L}^{-1}} g(t) = \sin t$$

We are now ready to work an important initial-boundary-value problem.

Heat Conduction in a Semi Infinite Medium

Consider a large (deep) container of liquid that is insulated on the sides. Suppose the liquid has an initial temperature of u_0 and that the temperature of the air above the liquid is zero (some reference temperature). Our goal is to find the temperature of the liquid at various depths of the container at different values of time. To do so, we must solve the problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx} & 0 < x < \infty \quad 0 < t < \infty \\ \text{BC} & u_x(0,t) - u(0,t) = 0 & 0 < t < \infty \\ \text{IC} & u(x,0) = u_0 & 0 < x < \infty \end{array} \tag{13.6}$$

See Figure 13.2.

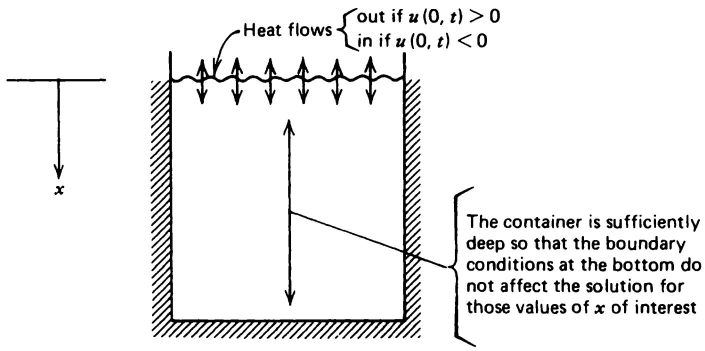


FIGURE 13.2 Diagram illustrating the heat-flow problem.

To solve this problem, we transform the *time variable t* by means of the Laplace transform (conceivably, we could also transform x by means of the Laplace transform, since x also ranges from 0 to ∞). Transforming our problem, we arrive at an ODE in x

$$
(13.7) \qquad \begin{array}{ll} \text{ODE} & sU(x) - u_0 = \dfrac{d^2U}{dx^2}(x) \qquad 0 < x < \infty \\ \text{BC} & \dfrac{dU}{dx}(0) = U(0) \end{array}
$$

(we transform the PDE and the BC—not the IC). This is a second-order ODE with one BC at $x = 0$ [for physical reasons, we *really* have a second, implied BC that says $U(x)$ is bounded]. Note that we have dropped the s-notation in $U(x,s)$ in favor of the simpler notation $U(x)$, since the differential equation in (13.7) depends only on x.

To solve (13.7), we first find the general solution (homogeneous + a particular solution), which is

$$
U(x) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{u_0}{s}
$$

Substituting this expression into the BCs of (13.7) allows us to find the constants c_1 and c_2 (first note that $c_1 = 0$ or else the temperature will go to infinity as x gets large). Finding c_2 from the BC at $x = 0$ gives us the answer for $U(x)$

$$
(13.8) \qquad U(x) = -u_0 \left\{ \frac{e^{-\sqrt{s}\,x}}{s(\sqrt{s} + 1)} \right\} + \frac{u_0}{s}
$$

Now for the last step. To find the temperature $u(x,t)$, we compute

$$
u(x,t) = \mathcal{L}^{-1}[U(x,s)]
$$

[we now put back s in $U(x,s)$]. To find this inverse transform, we must resort to the tables of inverse Laplace transforms in the appendix; they will give us

$$u(x,t) = u_0 - u_0\left[erfc(x/2\sqrt{t}) - erfc\left(\sqrt{t} + x/2\sqrt{t}\right) e^{(x+t)}\right] \tag{13.9}$$

where

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\xi^2}\, d\xi$$

is the *complementary-error function* whose graph is given in Figure 13.3.

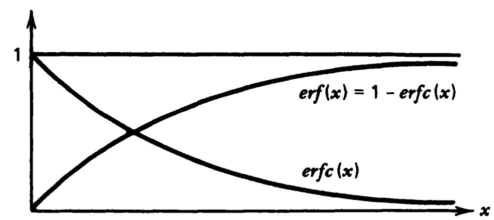


FIGURE 13.3 Graphs of the error (*erf*) and complementary-error (*erfc*) functions.

