DISCRETE DATA ANALYTIC STUDY OF THE TRAFFIC LIGHT PROBLEM *

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* **This work is instructed by Dr. Bikash Das**

ABSTRACT. It is clear as technology and humanity continues to advance, the role of traffic also grows. As such, we continue to push our understanding of traffic. We derived an algorithm to predict an individual's most likely path of travel from point A to point B by solving a system of differential equations. Data were collected randomly on a local neighborhood of UNG's Gainesville Campus about the different attributes of paths in contrast to the varying driving conditions. Finally we tested and compared the analytic model with a proposed ideal abstract graph theoretic model.

1. INTRODUCTION

When a student travels to school, given that they have a fixed amount of time to complete the trip to arrive at an expected time, they may make decisions under influence of emotion. This is considered under the ideas of emotive game theory and strategy making. In this paper, we discuss the impact of varying speed limit, differing number of traffic lights, and the distance to the campus as the three contributing factors in the emotive game. The constant payoff is the varying speed of the car, and hence, the driver's acceleration and deceleration of the car. We studied these three factors and their correlation effecting the driving of a student in a time sensitive situation. First, using a system of differential equations and then a graph theoretic approach. We interpreted our solution systems numerically using MatLab.

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Consider that a student at a distance y from the UNG Gainesville campus starts driving his or her car. In this paper, we primarily intend to answer the following three questions:

- If t is the amount of time before the student is expected to arrive on campus, how does the student relate the speed of the car with the varying speed limits x on the different stretches of the road?
- What will be the correlation of the speed of the car when the student knows the distance to be traveled is y?
- If the number of traffic lights on the student's route is l, what will be the varying speed of the student while coming to school?

To solve this, we use a systematic problem-solving method. The first step is to translate descriptions of this phenomena and data into mathematical models. In this case, we are finding a sufficient condition for required amount of time that the car takes to reach the school, constraint to speed limit of the road, number of traffic lights and distance to be covered with time sensitive travel phenomenon.

The models obtained from the above approach are nonlinear, first order, differential equations representing pair of these conditions as in a system of equations. Then, we use theory of differential equations to obtain an implicit solution for parameter. Due to the model being nonlinear, it is very difficult to find an explicit solution for the time t. To find an approximate solution, we have suggested the Bisection method. We have produced a MatLab code to solve for the other parameters discussed in this project.

In the final section we talked about the graph theoratic interpretation of the model. We use the fact that Two graphs G_1 and G_2 are isomorphic if and only if their (Pseudo)incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows and columns. So instead of working with the entire mess around the school we can work with a much simpler model of the described traffic model around the school.

2. Game Theoretic Setup

We are considering the speed limits and number of traffic lights on a given stretch of road from point A to the UNG Gainesville Campus and trying to understand how these factors are related with the driver's motor responses while accelerating or decelerating the car.

2.1. **Speed Limit.** We assume that while driving, if the driver is aware of the remaining travel time (has to be in class by 8), then as the speed limit lowers on the road the driver has the tendency of speeding (accelerating) the car. Conversely, if the speed limit is higher there is not much urge to accelerate, but deceleration does not make any sense either, as reaching five minutes early would provide the student a five minute breather before class.

On the other hand, we believe that while driving a driver thinks of the speed limit as a function of time, because he or she is aware of the speed limit on each stretch of the route. So, if the driver is driving fast enough through a higher speed limit region the varying speed limit will have a rather intense impact on his or her driving. On the contrary if the car is being driven at 30 miles per hour in a stretch of the road where the usual speed limit is 45 miles per hour, then the driver will show the least amount of respect towards the remaining stretch of the road and the varying speed limit.

Let v be the velocity of the car at time t when the distance traveled is s. Then we get the following payoff matrix to understand the interrelation between driving speed and speed limit of a stretch of road.

	increasing speed limit	decreasing speed limit
acceleration	0	2
deceleration	-1	-2
constant	0	0

TABLE 1. Speed Limit table

	acceleration	deceleration	constant
increasing speed limit	1	2	1
decreasing speed limit	2	-4	-1

Table 2. Acceleration table

So, if x is considered as the speed limit on the stretch of the road then the expected payoff of the accelerating car verses the speed limit will be $\frac{dv}{dt} \propto 0 \cdot x + 2 \cdot x - 1 \cdot x - 2 \cdot x + 0 \cdot x + 0 \cdot x = -x$.

The following payoff chart discusses the varying speed limit (rate of change of speed limit as a function of time) verses the acceleration or deceleration of the car.

So the expected payoff of the varying speed limit versus the speed of the car will be $\frac{dx}{dt} \propto 1 \cdot v + 2 \cdot v + 1 \cdot v + 2 \cdot v - 4 \cdot v + 1 \cdot v = 3v$.

From the data collected through the GPS signals it was established that while driving at a rate of 30 miles per hour a driver tends to choose a route that will have an average of 10 miles per hour higher speed limit. So we get $\frac{dx}{dt} = 3k_1 v$ transformed into $10 = 3k_1(30)$. So $k_1 = \frac{1}{9}$. Thus we will get the differential equation $\frac{dx}{dt} = \frac{1}{3}v$. We also found, when the speed limit is 45 miles per hour the average acceleration for the car is 5 miles per square second. Similarly, the differential expression $\frac{dv}{dt} = -k_2 x$ transforms into $5 = -k_2(45)$. Hence $k_2 = -\frac{1}{9}$. Thus the second differential equation of the system will be $\frac{dv}{dt} = \frac{1}{9}x$.

