Discrete Data Analytic Study of the Traffic Light Problem

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It is clear as technology and humanity continues to advance, the role of traffic also grows. As such, we continue to push our understanding of traffic. We derived an algorithm to predict an individual's most likely path of travel from point A to point B by solving a system of differential equations. Data were collected randomly on a local neighborhood of UNG's Gainesville Campus about the different attributes of paths in contrast to the varying driving conditions. Finally we tested and compared the analytic model with a proposed ideal abstract graph theoretic model.

We are considering the speed limits and number of traffic lights on a given stretch of road from point A to the UNG Gainesville Campus and trying to understand how these factors are related with the driver's motor responses while accelerating or decelerating the car.

We assume that while driving, if the driver is aware of the remaining travel time (has to be in class by 8), then as the speed limit lowers on the road the driver has the tendency of speeding (accelerating) the car. Conversely, if the speed limit is higher there is not much urge to accelerate, but deceleration does not make any sense either, as reaching five minutes early would provide the student a five minute breather before class.

On the other hand, we believe that while driving a driver thinks of the speed limit as a function of time, because he or she is aware of the speed limit on each stretch of the route. So, if the driver is driving fast enough through a higher speed limit region the varying speed limit will have a rather intense impact on his or her driving. On the contrary if the car is being driven at 30 miles per hour in a stretch of the road where the usual speed limit is 45 miles per hour, then the driver will show the least amount of respect towards the remaining stretch of the road and the varying speed limit.

Let v be the velocity of the car at time t when the distance traveled is s. Then we get the following payoff matrix to understand the interrelation between driving speed and speed limit of a stretch of road.

	increasing speed limit	decreasing speed limit
acceleration	0	2
deceleration	-1	-2
constant	0	0

Table: Speed Limit table

So, if x is considered as the speed limit on the stretch of the road then the expected payoff of the accelerating car verses the speed limit will be $\frac{dv}{dt} \propto 0 \cdot x + 2 \cdot x - 1 \cdot x - 2 \cdot x + 0 \cdot x + 0 \cdot x = -x$.

The following payoff chart discusses the varying speed limit (rate of change of speed limit as a function of time) verses the acceleration or deceleration of the car.

Speed Limit

	acceleration	deceleration	constant
increasing speed limit	1	2	1
decreasing speed limit	2	-4	-1

Table: Acceleration table

So the expected payoff of the varying speed limit versus the speed of the car will be $\frac{dx}{dt} \propto 1 \cdot v + 2 \cdot v + 1 \cdot v + 2 \cdot v - 4 \cdot v + 1 \cdot v = 3v$.

From the data collected through the GPS signals it was established that while driving at a rate of 30 miles per hour a driver tends to choose a route that will have an average of 10 miles per hour higher speed limit. So we get $\frac{dx}{dt} = 3k_1 v$ transformed into $10 = 3k_1(30)$. So $k_1 = \frac{1}{9}$. Thus we will get the differential equation $\frac{dx}{dt} = \frac{1}{3}v$. We also found, when the speed limit is 45 miles per hour the average acceleration for the car is 5 miles per square second. Similarly, the differential expression $\frac{dv}{dt} = -k_2 x$ transforms into $5 = -k_2(45)$. Hence $k_2 = -\frac{1}{9}$. Thus the second differential equation of the system will be $\frac{dv}{dt} = \frac{1}{9}x$.

The first system of equations can be written as

$$\Rightarrow \begin{bmatrix} v' \\ x' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{9} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix}$$

Let $A = \begin{bmatrix} 0 & \frac{1}{9} \\ \frac{1}{3} & 0 \end{bmatrix}$.