If we spend a little time analyzing this equation and graphing it by means of a computer with a plotter attachment, we will see that it looks like Figure 13.4.

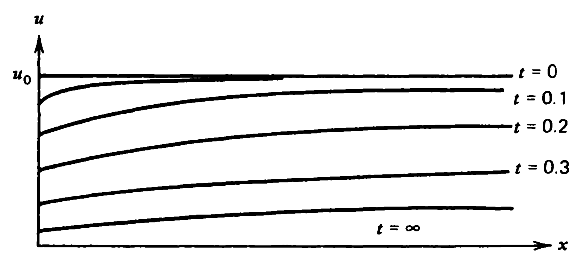


FIGURE 13.4 Temperatures inside the infinite medium for different values of time.

NOTES

1. The Laplace transform can also be applied to problems where the PDE is nonhomogeneous (in separation of variables, the equation had to be homogeneous), but the Laplace transform will generally work only if the equation has constant coefficients (in separation of variables, we could have variable coefficients). The following table lists the types of problems the two methods will handle.

TABLE 13.2 Comparison of Laplace Transform and Separation of Variables

	Method	
	Laplace Transform	Separation of Variables
Nonhomogeneous PDE	yes	no
Nonhomogeneous BC	yes	no
Variable coefficients	no	yes
Nonlinear equations	no	no

2. The Hankel and Mellin transforms are also used to solve IBVPs and BVPs but differ from the Laplace transform in one regard. The Laplace transform converts derivatives to multiplication operations by means of a formula like

$$\mathscr{L}[y'] = s\mathscr{L}[y] - y(0)$$

whereas the Hankel and Mellin transforms convert *differential operators* to multiplication; for example, the Hankel transform

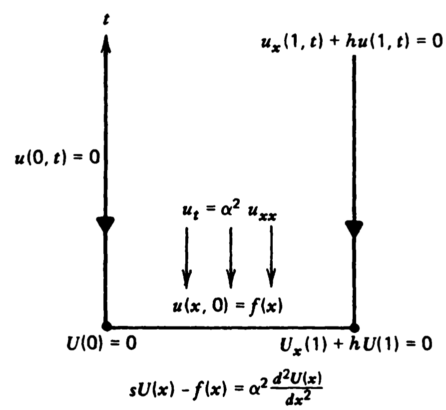
$$H[y] = \int_0^\infty rJ_0(\xi r)y(r)\,dr$$

transforms the differential operator

$$H[y''(r) + \frac{1}{r}y'(r)] = -\xi^2 H[y]$$

In this way, specific differential equations with variable coefficients (Bessel's equation) can be solved.

3. The Laplace transform (which transforms t) can be interpreted as projecting the xt-plane onto the x-axis, and the original BCs, PDE, and IC are transformed into a new differential equation and BCs. See the following diagram.



PROBLEMS

1. Verify the following formula for the transform of the partial derivative u_t

$$\mathscr{L}[u_t(x,t)] = sU(x,s) - u(x,0)$$

2. Solve the following initial-value problem by means of the Laplace transform

$$\begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} & -\infty < x < \infty \quad 0 < t < \infty \\ \text{IC} & u(x,0) = \sin x & -\infty < x < \infty \end{array}$$

3. Solve the problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx} & 0 < x < \infty \quad 0 < t < \infty \\ \text{BC} & u(0,t) = \sin t & 0 < t < \infty \\ \text{IC} & u(x,0) = 0 & 0 \leq x < \infty \end{array}$$

by means of the Laplace transform (transform t). What is the physical interpretation of this problem?

4. Solve the boundary-value problem

$$\text{ODE} \qquad \frac{d^2U}{dx^2} - sU = A \qquad 0 < x < 1$$

$$\text{BCs} \qquad \begin{cases} \dfrac{dU}{dx}(0) = 0 \\ U(1) = 0 \end{cases}$$

OTHER READING

1. *A First Course in PDE* by H. Weinberger. Ginn and Co., 1965. This text contains an extensive section on contour integration, which is the tool used for evaluating the inverse Laplace transform.
2. Almost any beginning text in ODE will contain a chapter on the Laplace transform.