3. Solutions Acceleration vs Speed Limit

The first system of equations can be written as

$$\Rightarrow \begin{bmatrix} v' \\ x' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{9} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix}$$

Let $A = \begin{bmatrix} 0 & \frac{1}{9} \\ \frac{1}{3} & 0 \end{bmatrix}$.

Solving then for the Eigen Values:

$$\begin{vmatrix} 0-\lambda & \frac{1}{9} \\ \frac{1}{3} & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda_1 = \frac{\sqrt{3}}{9} \lambda_2 = -\frac{\sqrt{3}}{9}$$

Let $\vec{w_1}$ and $\vec{w_2}$ are the two Eigen vectors corresponding to λ_1, λ_2.

$(A-\lambda_1 I)\vec{w_1}=0$

$$\Rightarrow \left(\begin{bmatrix} 0 & \frac{1}{9} \\ \frac{1}{3} & 0 \end{bmatrix} - \frac{\sqrt{3}}{9}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$-\sqrt{3}w_{11}+w_{21}=0$$

$$3w_{11}-\sqrt{3}w_{21}=0$$

Hence,

$$w_{11}=1, w_{21}=\sqrt{3}$$

$(A-\lambda_2 I)\vec{w_2}=0$

$$\Rightarrow \left(\begin{bmatrix} 0 & \frac{1}{9} \\ \frac{1}{3} & 0 \end{bmatrix} + \frac{\sqrt{3}}{9}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$\sqrt{3}w_{12}+w_{22}=0$$

$$3w_{12}+\sqrt{3}w_{22}=0$$

Hence,

$$w_{21}=1, w_{22}=-\sqrt{3}$$

So

$$\begin{cases} v(t)=c_1e^{\frac{\sqrt{3}}{9}t}+c_2e^{-\frac{\sqrt{3}}{9}t} \\ \\ x(t)=\sqrt{3}c_1e^{\frac{\sqrt{3}}{9}t}-\sqrt{3}c_2e^{-\frac{\sqrt{3}}{9}t} \end{cases} \tag{3.1}$$

Qualitative property of the solutions:

What happens when $t\to\infty$?

(1) if $c_1>0$, then $v\to\infty, x\to\infty$
(2) if $c_1<0$, then $v\to-\infty, x\to-\infty$

Asymptotic relation between v, x, looks like $\frac{v}{x}=\frac{c_1e^{\frac{\sqrt{3}}{9}t}+c_2e^{-\frac{\sqrt{3}}{9}t}}{\sqrt{3}c_1e^{\frac{\sqrt{3}}{9}t}-\sqrt{3}c_2e^{-\frac{\sqrt{3}}{9}t}}$

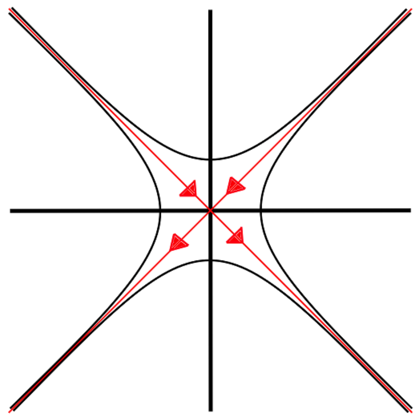
As $t\to\infty$, we have $\frac{v}{x}=\frac{c_1e^{\frac{2\sqrt{3}}{9}t}}{\sqrt{3}c_1e^{\frac{2\sqrt{3}}{9}t}}=\frac{1}{\sqrt{3}}$. This means $v\to\sqrt{3}x$.

Asymptotic relation between v, x, looks like $\frac{v}{x}=\frac{c_1e^{\frac{\sqrt{3}}{9}t}+c_2e^{-\frac{\sqrt{3}}{9}t}}{\sqrt{3}c_1e^{\frac{\sqrt{3}}{9}t}-\sqrt{3}c_2e^{-\frac{\sqrt{3}}{9}t}}$

As $t\to-\infty$, we have $\frac{v}{x}=\frac{c_2e^{\frac{-2\sqrt{3}}{9}t}}{-\sqrt{3}c_2e^{\frac{-2\sqrt{3}}{9}t}}=-\frac{1}{\sqrt{3}}$. This means $v\to-\sqrt{3}x$.

Phase portrait: Since the matrix was nonsingular $\vec{w} = \vec{0}$ is the only critical point such that $\vec{w}' = 0$.

If $c_1 = 0$, $v = -\sqrt{3}x$ is the trajectory going toward zero, and if $c_2 = 0$, $v = \sqrt{3}x$ is the trajectory going away from zero.