Solving then for the Eigen Values:

$$\begin{vmatrix} 0-\lambda & \frac{1}{9} \\ \frac{1}{3} & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda_1 = \frac{\sqrt{3}}{9} \lambda_2 = -\frac{\sqrt{3}}{9}$$

Let $\vec{w}_1$ and $\vec{w}_2$ are the two Eigen vectors corresponding to λ_1, λ_2.

$$(A - \lambda_1 I)\vec{w}_1 = 0$$

$$\Rightarrow \left(\begin{bmatrix} 0 & \frac{1}{9} \\ \frac{1}{3} & 0 \end{bmatrix} - \frac{\sqrt{3}}{9} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$-\sqrt{3}w_{11} + w_{21} = 0$$

$$3w_{11} - \sqrt{3}w_{21} = 0$$

Hence,

$$w_{11} = 1, w_{21} = \sqrt{3}$$

$(A - \lambda_2 I)\vec{w_2} = 0$

$$\Rightarrow \left(\begin{bmatrix} 0 & \frac{1}{9} \\ \frac{1}{3} & 0 \end{bmatrix} + \frac{\sqrt{3}}{9} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$\sqrt{3}w_{12} + w_{22} = 0$$

$$3w_{12} + \sqrt{3}w_{22} = 0$$

Hence,

$$w_{21} = 1, w_{22} = -\sqrt{3}$$

So

$$\begin{cases} v(t) = c_1 e^{\frac{\sqrt{3}}{9}t} + c_2 e^{-\frac{\sqrt{3}}{9}t} \\ \\ x(t) = \sqrt{3}c_1 e^{\frac{\sqrt{3}}{9}t} - \sqrt{3}c_2 e^{-\frac{\sqrt{3}}{9}t} \end{cases} \tag{1}$$

Qualitative property of the solutions:

What happens when $t \to \infty$?

1. if $c_1 > 0$, then $v \to \infty, x \to \infty$
2. if $c_1 < 0$, then $v \to -\infty, x \to -\infty$

Asymptotic relation between v, x, looks like $\frac{v}{x} = \frac{c_1 e^{\frac{\sqrt{3}}{9}t} + c_2 e^{-\frac{\sqrt{3}}{9}t}}{\sqrt{3}c_1 e^{\frac{\sqrt{3}}{9}t} - \sqrt{3}c_2 e^{-\frac{\sqrt{3}}{9}t}}$

As $t \to \infty$, we have $\frac{v}{x} = \frac{c_1 e^{\frac{2\sqrt{3}}{9}t}}{\sqrt{3}c_1 e^{\frac{2\sqrt{3}}{9}t}} = \frac{1}{\sqrt{3}}$. This means $v \to \sqrt{3}x$.

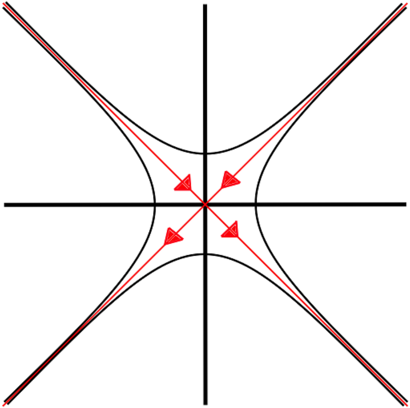
Asymptotic relation between v, x, looks like $\frac{v}{x} = \frac{c_1 e^{\frac{\sqrt{3}}{9}t} + c_2 e^{-\frac{\sqrt{3}}{9}t}}{\sqrt{3}c_1 e^{\frac{\sqrt{3}}{9}t} - \sqrt{3}c_2 e^{-\frac{\sqrt{3}}{9}t}}$

As $t \to -\infty$, we have $\frac{v}{x} = \frac{c_2 e^{\frac{-2\sqrt{3}}{9}t}}{-\sqrt{3}c_2 e^{\frac{-2\sqrt{3}}{9}t}} = -\frac{1}{\sqrt{3}}$. This means $v \to -\sqrt{3}x$.

Phase portrait: Since the matrix was nonsingular $\vec{w} = \vec{0}$ is the only critical point such that $\vec{w}' = 0$.

If $c_1 = 0$, $v = -\sqrt{3}x$ is the trajectory going toward zero, and if $c_2 = 0$, $v = \sqrt{3}x$ is the trajectory going away from zero.