LESSON 14

Duhamel's Principle

PURPOSE OF LESSON: To show how the Laplace transform can bring out interesting underlying phenomena concerning solutions of differential equations, in particular, by algebraically manipulating the Laplace transform of the solution of a PDE, we discover an interesting idea known as *Duhamel's principle.* This principle has interpretations in ODE, but we will illustrate how it works in the context of a specific initial-boundary-value problem.

In addition to providing a powerful tool for solving PDEs, the Laplace transform also provides insight into the nature of solutions to physical problems. With the help of the Laplace transform, we illustrate a very important and interesting concept known as *Duhamel's principle* in this lesson. Before getting to this principle, however, let's discuss a problem that occurs frequently in engineering.

Heat Flow within a Rod with Temperature Fixed on the Boundaries

Quite often, it is important to find the temperature inside a medium due to *time-varying boundary conditions*. For example, consider an insulated rod with temperature specified as $f(t)$ on the right end

$$\text{PDE} \quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$$

$$(14.1) \qquad \text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = f(t) \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 0 \quad 0 \leq x \leq 1$$

See Figure 14.1.

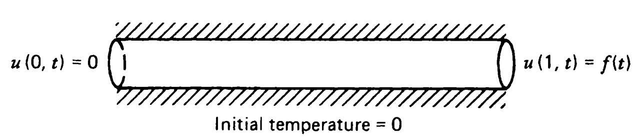


FIGURE 14.1 Time-varying boundary conditions.

We may think that the solution to problem (14.1) can be easily found once we know the solution to the simpler version (constant temperature on the boundaries)

$$
(14.2) \qquad
\begin{array}{lll}
\text{PDE} & w_t = w_{xx} & 0 < x < 1 \quad 0 < t < \infty \\[2ex]
\text{BCs} & \begin{cases} w(0,t) = 0 \\ w(1,t) = 1 \end{cases} & 0 < t < \infty \\[2ex]
\text{IC} & w(x,0) = 0 & 0 \leqslant x \leqslant 1
\end{array}
$$

In fact, if we solve problems (14.1) and (14.2) side by side by the Laplace transform, we will see a striking result (Duhamel's principle) that will give us the solution to (14.1) *in terms of the solution of (14.2).*

So, solving (14.1) and (14.2) at the same time, we have

Easy Problem (14.2)
(constant BC)

Transform problem (14.2) by the Laplace transform

$\downarrow$

$$\frac{d^2W}{dx^2} - sW(x) = 0$$
$$W(0) = 0$$
$$W(1) = 1/s$$

Solve the ODE

$\downarrow$

$$W(x,s) = \frac{1}{s}\left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\right]$$

Find the inverse transform

$\downarrow$

$$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2t}\sin(n\pi x)$$

(solution to the constant BC problem)

Hard Problem (14.1)
(time-varying BC)

Transform problem (14.1) by the Laplace transform

$\downarrow$

$$\frac{d^2U}{dx^2} - sU(x) = 0$$
$$U(0) = 0$$
$$U(1) = F(s)$$

Solve the ODE

$\downarrow$

$$U(x,s) = F(s)\left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\right]$$

Don't invert yet—first multiply and divide by s

$\downarrow$

$$U(x,s) = F(s)\left\{s\left[\frac{\sinh(x\sqrt{s})}{s\sinh(\sqrt{s})}\right]\right\}$$

Easy Problem (14.2) (Cont.)
(constant BC)

Hard Problem (14.1) (Cont.)
(time-varying BC)

Using the relationship

$$\mathscr{L}[w_t] = sW - w(x,0)$$

we have

$$U(x,s) = F(s)\,\mathscr{L}[w_t]$$

Hence

$$\begin{aligned} u(x,t) &= \mathscr{L}^{-1}\{F(s)\mathscr{L}[w_t]\} \\ &= \mathscr{L}^{-1}[F(s)] * [w_t] \\ &= f(t) * w_t(t) \\ &= \int_0^t f(\tau)\, w_\tau(x,t-\tau)\, d\tau \end{aligned}$$