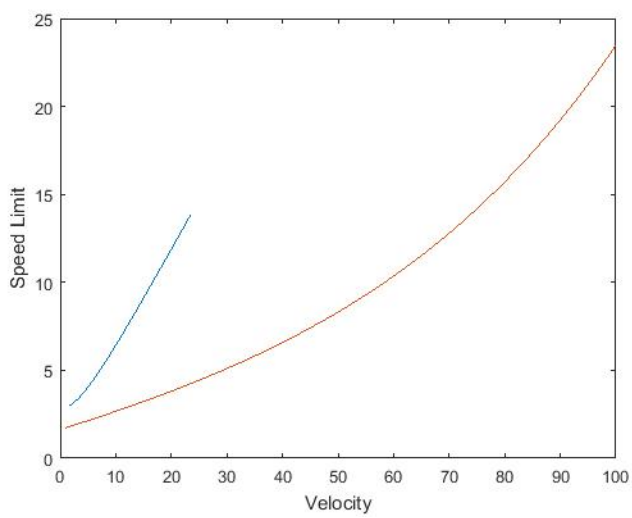


The approximate time t for the vehicle to reach 45 miles per hour after starting from rest = 16.172905 secs

The error in calculation is = 6.866455e-04

Number of steps required are = 17

A constant acceleration will require the speed limit of the road at time 16.1729 secs = 77.784341 miles.



3.1. **Distance yet to be covered.** Now we study the correspondence between the distance yet to be covered by the driver impacting the acceleration and deceleration of the car. For example, if the driver has to take a class at 10 am and at 9 : 55 am the driver finds that he or she still needs to traverse 2 more miles to reach school, then irrespective of the speed limit the car will be accelerated. But, if there is enough time to travel the remaining distance the car either maintains the speed or decelerates as required. As previously discussed there is a relation between the remaining distance to the destination and the varying speed of the car. If the destination is close enough the car is decelerated and on the other hand the driver will accelerate the car.

Let C be the distance from point A to the UNG-Gainesville campus. Since at time t distance s has already been covered, the distances yet to be covered is, $C - s$. For the time being, we are going to call, $C - s = y$.The following table represents the payoff matrix between acceleration and distance yet to be traveled.

	Higher Distance	Shorter Distance
acceleration	2	1
deceleration	-2	-1
constant	0	1

TABLE 3. Speed Limit table

So, if y is considered the remaining distance of the road then the expected payoff of the accelerating car versus the speed limit will be $\frac{dv}{dt} \propto 2{\cdot}y{+}1{\cdot}y{-}2{\cdot}y{-}1{\cdot}y{+}0{\cdot}y{+}1{\cdot}y = y$.

The following payoff chart discusses the varying distance towards the destination(rate of change in remaining distance to the destination as a function of time) versus the acceleration or deceleration of the car.

So the expected payoff of the distance remaining verses the the speed of the car will be $\frac{dy}{dt} \propto 4 \cdot v + -1 \cdot v + 2 \cdot v + 0 \cdot v + 1 \cdot v + 2 \cdot v = 8v$.

From the data collected through the GPS signals it was established on an average when the distance from the destination is 2 miles a driver tends to accelerate the car

	acceleration	deceleration	constant
Higher Distance	4	-1	2
Shorter Distance	0	1	2

TABLE 4. Acceleration table

at 10 miles per hours. So we get $\frac{dv}{dt} = k_3 y$ transformed into $10 = k_3(2)$. So $k_3 = 5$. Thus we will get the differential equation $\frac{dv}{dt} = 5y$.We also found, when the the car is running at 40 miles per hour the average distance to the school is 1 mile. Similarly, the differential expression $\frac{dy}{dt} = k_4(8v)$ transforms into $1 = k_4(320)$. Hence $k_4 = \frac{1}{320}$. Thus the second differential equation of the system will be $\frac{dy}{dt} = \frac{1}{40}v$.

So we get a system of equation:

$$\begin{cases} Acceleration: \dfrac{dv}{dt} = 5y \\ \\ Distance\ to\ be\ covered: \dfrac{dy}{dt} = \dfrac{1}{40}v \end{cases} \tag{3.2}$$

4. Solutions Acceleration vs Speed Limit

5. Solutions Acceleration vs Distance to Cover

The second system of equations can be written as

$$\Rightarrow \begin{bmatrix} v' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ \frac{1}{40} & 0 \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix}$$

Let $B = \begin{bmatrix} 0 & 5 \\ \frac{1}{40} & 0 \end{bmatrix}$.

Solving then for the Eigen Values:

$$\begin{vmatrix} 0-\beta & 5 \\ \frac{1}{40} & 0-\beta \end{vmatrix} = 0$$

$$\Rightarrow \beta_1 = \frac{1}{2\sqrt{2}} \beta_2 = -\frac{1}{2\sqrt{2}}$$

Let $\vec{u_1}$ and $\vec{u_2}$ are the two Eigen vectors corresponding to β_1, β_2.