Distance yet to be covered



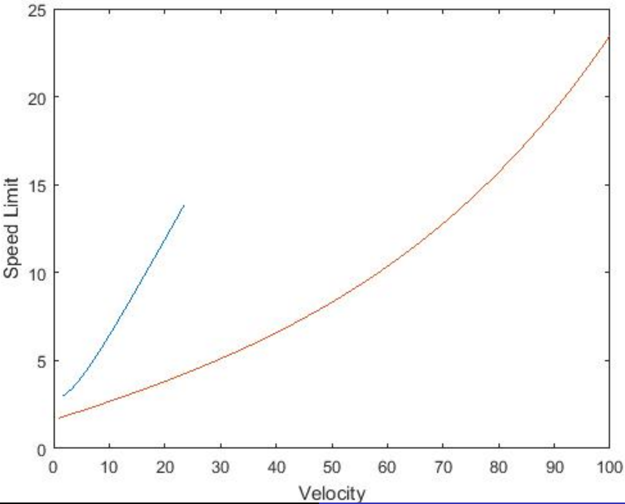
The approximate time t for the vehicle to reach 45 miles per hour after starting from rest = 16.172905 secs

The error in calculation is = 6.866455e-04

Number of steps required are = 17

A constant acceleration will require the speed limit of the road at time 16.1729 secs = 77.784341 miles.

Distance yet to be covered



Distance yet to be covered

Now we study the correspondence between the distance yet to be covered by the driver impacting the acceleration and deceleration of the car. For example, if the driver has to take a class at 10 am and at 9 : 55 am the driver finds that he or she still needs to traverse 2 more miles to reach school, then irrespective of the speed limit the car will be accelerated. But, if there is enough time to travel the remaining distance the car either maintains the speed or decelerates as required. As previously discussed there is a relation between the remaining distance to the destination and the varying speed of the car. If the destination is close enough the car is decelerated and on the other hand the driver will accelerate the car.

Let C be the distance from point A to the UNG-Gainesville campus. Since at time t distance s has already been covered, the distances yet to be covered is, $C - s$. For the time being, we are going to call, $C - s = y$.The following table represents the payoff matrix between acceleration and distance yet to be traveled.

	Higher Distance	Shorter Distance
acceleration	2	1
deceleration	-2	-1
constant	0	1

Table: Speed Limit table

So, if y is considered the remaining distance of the road then the expected payoff of the accelerating car versus the speed limit will be $\frac{dv}{dt} \propto 2 \cdot y + 1 \cdot y - 2 \cdot y - 1 \cdot y + 0 \cdot y + 1 \cdot y = y$.

Distance yet to be covered

The following payoff chart discusses the varying distance towards the destination(rate of change in remaining distance to the destination as a function of time) versus the acceleration or deceleration of the car.

Distance yet to be covered

	acceleration	deceleration	constant
Higher Distance	4	-1	2
Shorter Distance	0	1	2

Table: Acceleration table

So the expected payoff of the distance remaining verses the the speed of the car will be

$\frac{dy}{dt} \propto 4 \cdot v + -1 \cdot v + 2 \cdot v + 0 \cdot v + 1 \cdot v + 2 \cdot v = 8v.$

Distance yet to be covered

From the data collected through the GPS signals it was established on an average when the distance from the destination is 2 miles a driver tends to accelerate the car at 10 miles per hours. So we get $\frac{dv}{dt} = k_3 y$ transformed into $10 = k_3(2)$. So $k_3 = 5$. Thus we will get the differential equation $\frac{dv}{dt} = 5y$.We also found, when the the car is running at 40 miles per hour the average distance to the school is 1 mile. Similarly, the differential expression $\frac{dy}{dt} = k_4(8v)$ transforms into $1 = k_4(320)$. Hence $k_4 = \frac{1}{320}$. Thus the second differential equation of the system will be $\frac{dy}{dt} = \frac{1}{40}v$.