(or by integration by parts)

$$= \int_0^t w(x,t-\tau)\, f'(\tau)\, d\tau + f(0)w(x,t)$$

(solution to the time-varying problem in terms of the solution of the constant BC problem)

In other words, we have found the solution $u(x,t)$ to the *time-varying* problem in terms of the solution to the easy problem (constant BCs); that is,

$$\begin{aligned} u(x,t) &= \int_0^t w_t(x,t-\tau)f(\tau)\, d\tau \\ &= \int_0^t w(x,t-\tau)f'(\tau)\, d\tau + f(0)w(x,t) \end{aligned} \tag{14.3}$$

Equations (14.3) are known as **Duhamel's principle.** We can now take the solution $w(x,t)$ to the constant BC problem

$$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(n\pi)^2 t} \sin(n\pi x)$$

and substitute it into equation (14.3) to obtain the solution to the time-varying problem [we must use the second equation in (14.3), since if we differentiate the infinite series representation for $w(x,t)$ term by term (with respect to t), it results in a *divergent* series].

There is another aspect of Duhamel's formulas that makes them very useful.

The Importance of Duhamel's Principle

In the problem just discussed, we were able to solve the easy problem with constant BCs, so we used Duhamel's formulas (14.3) to obtain the solution to the time-varying BCs. Quite often, however, even the easy problem (constant BCs) cannot be solved analytically. What we can do, however, is observe the solution *experimentally*; in other words, we can rig a device that has constant BCs and experimentally measure the response. We can then use Duhamel's principle to find the solution for *any* time-varying BC. In fact, we have only to observe the response $w(x,t)$ to the constant BC problem *once*. When we have this data, we can then solve the problem with *arbitrary* BC $f(t)$ by substituting into Duhamel's formulas (14.3).

NOTES

There is another interesting version of Duhamel's principle that gives the answer to the problem

$$\text{(14.4)}\qquad \begin{array}{lll} \text{PDE} & u_t = u_{xx} & 0 < x < 1 \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = f(t) \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = 0 & 0 \leq x \leq 1 \end{array}$$

in terms of the solution $w(x,t)$ of the *alternative simple problem*

$$\text{(14.5)}\qquad \begin{array}{lll} \text{PDE} & w_t = w_{xx} & 0 < x < 1 \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} w(0,t) = 0 \\ w(1,t) = \delta(t) \end{cases} & \text{(temperature impulse at } t = 0 \\ \text{IC} & w(x,0) = 0 & 0 \leq x \leq 1 \end{array}$$

Knowing this formula, which is

$$\text{(14.6)}\qquad u(x,t) = \int_0^t w(x,t-\tau)f(\tau)\,d\tau$$

allows us to find the temperature response $u(x,t)$ to an arbitrary boundary temperature $f(t)$ once we have carried out an experiment to determine the temperature response $w(x,t)$ from an impulse temperature.

PROBLEMS

1. Prove the Duhamel principle (14.6) by transforming both problems (14.4) and (14.5) and using an argument similar to the one for finding (14.3) in the lesson.
   HINT The Laplace transform of the impulse function $\delta(t)$ is $\mathscr{L}[\delta(t)] = 1$.

2. Show that the partial derivative w_t of

$$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(n\pi)^2 t} \sin (n\pi x)$$

   diverges for all x if we differentiate the series term by term.
3. Suppose we have a metal rod (laterally insulated) and we supply an *initial impulse of heat* at the right-hand side (the left-hand side is fixed at zero). Suppose the initial temperature of the rod is zero (some reference temperature) and the temperature at the midpoint $x = 0.5$ is measured at various values of time, so that we have the following table:

Values of Time	Midpoint Temperature
t_1	w_1
$t_2 = 2t_1$	w_2
$t_3 = 3t_1$	w_3
.	.
.	.
.	.
$t_n = nt_1$	w_n