$(B - \beta_1 I)\vec{u_1} = 0$

$$\Rightarrow \left(\begin{bmatrix} 0 & 5 \\ \frac{1}{40} & 0 \end{bmatrix} - \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$-\frac{1}{2\sqrt{2}} u_{11} + 5u_{21} = 0$$

$$\frac{1}{40} u_{11} - \frac{1}{2\sqrt{2}} u_{21} = 0$$

Hence,

$$u_{11} = 2\sqrt{2}, u_{21} = \frac{1}{5}$$

$(A - \beta_2 I)\vec{u_2} = 0$

$$\Rightarrow \left(\begin{bmatrix} 0 & 5 \\ \frac{1}{40} & 0 \end{bmatrix} + \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$\frac{1}{2\sqrt{2}} u_{12} + 5u_{22} = 0$$
$$\frac{1}{40} u_{12} + \frac{1}{2\sqrt{2}} u_{22} = 0$$

Hence,

$$u_{12} = 2\sqrt{2}, u_{22} = -\frac{1}{5}$$

So

$$\begin{cases} v(t) = 2\sqrt{2} d_1 e^{\frac{1}{2\sqrt{2}}t} + 2\sqrt{2} d_2 e^{-\frac{1}{2\sqrt{2}}t} \\ \\ y(t) = 5 d_1 e^{\frac{1}{2\sqrt{2}}t} - 5 d_2 e^{-\frac{1}{2\sqrt{2}}t} \end{cases} \tag{5.1}$$

Qualitative property of the solutions:

What happens when $t \to \infty$?

(1) if $d_1 > 0$, then $v \to \infty, y \to \infty$
(2) if $d_1 < 0$, then $v \to -\infty, y \to -\infty$

Asymptotic relation between v, y, looks like $\frac{v}{y} = \frac{2\sqrt{2} d_1 e^{\frac{1}{2\sqrt{2}}t} + 2\sqrt{2} d_2 e^{-\frac{1}{2\sqrt{2}}t}}{5 d_1 e^{\frac{1}{2\sqrt{2}}t} - 5 d_2 e^{-\frac{1}{2\sqrt{2}}t}}$

As $t \to \infty$, we have $\frac{v}{y} = \frac{2\sqrt{2} d_1 e^{\frac{1}{\sqrt{2}}t}}{5 d_1 e^{\frac{1}{\sqrt{2}}t}} = \frac{2\sqrt{2}}{5}$. This means $2\sqrt{2}v \to 5y$.

Asymptotic relation between v, y, looks like $\frac{v}{y} = \frac{2\sqrt{2} d_1 e^{\frac{1}{2\sqrt{2}}t} + 2\sqrt{2} d_2 e^{-\frac{1}{2\sqrt{2}}t}}{5 d_1 e^{\frac{1}{2\sqrt{2}}t} - 5 d_2 e^{-\frac{1}{2\sqrt{2}}t}}$

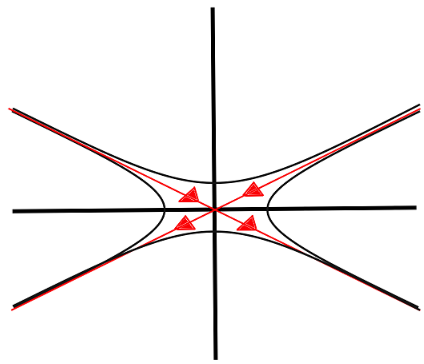
As $t \to -\infty$, we have $\frac{v}{y} = \frac{2\sqrt{2} d_2 e^{\frac{1}{\sqrt{2}}t}}{-5 d_2 e^{\frac{1}{\sqrt{2}}t}} = \frac{2\sqrt{2}}{-5}$. This means $2\sqrt{2}v \to -5y$.

Phase portrait: Since the matrix was nonsingular $\vec{u} = \vec{0}$ is the only critical point such that $\vec{u}' = 0$.

If $d_1 = 0$, $2\sqrt{2}v = -5y$ is the trajectory going toward zero, and if $d_2 = 0$, $2\sqrt{2}v = 5y$ is the trajectory going away from zero.

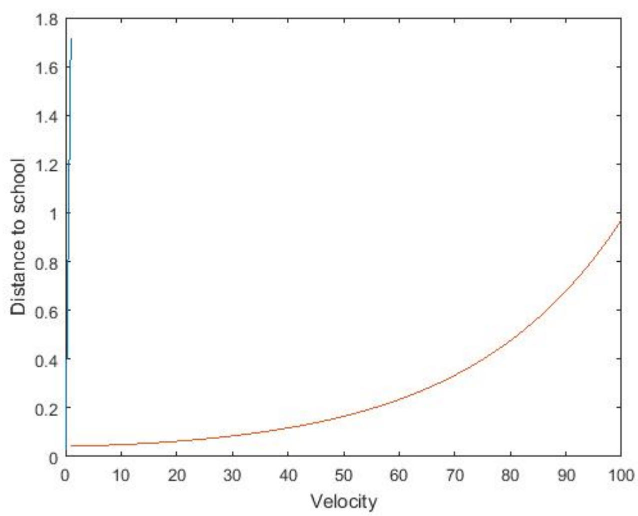
The approximate time t for the vehicle to reach school when at a distance of 1 miles from school is = 32.717628 secs

The error in calculation is = 6.866455e-04



Number of steps required are = 17

The average velocity will be 33.941629 miles/hour.



5.1. **Number of Traffic Lights.** We assume that while driving, if the driver is aware of the remaining travel time (has to be in class by 8), then as the number of remaining traffic lights decreases on the road the driver has the tendency of maintaining their speed. Conversely, if the number of remaining traffic lights increases the driver tends to accelerate.

On the other hand, we believe that while driving a driver thinks of the number of remaining traffic lights as a function of time, because he or she is aware of the traffic lights on each stretch of the route. So, if the driver is driving fast enough through a remaining stretch of road containing a high number of traffic lights, the driver will tend not to accelerate the car, as they will already be driving quickly.