So we get a system of equation:

$$\begin{cases} Acceleration: \dfrac{dv}{dt} = 5y \\ \\ Distance\ to\ be\ covered: \dfrac{dy}{dt} = \dfrac{1}{40}v \end{cases} \tag{2}$$

The second system of equations can be written as

$$\Rightarrow \begin{bmatrix} v' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ \frac{1}{40} & 0 \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix}$$

Let $B = \begin{bmatrix} 0 & 5 \\ \frac{1}{40} & 0 \end{bmatrix}$.

Solving then for the Eigen Values:

$$\begin{vmatrix} 0-\beta & 5 \\ \frac{1}{40} & 0-\beta \end{vmatrix} = 0$$

$$\Rightarrow \beta_1 = \frac{1}{2\sqrt{2}} \beta_2 = -\frac{1}{2\sqrt{2}}$$

Let $\vec{u_1}$ and $\vec{u_2}$ are the two Eigen vectors corresponding to β_1, β_2.

$$(B - \beta_1 I)\vec{u_1} = 0$$

$$\Rightarrow \left(\begin{bmatrix} 0 & 5 \\ \frac{1}{40} & 0 \end{bmatrix} - \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$-\frac{1}{2\sqrt{2}}u_{11} + 5u_{21} = 0$$

$$\frac{1}{40}u_{11} - \frac{1}{2\sqrt{2}}u_{21} = 0$$

Hence,

$$u_{11} = 2\sqrt{2}, u_{21} = \frac{1}{5}$$

$$(A - \beta_2 I)\vec{u_2} = 0$$

$$\Rightarrow \left(\begin{bmatrix} 0 & 5 \\ \frac{1}{40} & 0 \end{bmatrix} + \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$\frac{1}{2\sqrt{2}}u_{12} + 5u_{22} = 0$$

$$\frac{1}{40}u_{12} + \frac{1}{2\sqrt{2}}u_{22} = 0$$

Hence,

$$u_{12} = 2\sqrt{2}, u_{22} = -\frac{1}{5}$$

So

$$\begin{cases} v(t) = 2\sqrt{2}d_1 e^{\frac{1}{2\sqrt{2}}t} + 2\sqrt{2}d_2 e^{-\frac{1}{2\sqrt{2}}t} \\ \\ y(t) = 5d_1 e^{\frac{1}{2\sqrt{2}}t} - 5d_2 e^{-\frac{1}{2\sqrt{2}}t} \end{cases} \tag{3}$$

Qualitative property of the solutions:

What happens when $t \to \infty$?

1. if $d_1 > 0$, then $v \to \infty, y \to \infty$
2. if $d_1 < 0$, then $v \to -\infty, y \to -\infty$

Asymptotic relation between v, y, looks like

$$\frac{v}{y} = \frac{2\sqrt{2}d_1 e^{\frac{1}{2\sqrt{2}}t} + 2\sqrt{2}d_2 e^{-\frac{1}{2\sqrt{2}}t}}{5d_1 e^{\frac{1}{2\sqrt{2}}t} - 5d_2 e^{-\frac{1}{2\sqrt{2}}t}}$$

As $t \to \infty$, we have $\frac{v}{y} = \frac{2\sqrt{2}d_1 e^{\frac{1}{\sqrt{2}}t}}{5d_1 e^{\frac{1}{\sqrt{2}}t}} = \frac{2\sqrt{2}}{5}$. This means $2\sqrt{2}v \to 5y$.