   Using this data, how could we approximate the temperature response at the point u(0.5, t_n) due to the BCs

   (a) $u(1,t) = \sin t$
   (b) $u(1,t) = f(t)$ (arbitrary $f(t)$)
4. Using Duhamel's principle, what is the solution of the IBVP

$$\text{PDE} \quad u_t = \alpha^2 u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = \sin t \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = 0 \quad 0 \leq x \leq 1$$

OTHER READING

1. *Differential Equations* by C. Wylie. McGraw-Hill, 1979. Duhamel's principle is discussed in conjunction with problems in ODE in Chapter 6 of this text.

2. *Equations of Mathematical Physics* by A. N. Tikhonov and A. A. Samarskii. Macmillan, 1963; Dover, 1990. An excellent source of all kinds of applied problems; the Duhamel principle is discussed on page 261.

LESSON 15

The Convection Term u_x in the Diffusion Problems

PURPOSE OF LESSON: To show how the term u_x in the diffusion equation

$$u_t = \underbrace{Du_{xx}}_{\substack{\text{Diffusion}\\ \text{term}}} - \underbrace{Vu_x}_{\substack{\text{Convection}\\ \text{term}}}$$

represents the phenomenon of convection. Phenomena described by this convection-diffusion equation exhibit both diffusion and convection properties and are common in many situations. How much diffusion and convection takes place depends on the relative size of the two coefficients D and V.

Inasmuch as the convection of a substance represents material moving with the medium, it is possible to pick a moving coordinate system that moves with the medium. In this way, the convection term is eliminated and the equation can be solved in terms of the moving coordinate and then transformed back into the original variable x.

So far, we have been concerned with heat flow (or diffusion of some kind) in a one-dimensional domain. Suppose now we consider the problem of finding the *concentration* of a substance upwards from the surface of the earth where the substance both diffuses through the air and is *carried upward* (convected) by moving currents (moving with velocity V). Clearly, it is possible for the convection of the substance to contribute more of a movement in the substance than the diffusion itself. (It would depend on the relative size of the diffusion coefficient and the velocity of the air.) **Diffusion** is mixing the substance through the air, while **convection** is the movement of the substance *by means* of the air (the movement of the medium); in any case, it is our purpose here to solve the diffusion-convection equation

$$\boxed{u_t = Du_{xx} - Vu_x}$$

and to show how it is derived.

To verify this equation for a concentration $u(x,t)$ of a substance, we use *two basic facts*

1. *Flux due to convection*
   The flux of material (from left to right) across a point due to *convection* is given by $Vu(x,t)$, where V is the velocity of the medium (cm/sec) and $u(x,t)$ is the linear concentration (g/cm) (Figure 15.1).

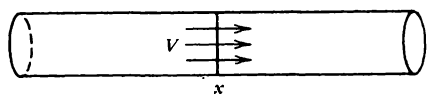


FIGURE 15.1 Amount of material across x (per second) due to convection is $Vu(x,t)$.

2. *Flux due to diffusion*
   The *flux* of material (from left to right) across a point x due to *diffusion* is given by $-Du_x(x,t)$, where D is the diffusion coefficient.

If we substitute these two expressions into the *conservation equation* in Lesson 4, we can prove that the basic PDE is

$$\boxed{u_t = Du_{xx} - Vu_x}$$

To get an idea of what solutions look like or how they behave with the convection term included, let's first work a problem that is pure convection (the diffusion term is zero). A typical problem would be dumping a substance into a clean river (moving with velocity V) and observing the concentration of the substance downstream. For example, if x measures the distance downstream from where the substance is added and if the substance *does not diffuse* with the running water, then the concentration of the substance $u(x,t)$ can be found by solving the following mathematical model:

	PDE	$u_t = -Vu_x$	$0 < x < \infty \quad 0 < t < \infty$
(15.1)	BC	$u(0,t) = P$	$\longleftarrow$ Constant input of the substance
	IC	$u(x,0) = 0$	$\longleftarrow$ Initially a clean river

This problem is illustrated in Figure 15.2.