On the contrary if the car is being driven at a slower rate in a stretch of the road where the number of remaining traffic lights is low, then the driver will slow down (decelerate) or maintain the same speed.

Let v be the velocity of the car at time t when the remaining number of traffic lights is l. Then we get the following payoff matrix to understand the interrelation between driving speed and number of remaining traffic lights on a stretch of road.

	high number of traffic lights	low number of traffic lights
acceleration	1	0
deceleration	-1	1
constant	1	2

Table 5. Speed Limit table

So, if l is considered as the number of remaining traffic lights on the stretch of the road then the expected payoff of the accelerating car versus the speed limit will be $\frac{dv}{dt} \propto 1 \cdot l + 0 \cdot l - 1 \cdot l + 1 \cdot l + 1 \cdot l + 2 \cdot l = 4l$.

The following payoff chart discusses the varying number of remaining traffic lights (rate of change of number of traffic lights as a function of time) versus the acceleration or deceleration of the car.

So the expected payoff of the varying number of remaining traffic lights versus the speed of the car will be $\frac{dl}{dt} \propto 2 \cdot v - 1 \cdot v + 1 \cdot v + 0 \cdot v + 1 \cdot v + 1 \cdot v = 4v$.

From the data collected through the GPS signals it was established that while there are four traffic lights remaining a driver will drive at 50 miles per hour. So we get $\frac{dl}{dt} = 4k_5 v$ transformed into $4 = 4k_5(50)$. So $k_5 = \frac{1}{50}$. Thus we will get the differential equation $\frac{dl}{dt} = \frac{2}{25}v$. We also found, when the driver is driving at 40 miles per hour, the remaining number of traffic lights is 2. Similarly, the differential expression $\frac{dv}{dt} = k_6 l$ transforms into $40 = k_6(2)$. Hence $k_6 = \frac{40}{2} = 20$. Thus the second differential equation of the system will be $\frac{dv}{dt} = 20l$.

	acceleration	deceleration	constant
high number of traffic lights	2	-1	1
low number of traffic lights	0	1	1

Table 6. Acceleration table

6. Solutions Acceleration vs Speed Limit

The first system of equations can be written as

$$\Rightarrow \begin{bmatrix} v' \\ l' \end{bmatrix} = \begin{bmatrix} 0 & 20 \\ \frac{2}{25} & 0 \end{bmatrix} \begin{bmatrix} v \\ l \end{bmatrix}$$

Let $C = \begin{bmatrix} 0 & 20 \\ \frac{2}{25} & 0 \end{bmatrix}$.

Solving then for the Eigen Values:

$$\begin{vmatrix} 0-\gamma & 20 \\ \frac{2}{25} & 0-\gamma \end{vmatrix} = 0$$

$$\Rightarrow \gamma_1 = \frac{2\sqrt{10}}{5} \gamma_2 = -\frac{2\sqrt{10}}{5}$$

Let $\vec{q_1}$ and $\vec{q_2}$ are the two Eigen vectors corresponding to γ_1, γ_2.

$(C - \gamma_1 I)\vec{q_1} = 0$

$$\Rightarrow \left(\begin{bmatrix} 0 & 20 \\ \frac{2}{25} & 0 \end{bmatrix} - \frac{2\sqrt{10}}{5} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$-2\frac{\sqrt{10}}{5} q_{11} + 20 q_{21} = 0$$

$$\frac{2}{25} q_{11} - \frac{2\sqrt{10}}{5} q_{21} = 0$$

Hence,

$$q_{11} = \frac{\sqrt{10}}{4}, q_{21} = \frac{1}{20}$$

$(C - \gamma_2 I)\vec{q_1} = 0$

$$\Rightarrow \left(\begin{bmatrix} 0 & 20 \\ \frac{2}{25} & 0 \end{bmatrix} + \frac{2\sqrt{10}}{5} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Giving us the equations:

$$\frac{2\sqrt{10}}{5} q_{12} + 20 q_{22} = 0$$

$$\frac{2}{25} q_{12} + \frac{2\sqrt{10}}{5} q_{22} = 0$$

Hence,

$$q_{12} = \frac{\sqrt{10}}{4}, q_{22} = -\frac{1}{20}$$

So

$$\begin{cases} v(t) = \frac{\sqrt{10}}{4} e_1 e^{\frac{2\sqrt{10}}{5}t} + \frac{\sqrt{10}}{4} e_2 e^{-\frac{2\sqrt{10}}{5}t} \\ \\ l(t) = \frac{1}{20} e_1 e^{\frac{2\sqrt{10}}{5}t} - \frac{1}{20} e_2 e^{-\frac{2\sqrt{10}}{5}t} \end{cases} \tag{6.1}$$

Qualitative property of the solutions:

What happens when $t \to \infty$?