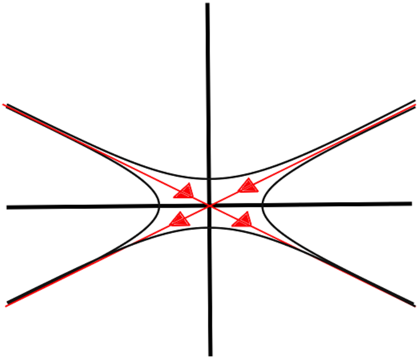
Asymptotic relation between v, y, looks like

$$\frac{v}{y} = \frac{2\sqrt{2}d_1 e^{\frac{1}{2\sqrt{2}}t} + 2\sqrt{2}d_2 e^{-\frac{1}{2\sqrt{2}}t}}{5d_1 e^{\frac{1}{2\sqrt{2}}t} - 5d_2 e^{-\frac{1}{2\sqrt{2}}t}}$$

As $t \to -\infty$, we have $\frac{v}{y} = \frac{2\sqrt{2}d_2 e^{\frac{1}{\sqrt{2}}t}}{-5d_2 e^{\frac{1}{\sqrt{2}}t}} = \frac{2\sqrt{2}}{-5}$. This means $2\sqrt{2}v \to -5y$.

Phase portrait: Since the matrix was nonsingular $\vec{u} = \vec{0}$ is the only critical point such that $\vec{u}' = 0$.

If $d_1 = 0$, $2\sqrt{2}v = -5y$ is the trajectory going toward zero, and if $d_2 = 0$, $2\sqrt{2}v = 5y$ is the trajectory going away from zero.

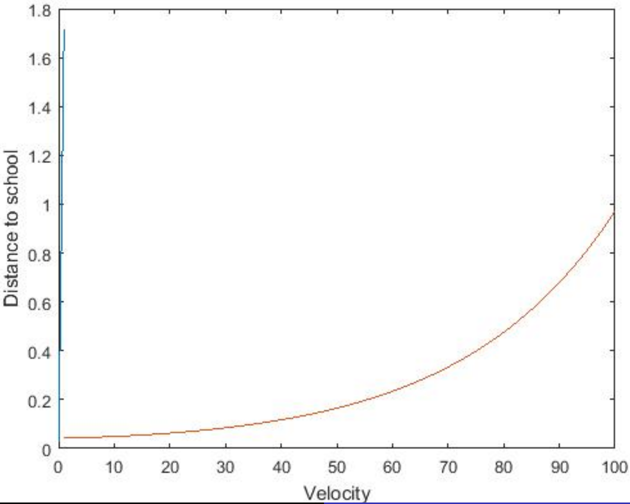


The approximate time t for the vehicle to reach school when at a distance of 1 miles from school is = 32.717628 secs

The error in calculation is = 6.866455e-04

Number of steps required are = 17

The average velocity will be 33.941629 miles/hour.


Distance to school
Velocity
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
100

We assume that while driving, if the driver is aware of the remaining travel time (has to be in class by 8), then as the number of remaining traffic lights decreases on the road the driver has the tendency of maintaining their speed. Conversely, if the number of remaining traffic lights increases the driver tends to accelerate.

On the other hand, we believe that while driving a driver thinks of the number of remaining traffic lights as a function of time, because he or she is aware of the traffic lights on each stretch of the route. So, if the driver is driving fast enough through a remaining stretch of road containing a high number of traffic lights, the driver will tend not to accelerate the car, as they will already be driving quickly. On the contrary if the car is being driven at a slower rate in a stretch of the road where the number of remaining traffic lights is low, then the driver will slow down (decelerate) or maintain the same speed.

Let v be the velocity of the car at time t when the remaining number of traffic lights is l. Then we get the following payoff matrix to understand the interrelation between driving speed and number of remaining traffic lights on a stretch of road.

	high number of traffic lights	low number of traffic lights
acceleration	1	0
deceleration	-1	1
constant	1	2

Table: Speed Limit table

So, if l is considered as the number of remaining traffic lights on the stretch of the road then the expected payoff of the accelerating car versus the speed limit will be

$\frac{dv}{dt} \propto 1 \cdot l + 0 \cdot l - 1 \cdot l + 1 \cdot l + 1 \cdot l + 2 \cdot l = 4l.$

The following payoff chart discusses the varying number of remaining traffic lights (rate of change of number of traffic lights as a function of time) versus the acceleration or deceleration of the car.

	acceleration	deceleration	constant
high number of traffic lights	2	-1	1
low number of traffic lights	0	1	1