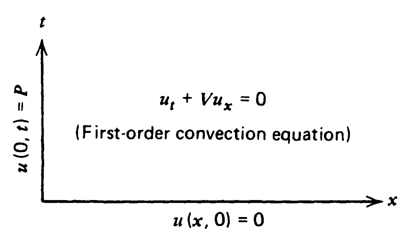


FIGURE 15.2 Pure convection problem.

Before solving this problem, we should think a little about what the solution should be. It's obvious that the pollutant (substance in the river) will initially be zero, but once it is added at a constant rate at $x = 0$, it will move downstream with velocity V. To see this mathematically, let's solve (15.1). Since it is a linear PDE with linear BCs, we should think in terms of separation of variables and integral transforms; however, since the x-variable is unbounded, separation of variables is out. Let's use the Laplace transform on t.

Laplace Transform Solution to the Convection Problem

The convection problem (15.1) can be replaced by

$$\begin{aligned} sU(x) &= -V\frac{dU}{dx} \qquad 0 < x < \infty \\ U(0) &= \frac{P}{s} \end{aligned}$$

by using the Laplace transform

$$U(x) = \int_0^\infty u(x,t)e^{-st}\,dt$$

Solving this very simple initial-value problem, we get

$$U(x) = \frac{P}{s}e^{-(sx/V)}$$

Looking up the inverse transform in the tables, we see

$$u(x,t) = \mathcal{L}^{-1}[U] = PH(t - x/V)$$

where $H(\xi)$ is the Heaviside function (step function)

$$H(\xi) = \begin{cases} 0 & \xi < 0 \\ 1 & \xi \geqslant 0 \end{cases}$$

Hence, the solution of our problem is just

$$u(x,t) = \begin{cases} 0 & t < x/V \\ P & t \geqslant x/V \end{cases}$$

This was pretty simple; certainly, it isn't any more complicated than dumping something on a conveyor belt and watching it move along. It does, however, become more interesting when the solute (pollutant) *diffuses* with the medium.

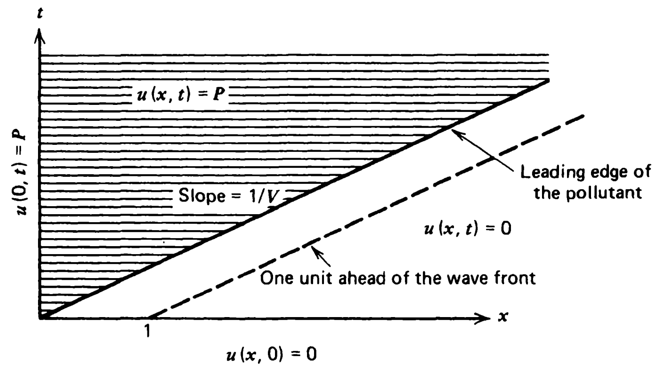


FIGURE 15.3 Pure convection wave.

To see what happens when a moving wave diffuses, we solve the following problem:

$$\text{(15.2)} \qquad \begin{array}{lll} \text{PDE} & u_t = Du_{xx} - Vu_x & -\infty < x < \infty \\ \text{IC} & u(x,0) = 1 - H(x) & -\infty < x < \infty \end{array}$$

where, as usual, $H(x)$ is the Heaviside function. The initial concentration is shown in Figure 15.4.

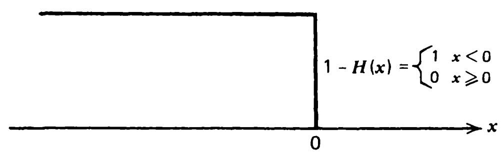


FIGURE 15.4 Initial condition for the diffusion-convection equation.