(1) if $e_1 > 0$, then $v \to \infty, l \to \infty$
(2) if $e_1 < 0$, then $v \to -\infty, l \to -\infty$

Asymptotic relation between v, l, looks like $\frac{v}{l} = \frac{\frac{\sqrt{10}}{4} e_1 e^{\frac{2\sqrt{10}}{5}t} + \frac{\sqrt{10}}{4} e_2 e^{-\frac{2\sqrt{10}}{5}t}}{\frac{1}{20} e_1 e^{\frac{2\sqrt{10}}{5}t} - \frac{1}{20} e_2 e^{-\frac{2\sqrt{10}}{5}t}}$

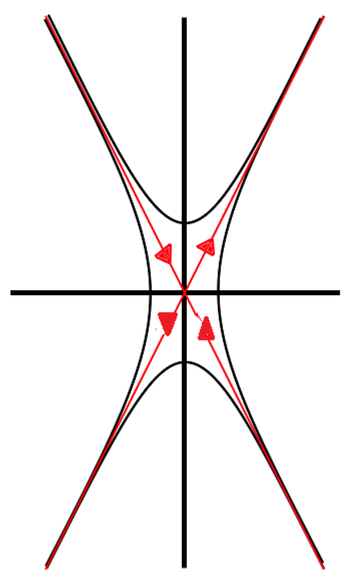
As $t \to \infty$, we have $\frac{v}{l} = \frac{\frac{\sqrt{10}}{4} e_1 e^{\frac{4\sqrt{10}}{5}t}}{\frac{1}{20} e_1 e^{\frac{4\sqrt{10}}{5}t}} = 5\sqrt{10}$. This means $5\sqrt{10}v \to l$.

Asymptotic relation between v, x, looks like $\frac{v}{l} = \frac{\frac{\sqrt{10}}{4} e_1 e^{\frac{2\sqrt{10}}{5}t} + \frac{\sqrt{10}}{4} e_2 e^{-\frac{2\sqrt{10}}{5}t}}{\frac{1}{20} e_1 e^{\frac{2\sqrt{10}}{5}t} - \frac{1}{20} e_2 e^{-\frac{2\sqrt{10}}{5}t}}$

As $t \to -\infty$, we have $\frac{v}{l} = \frac{\sqrt{10}4 e_2 e^{\frac{2\sqrt{10}}{5}t}}{-\frac{1}{20} e_2 e^{\frac{2\sqrt{10}}{5}t}} = -5\sqrt{10}$. This means $-5\sqrt{10}v \to l$.

Phase portrait: Since the matrix was nonsingular $\vec{q} = \vec{0}$ is the only critical point such that $\vec{q}' = 0$.

If $e_1 = 0$, $-5\sqrt{10}v = l$ is the trajectory going toward zero, and if $e_2 = 0$, $5\sqrt{10}v = l$ is the trajectory going away from zero.

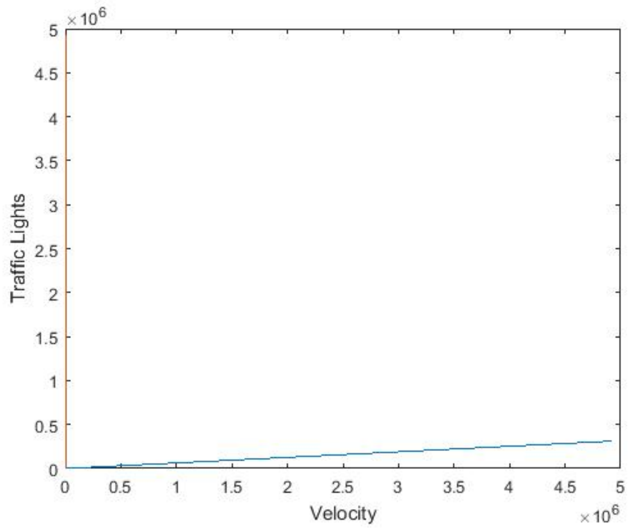


The approximate time t for the vehicle to reach school when 5 traffic lights to cross is = 1.272697 secs

The error in calculation is = 6.866455e-04

Number of steps required are = 17

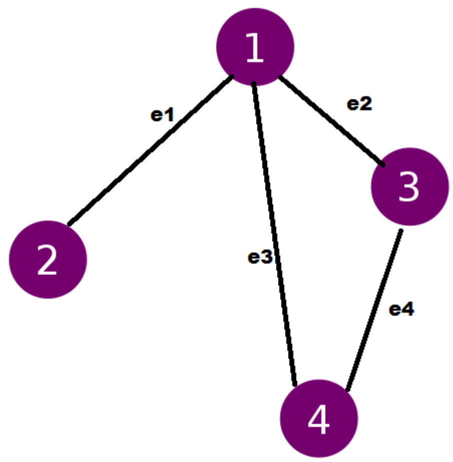
The average velocity will be 80.669943 miles/hour.



7. Graphical Interpretation

In graph theory an undirected graph has two kinds of incidence matrices: unoriented and oriented.

The unoriented incidence matrix (or simply incidence matrix) of an undirected graph is a $n \times m$ matrix B, where n and m are the numbers of vertices and edges respectively, such that $B_{i,j} = 1$ if the vertex vi and edge ej are incident and 0 otherwise.