Table: Acceleration table

So the expected payoff of the varying number of remaining traffic lights versus the speed of the car will be

$\frac{dl}{dt} \propto 2 \cdot v - 1 \cdot v + 1 \cdot v + 0 \cdot v + 1 \cdot v + 1 \cdot v = 4v.$

From the data collected through the GPS signals it was established that while there are four traffic lights remaining a driver will drive at 50 miles per hour. So we get $\frac{dl}{dt} = 4k_5 v$ transformed into $4 = 4k_5(50)$. So $k_5 = \frac{1}{50}$. Thus we will get the differential equation $\frac{dl}{dt} = \frac{2}{25}v$. We also found, when the driver is driving at 40 miles per hour, the remaining number of traffic lights is 2. Similarly, the differential expression $\frac{dv}{dt} = k_6 l$ transforms into $40 = k_6(2)$. Hence $k_6 = \frac{40}{2} = 20$. Thus the second differential equation of the system will be $\frac{dv}{dt} = 20l$.

The first system of equations can be written as

$$\Rightarrow \begin{bmatrix} v' \\ l' \end{bmatrix} = \begin{bmatrix} 0 & 20 \\ \frac{2}{25} & 0 \end{bmatrix} \begin{bmatrix} v \\ l \end{bmatrix}$$

Let $C = \begin{bmatrix} 0 & 20 \\ \frac{2}{25} & 0 \end{bmatrix}$.

Solving then for the Eigen Values:

$$\begin{vmatrix} 0-\gamma & 20 \\ \frac{2}{25} & 0-\gamma \end{vmatrix} = 0$$

$$\Rightarrow \gamma_1 = \frac{2\sqrt{10}}{5} \gamma_2 = -\frac{2\sqrt{10}}{5}$$

Let $\vec{q_1}$ and $\vec{q_2}$ are the two Eigen vectors corresponding to γ_1, γ_2.

$$(C - \gamma_1 I)\vec{q_1} = 0$$

$$\Rightarrow \left(\begin{bmatrix} 0 & 20 \\ \frac{2}{25} & 0 \end{bmatrix} - \frac{2\sqrt{10}}{5} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$-2\frac{\sqrt{10}}{5}q_{11} + 20q_{21} = 0$$

$$\frac{2}{25}q_{11} - \frac{2\sqrt{10}}{5}q_{21} = 0$$

Hence,

$$q_{11} = \frac{\sqrt{10}}{4}, q_{21} = \frac{1}{20}$$

$$(C - \gamma_2 I)\vec{q_1} = 0$$

$$\Rightarrow \left(\begin{bmatrix} 0 & 20 \\ \frac{2}{25} & 0 \end{bmatrix} + \frac{2\sqrt{10}}{5} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$\frac{2\sqrt{10}}{5}q_{12} + 20q_{22} = 0$$

$$\frac{2}{25}q_{12} + \frac{2\sqrt{10}}{5}q_{22} = 0$$

Hence,

$$q_{12} = \frac{\sqrt{10}}{4}, q_{22} = -\frac{1}{20}$$

So

$$\begin{cases} v(t) = \dfrac{\sqrt{10}}{4} e_1 e^{\frac{2\sqrt{10}}{5}t} + \dfrac{\sqrt{10}}{4} e_2 e^{-\frac{2\sqrt{10}}{5}t} \\ \\ l(t) = \dfrac{1}{20} e_1 e^{\frac{2\sqrt{10}}{5}t} - \dfrac{1}{20} e_2 e^{-\frac{2\sqrt{10}}{5}t} \end{cases} \tag{4}$$

Qualitative property of the solutions:

What happens when $t \to \infty$?