Note that in the new problem (15.2), we have moved the boundary to $-\infty$ (we now have an *initial-value problem*), so that it doesn't confuse the real issue of measuring the relative effects of D versus V (just a technicality). To solve (15.2), we could use the Laplace transform on t or the Fourier transform on x; however, in this case, there is another alternative that is very interesting. That is, instead of measuring the concentration $u(x,t)$ as a function of x, we introduce a new coordinate ξ, which moves along the x-axis with velocity V. In other words, instead of placing our coordinate system along the bank of the river (so to speak), we now place our coordinate system so that it moves with the *wave front* (of course, now when we have diffusion in addition to convection, we won't have a sharp wave front). Mathematically this says that we change our space coordinate x to $\xi = x - Vt$. It's now clear that

when $\xi = 0$	we are on the wave front
when $\xi = 1$	we are one unit ahead of the front
when $\xi = -1$	we are one unit behind the front

So our goal is to transform the initial-value problem (IVP)

$$\begin{array}{rll} \text{PDE} & u_t = Du_{xx} - Vu_x & -\infty < x < \infty \\ \text{IC} & u(x,0) = 1 - H(x) & -\infty < x < \infty \end{array}$$

into a *new one* in the moving coordinate system, solve it, and then transform back to get the solution in terms of the original coordinates (x,t). To begin, we make what is called a *change of variables* (change of *independent* variables). In place of the old coordinates (x,t), we introduce new ones (ξ,τ)

$$\begin{aligned} \xi &= x - Vt \\ \tau &= t \end{aligned}$$

The reader should note that τ is the same as t, but it is less confusing if we give it a new name. To rewrite the PDE in terms of (ξ,τ), we use the chain rule

$$\begin{aligned} u_t &= u_\xi \xi_t + u_\tau \tau_t = -Vu_\xi + u_\tau \\ u_x &= u_\xi \xi_x = u_\xi \\ u_{xx} &= (u_\xi)_x = u_{\xi\xi}\xi_x = u_{\xi\xi} \end{aligned}$$

Using *functional diagrams*, as in Figure 15.5, makes these chain-rule arguments clearer.

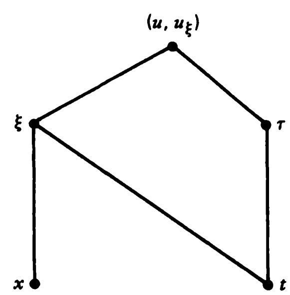


FIGURE 15.5 Diagram illustrating functional dependence of variables.

The diagram in Figure 15.5 is useful for computing the partial derivatives of u, u_ξ with respect to x and t, since it shows exactly how u and u_ξ depend, in general, on ξ and τ and that ξ depends, in turn, on both x and t. The variable τ, on the other hand, depends only on t.

So much for the transformation. We now substitute our computed u_t, u_x, and u_{xx} into the PDE to get

$$-Vu_\xi + u_\tau = Du_{\xi\xi} - Vu_\xi$$

or

$$u_\tau = Du_{\xi\xi}$$

Hence, our new IVP in terms of ξ and τ is

$$\begin{array}{lll} \text{PDE} & u_\tau = Du_{\xi\xi} & -\infty < \xi < \infty \\ \text{IC} & u(\xi,0) = 1 - H(\xi) & -\infty < \xi < \infty \end{array}$$

(Note that $\xi = x$ when $t = 0$, so our ICs are both the same.) This problem has already been solved in Lesson 12 by the Fourier transform and has the solution

$$u(\xi,\tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{\infty} \phi(\beta) e^{-(\xi-\beta)^2/4D\tau}\, d\beta$$

where $\phi(\beta)$ is the initial condition. Hence, in our case, we have

$$u(\xi,\tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{0} e^{-(\xi-\beta)^2/4D\tau}\, d\beta$$

By letting

$$\bar{\beta} = \frac{\xi-\beta}{2\sqrt{D\tau}} \qquad d\bar{\beta} = \frac{-1}{2\sqrt{D\tau}}\, d\beta$$

we get the interesting result

$$(15.3) \quad u(\xi,\tau) = \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{\xi}{2\sqrt{D\tau}}}^{\infty} e^{-\bar{\beta}^2}\, d\bar{\beta}\right]$$

$$= \begin{cases} \frac{1}{2}\left[1 + erf\left(\frac{-\xi}{2\sqrt{D\tau}}\right)\right] & \xi < 0 \\ \frac{1}{2}\, erfc\left(\frac{\xi}{2\sqrt{D\tau}}\right) & \xi \geq 0 \end{cases}$$

The graph of this function is plotted for various values of t in Figure 15.6.