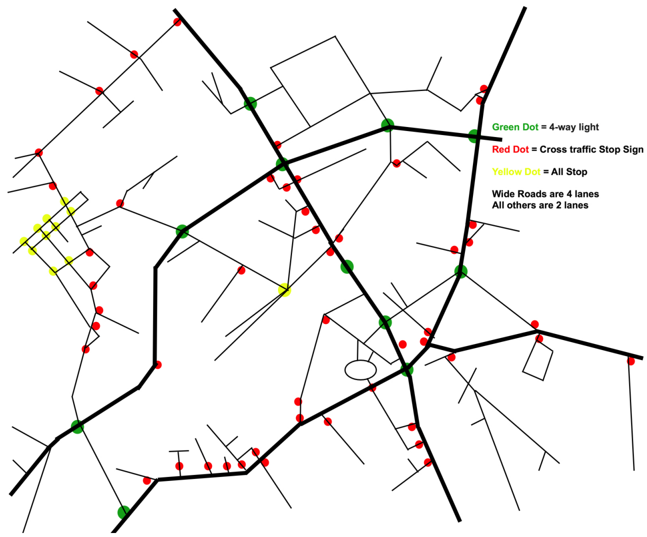


For example the incidence matrix of the undirected graph shown on the right is a matrix consisting of 4 rows (corresponding to the four vertices, 1-4) and 4 columns (corresponding to the four edges, e1-e4):

Theorem 7.1. ***Theorem (Das, Tucker):*** *Two graphs G_1 and G_2 are isomorphic if and only if their (Pseudo)incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows and columns.*

Proof. Let the graphs G_1 and G_2 be isomorphic. Then there is a one-one correspondence between the vertices and edges in G_1 and G_2 such that the incidence relation is preserved. Thus $A(G_1)$ and $A(G_2)$ are either same or differ only by permutation of rows and columns.

□



Pseudo Incident Matrix

3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0
1	4	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	2	6	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
0	0	0	2	8	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0
0	2	0	0	2	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	1	6	1	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	2	0	1	7	2	0	0	0	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	2	6	2	2	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	3	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	2	0	5	1	0	0	0	2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	6	2	0	2	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	2	5	2	0	0	0	1	0	0
0	0	0	0	0	0	2	0	0	0	0	0	2	4	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	2	0	0	5	2	0	0	1	0
0	0	0	0	0	0	0	0	0	0	2	0	0	0	2	6	2	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	5	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	3	1	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	3	0
0	0	0	2	0	0	0	2	0	1	0	0	0	0	0	0	0	0	0	5

Row Echelon form Pseudo Incident Matrix

-5.01518	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-4.39035	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-3.60649	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-3.02458	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-2.73294	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-2.12559	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-1.97905	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1.004	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-0.52005	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-0.28503	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1.29E-16	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0.410275	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1.223695	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1.631664	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1.95005	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2.528264	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.100822	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.568336	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4.612893	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5.657266

8. Numerical results

In this section, we will discuss the answer to the question *Determine the time of travel, speed limit, distance to school, number of traffic lights.* Let us recall the bisection method (c.f R. Burden, J. Faires, A. Reynolds, *Numerical Analysis*, Prindle, Weber and Schmidt, 1981.) for approximating solutions. Bisection method is the simplest among all the numerical schemes to solve the transcendental equations. This scheme is based on Intermediate Value Theorem for continuous functions.

Consider a transcendental equation $f(x) = 0$ which has a zero in the interval $[a, b]$ and that $f(a)f(b) < 0$. Bisection scheme computes the zero, say x, by repeatedly halving the interval $[a, b]$. That is, starting with $x = (a + b)/2$ the interval $[a, b]$ is replaced either with $[x, b]$ or with $[a, x]$ depending on the sign of $f(a)f(x)$. This process is continued until the zero is obtained. Since the zero is obtained numerically, the value of x may not exactly match with all the decimal places of the analytical solution of $f(x) = 0$ in the interval $[a, b]$. Hence any one of the following mechanisms can be used to stop the bisection iterations:

(1) Fixing a priori the total number of bisection iterations N.
(2) Testing the condition $|a - b| <$tol.
(3) Testing the condition $\left|f\left(\frac{a+b}{2}\right)\right| <$tol.

Algorithm 1 Bisection method

procedure BISECTION
    Input $\leftarrow a, b, \varepsilon$
Repeat:
    $x \leftarrow \frac{a+b}{2}$
    if $f(x) = 0$ **then**
        Root is x
        Stop;
    if $f(a)f(x) > 0$ **then**
        $a \leftarrow x$
        else
        $b \leftarrow x$
    Until $|b - a| < \varepsilon$.
    Print 'Approximate to root:', x
    Stop;