1. if $e_1 > 0$, then $v \to \infty$, $l \to \infty$
2. if $e_1 < 0$, then $v \to -\infty$, $l \to -\infty$

Asymptotic relation between v, l, looks like

$$\frac{v}{l} = \frac{\frac{\sqrt{10}}{4} e_1 e^{\frac{2\sqrt{10}}{5}t} + \frac{\sqrt{10}}{4} e_2 e^{-\frac{2\sqrt{10}}{5}t}}{\frac{1}{20} e_1 e^{\frac{2\sqrt{10}}{5}t} - \frac{1}{20} e_2 e^{-\frac{2\sqrt{10}}{5}t}}$$

As $t \to \infty$, we have $\frac{v}{l} = \frac{\frac{\sqrt{10}}{4} e_1 e^{\frac{4\sqrt{10}}{5}t}}{\frac{1}{20} e_1 e^{\frac{4\sqrt{10}}{5}t}} = 5\sqrt{10}$. This means $5\sqrt{10}v \to l$.

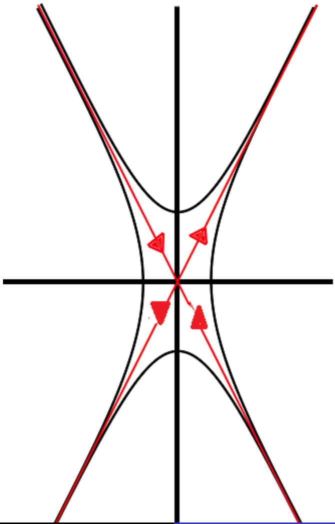
Asymptotic relation between v, x, looks like

$$\frac{v}{l} = \frac{\frac{\sqrt{10}}{4} e_1 e^{\frac{2\sqrt{10}}{5}t} + \frac{\sqrt{10}}{4} e_2 e^{-\frac{2\sqrt{10}}{5}t}}{\frac{1}{20} e_1 e^{\frac{2\sqrt{10}}{5}t} - \frac{1}{20} e_2 e^{-\frac{2\sqrt{10}}{5}t}}$$

As $t \to -\infty$, we have $\frac{v}{l} = \frac{\sqrt{10}4 e_2 e^{\frac{2\sqrt{10}}{5}t}}{-\frac{1}{20} e_2 e^{\frac{2\sqrt{10}}{5}t}} = -5\sqrt{10}$. This means $-5\sqrt{10}v \to l$.

Phase portrait: Since the matrix was nonsingular $\vec{q} = \vec{0}$ is the only critical point such that $\vec{q}' = 0$.

If $e_1 = 0$, $-5\sqrt{10}v = l$ is the trajectory going toward zero, and if $e_2 = 0$, $5\sqrt{10}v = l$ is the trajectory going away from zero.