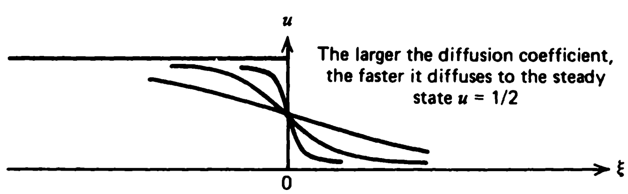


FIGURE 15.6 Simple diffusion from high to low concentrations.

Finally, to get the solution of our problem in terms of the coordinates x and t, we substitute

$$\begin{aligned} \xi &= x - Vt \\ \tau &= t \end{aligned}$$

into equation (15.3) to get

$$u(x,t) = \begin{cases} \dfrac{1}{2}\left[1 + erf\left(\dfrac{Vt - x}{2\sqrt{D\tau}}\right)\right] & Vt > x \\ erfc\left(\dfrac{x - \mathrm{Vt}}{2\sqrt{D\tau}}\right) & Vt \leq x \end{cases} \tag{15.4}$$

This is the solution of our diffusion-convection problem (15.2), and it is really very easy to interpret; it's just a moving version of Figure 15.6. In other words, depending on the relative size of D (diffusion coefficient) and V (velocity of the stream), the solution moves to the right with velocity V while, at the same time, the leading edge is diffusing at a rate defined by D (Figure 15.7 shows the break up of the leading edge).

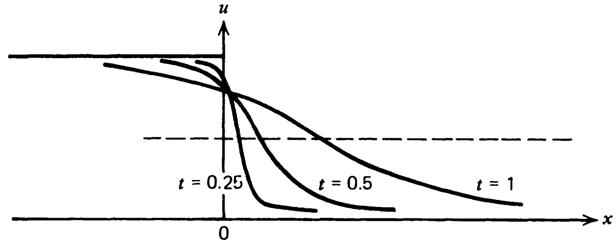


FIGURE 15.7 Diffusion-convection solution moving and diffusing at the same time.

NOTES

Changing coordinates is a very important technique in PDEs. By looking at physical systems with different coordinates, the equations are sometimes simplified.

PROBLEMS

1. Solve the initial-value problem

$$\begin{aligned} u_t &= u_{xx} - 2u_x & -\infty < x < \infty \quad & 0 < t < \infty \\ u(x,0) &= \sin x & -\infty < x < \infty \quad & \end{aligned}$$

2. Solve the following diffusion-convection problem by making a transformation as shown in Lesson 8:

$$u_t = u_{xx} - 2u_x \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$u(x,0) = e^x \sin x \qquad -\infty < x < \infty$$

3. What is the solution of the following *convection* problem:

$$\text{PDE} \quad u_t = -2u_x \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$\text{IC} \quad u(x,0) = e^{-x^2}$$

Check your answer.

4. Solve

$$u_t = Du_{xx} - Vu_x \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$u(x,0) = e^{-x^2} \qquad -\infty < x < \infty$$

Does the solution check? What does the solution look like for various values of time?

HINT Note that our transformation to moving coordinates allows us to essentially neglect the term Vu_x in the PDE. After solving the *new* problem,

$$\text{PDE} \quad u_\tau = Du_{\xi\xi} \qquad -\infty < \xi < \infty \qquad 0 < \tau < \infty$$
$$\text{IC} \quad u(\xi,0) = e^{-\xi^2} \qquad -\infty < \xi < \infty$$

we merely set $\xi = x - Vt$ and $\tau = t$. In this particular problem, it is possible to *evaluate* the integral

$$u(\xi,\tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{\infty} e^{-\beta^2} e^{-(\xi-\beta)^2/4D\tau} \, d\beta$$

This is the Fourier transform solution from Lesson 12. It may be more convenient for the reader to rewrite this integrand and then look it up in a table of integrals.