9. Matlab code

The first code is for the situation Velocity and Speed limit. We are running bisection code written in mat lab with accuracy of 0.001.

```
function BisectionVS(a, b, tol)
%UNTITLED Summary of this function goes here
%   Input the lower boundary as a, upper boundary as b, function and the
%   tolerance in decimal.


v = @(t) 2*exp(sqrt(3)/9*t) + exp(-sqrt(3)/9*t)-45;

x = @(t) 2*sqrt(3)*exp(sqrt(3)/9*t) - sqrt(3)*exp(-sqrt(3)/9*t);

va = feval(v,a);
vb = feval (v,b);
if va*vb>0
count=1;
fprintf('Wrong choice of a and b buddy,
given equation may have no root in interval [%f, %f].\n',a,b);
else
max1=1+round((log(abs(b-a))-log(tol))/log(2)); % take maximun iteration steps
for count= 1:max1
mid = (a+b)/2.0;
vm = feval(v,mid);
if vm*vb<0
a = mid;
else
b = mid;
end
if abs(b-a)<tol,break,end
end
format longEng
midpts=(a+b)/2;
err = abs(b-a);
```

```
vt = feval(v,midpts);
xt = feval (x,midpts);
fprintf('The approximate time t for the vehicle to reach 45 miles per hour
after starting from rest = %f\n\n secs',midpts);
fprintf('The error in calculation is = %e \n\n', err);
fprintf('Number of steps required are = %d \n\n',count);
fprintf('A constant acceleration will require the speed limit of the
road at time %12.4f secs = %f miles.\n\n', midpts, xt);

T = linspace(0,10,100);

ycoord = 2*exp(sqrt(3)/9*T) + exp(-sqrt(3)/9*T);
xcoord = 2*sqrt(3)*exp(sqrt(3)/9*T) - sqrt(3)*exp(-sqrt(3)/9*T);
set(gca,'YDir','reverse');
plot(xcoord,ycoord);
hold all
plot(xcoord);
xlabel('Velocity')
ylabel('Speed Limit')
```

Now we will introduce code is for the situation Velocity and Distance to the school. We are running bisection code written in mat lab with accuracy of 0.001.

```
function BisectionVD(a, b, tol)
%UNTITLED Summary of this function goes here
%   Input the lower boundary as a, upper boundary as b, function and the
%   tolerance in decimal.


v = @(t) 1/50*sqrt(2)*exp(1/(2*sqrt(2))*t) + 1/100*sqrt(2)*exp(-1/(2*sqrt(2))*t);

y = @(t) 1/20*exp(1/(2*sqrt(2))*t) - 1/40*exp(-1/(2*sqrt(2))*t)-1760*3;
ya = feval(y,a);
yb = feval (y,b);
if ya*yb>0
count=1;
fprintf('Wrong choice of a and b buddy,
given equation may have no root in interval [%f, %f].\n',a,b);
```

```
else
max1=1+round((log(abs(b-a))-log(tol))/log(2)); % take maximun iteration steps
for count= 1:max1
mid = (a+b)/2.0;
ym = feval(y,mid);
if ym*yb<0
a = mid;
else
b = mid;
end
if abs(b-a)<tol,break,end
end
format longEng
midpts=(a+b)/2;
err = abs(b-a);
yt = feval(y,midpts);
vt = feval (v,midpts)/1760/3*60;
fprintf('The approximate time t for the
vehicle to reach school when at a distance of 1 miles
from school is = %f secs\n\n',midpts);
fprintf('The error in calculation is = %e \n\n', err);
fprintf('Number of steps required are = %d \n\n',count);
fprintf('The average velocity will be %f miles/hour.\n\n',vt);

T = linspace(0,10,100);

ycoord = 1/20*exp(1/(2*sqrt(2))*T) - 1/40*exp(-1/(2*sqrt(2))*T);
xcoord = 1/50*sqrt(2)*exp(1/(2*sqrt(2))*T)
+ 1/100*sqrt(2)*exp(-1/(2*sqrt(2))*T);
set(gca,'YDir','reverse');
plot(xcoord,ycoord);
hold all
plot(xcoord);
xlabel('Velocity')
ylabel('Distance to school')
```

```
end
```

Our final binomial code is for the situation Velocity and Number of Traffic Lights. We are running bisection code written in mat lab with accuracy of 0.001.

```
function BisectionVL(a, b, tol)
%UNTITLED Summary of this function goes here
%   Input the lower boundary as a, upper boundary as b, function and the
%   tolerance in decimal.


v = @(t) 20*sqrt(10)/4*exp((2*sqrt(10)/5)*t)
+ 10*sqrt(10)/4*exp(-(2*sqrt(10)/5)*t);

y = @(t) 2*exp((2*sqrt(10)/5)*t) - exp((2*sqrt(10)/5)*t)-5;
ya = feval(y,a);
yb = feval (y,b);
if ya*yb>0
count=1;
fprintf('Wrong choice of a and b buddy, given equation
may have no root in interval [%f, %f].\n',a,b);
else
max1=1+round((log(abs(b-a))-log(tol))/log(2));
% take maximun iteration steps
for count= 1:max1
mid = (a+b)/2.0;
ym = feval(y,mid);
if ym*yb<0
a = mid;
else
b = mid;
end
if abs(b-a)<tol,break,end
end
format longEng
midpts=(a+b)/2;
err = abs(b-a);
yt = feval(y,midpts);
vt = feval (v,midpts);
```

```
fprintf('The approximate time t for the vehicle to reach
school when 5 traffic lights to cross is = %f secs\n\n',midpts);
fprintf('The error in calculation is = %e \n\n', err);
fprintf('Number of steps required are = %d \n\n',count);
fprintf('The average velocity will be %f miles/hour.\n\n',vt);

T = linspace(0,10,100);

ycoord = 2*exp((2*sqrt(10)/5)*T) - exp((2*sqrt(10)/5)*T);
xcoord = 20*sqrt(10)/4*exp((2*sqrt(10)/5)*T)
+ 10*sqrt(10)/4*exp(-(2*sqrt(10)/5)*T);
set(gca,'YDir','reverse');
plot(xcoord,ycoord);
hold all
plot(xcoord);
xlabel('Velocity')
ylabel('Traffic Lights')


end
```

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