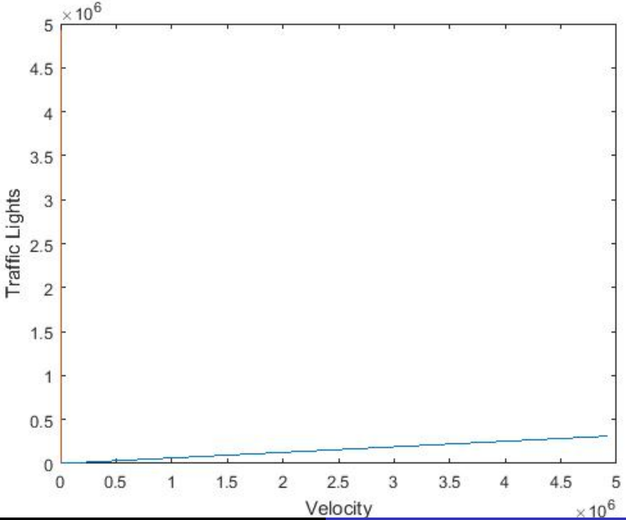


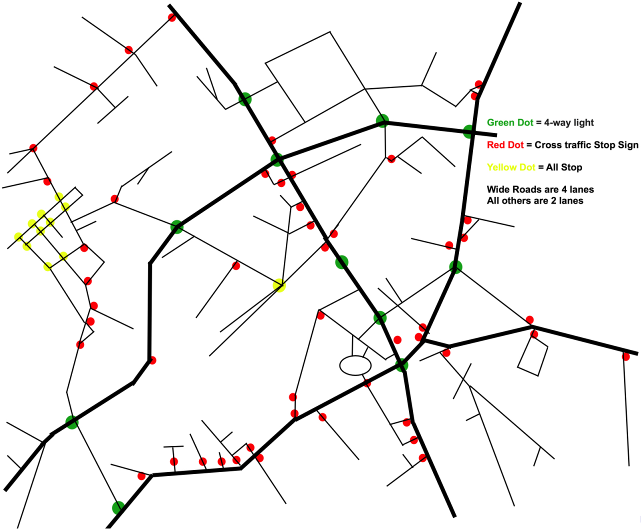
The approximate time t for the vehicle to reach school when 5 traffic lights to cross is = 1.272697 secs

The error in calculation is = 6.866455e-04

Number of steps required are = 17

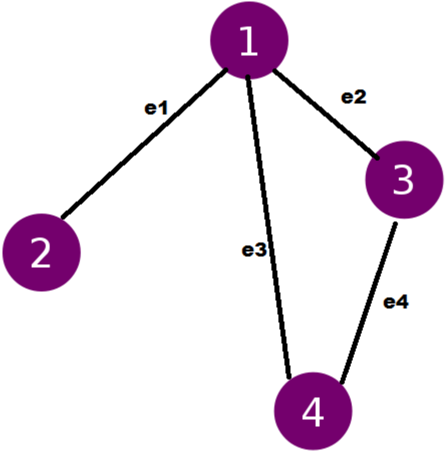
The average velocity will be 80.669943 miles/hour.


×10^6
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Traffic Lights
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Velocity
×10^6


Green Dot = 4-way light
Red Dot = Cross traffic Stop Sign
Yellow Dot = All Stop
Wide Roads are 4 lanes
All others are 2 lanes

In graph theory an undirected graph has two kinds of incidence matrices: unoriented and oriented.

The unoriented incidence matrix (or simply incidence matrix) of an undirected graph is a $n \times m$ matrix B, where n and m are the numbers of vertices and edges respectively, such that $B_{i,j} = 1$ if the vertex vi and edge ej are incident and 0 otherwise.


1
2
3
4
e1
e2
e3
e4

For example the incidence matrix of the undirected graph shown on the right is a matrix consisting of 4 rows (corresponding to the four vertices, 1-4) and 4 columns (corresponding to the four edges, e1-e4):

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Theorem (Das, Tucker): Two graphs G_1 and G_2 are isomorphic if and only if their (Pseudo)incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows and columns.

Proof: Let the graphs G_1 and G_2 be isomorphic. Then there is a one-one correspondence between the vertices and edges in G_1 and G_2 such that the incidence relation is preserved. Thus $A(G_1)$ and $A(G_2)$ are either same or differ only by permutation of rows and columns.

The converse follows, since permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Pseudo Incident Matrix

3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0
1	4	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	2	6	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
0	0	0	2	8	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0
0	2	0	0	2	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	1	6	1	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	2	0	1	7	2	0	0	0	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	2	6	2	2	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	3	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	2	0	5	1	0	0	0	2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	6	2	0	2	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	2	5	2	0	0	0	1	0	0
0	0	0	0	0	0	2	0	0	0	0	0	2	4	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	2	0	0	5	2	0	0	1	0
0	0	0	0	0	0	0	0	0	0	2	0	0	0	2	6	2	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	5	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	3	1	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	3	0
0	0	0	2	0	0	0	2	0	1	0	0	0	0	0	0	0	0	0	5

Row Echelon form Pseudo Incident Matrix

-5.01518	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-4.39035	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-3.60649	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-3.02458	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-2.73294	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-2.12559	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-1.97905	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1.004	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-0.52005	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-0.28503	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1.29E-16	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0.410275	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1.223695	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1.631664	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1.95005	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2.528264	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.100822	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.568336	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4.612893	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5.657266

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Thank You